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On effective mean-values of arithmetic functions

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*To Krishna Alladi,
inspired founder of
The Ramanujan Journal*

Abstract. Let r, f be multiplicative functions with $r \geqslant 0$, f is complex valued, $|f| \leqslant r$, and r satisfies some standard growth hypotheses. Let x be large, and assume that, for some real number τ, the quantities $r(p) - \Re e\,\{f(p)/p^{i\tau}\}$ are small in various appropriate average senses over the set of prime numbers not exceeding x. We derive from recent effective mean-value estimates an effective comparison theorem between the mean-values of f and of r on the set of integers $\leqslant x$. We also provide effective estimates for certain weighted moments of additive functions and for sifted mean-values of non-negative multiplicative functions.

Keywords: Quantitative estimates, comparison theorems, multiplicative functions, effective mean-value estimates, Wirsing's theorem, Hálasz' theorem, weighted distribution of additive functions, weighted moments of additive functions..

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1. Introduction

There is an abundant literature on estimates for mean-values multiplicative functions, usually appearing in the form of summatory functions

$$M(x;f) := \sum_{n\leqslant x} f(n) \qquad (x \geqslant 1).$$

One of the most useful class of results in this topic is that of *comparison theorems*, evaluating ratios $M(x;f)/M(x;r)$ where r is a majorant of $|f|$. The first general theorems of this type are due to Wirsing [10] and Halász [4], and we refer to [9] for a more complete account of the literature.

This note is devoted to describing some consequences of effective estimates of the above kind obtained in [9] and which we now state.

Let $\mathcal{M}(A,B)$ designate the class of those complex multiplicative functions f satisfying

$$\max_p |f(p)| \leqslant A, \quad \sum_{p,\,\nu\geqslant 2} \frac{|f(p^\nu)|\log p^\nu}{p^\nu} \leqslant B, \tag{1·1}$$

where, here and in the sequel, the letter p denotes a prime number. Define furthermore $w_f := 1$ if f is real, $w_f := \frac{1}{2}$ if f assumes some non real values, and write

$$Z(x;f) := \sum_{p\leqslant x} \frac{f(p)}{p}\cdot$$

The following two statements are established in [9]. Here and throughout the notation $]u,v]$ indicates a semi-open interval, including v but not u.

Theorem 1.1 ([9]). *Let*

$$\begin{aligned} &\mathfrak{a} \in]0,\tfrac{1}{2}],\ \mathfrak{b} \in [\mathfrak{a},1[,\ A \geqslant 2\mathfrak{b},\ B > 0,\ x \geqslant \mathrm{e}^4,\ \varepsilon = \varepsilon_x \in \big]1/\sqrt{\log x}, \tfrac{1}{2}\big],\\ &\varrho = \varrho_x \in [2\mathfrak{b}, A],\ \mathfrak{p} := \frac{\pi\varrho}{A},\ \beta := 1 - \frac{\sin\mathfrak{p}}{\mathfrak{p}}, \quad \mathfrak{h} := \frac{1-\mathfrak{b}}{\min(1,\varrho) - \mathfrak{b}}\cdot \end{aligned} \tag{1·2}$$

Assume that the multiplicative functions f, r, satisfy $r \in \mathcal{M}(A,B)$, $|f| \leqslant r$, and

$$\sum_{p\leqslant x} \frac{r(p) - \Re e\, f(p)}{p} \leqslant \tfrac{1}{2}\beta\mathfrak{b}\log(1/\varepsilon), \tag{1·3}$$

$$\sum_{x^\varepsilon < p \leqslant y} \frac{\{r(p) - \Re e\, f(p)\}^{\mathfrak{h}}\log p}{p} \ll \varepsilon^{\delta_1\mathfrak{h}}\log y \qquad (x^\varepsilon < y \leqslant x), \tag{1·4}$$

$$\sum_{p\leqslant y} \frac{\{r(p) - \varrho\}\log p}{p} \ll \varepsilon\log y \qquad (x^\varepsilon < y \leqslant x), \tag{1·5}$$

where $\delta_1\in]0,\frac23\beta\mathfrak{b}]$. *Then we have*

$$M(x;f)=\frac{\mathrm{e}^{-\gamma\varrho}x}{\Gamma(\varrho)\log x}\Big\{\prod_p\sum_{p^\nu\leqslant x}\frac{f(p^\nu)}{p^\nu}+O\Big(\varepsilon^\delta\,\mathrm{e}^{Z(x;f)}\Big)\Big\},\tag{1·6}$$

with $\delta:=w_f\delta_1$, *and where the implicit constant in* (1·6) *depends at most on* A, B, $\mathfrak{a}$, $\mathfrak{b}$, *and on the implicit constants in* (1·4) *and* (1·5).

As noted in [9], hypothesis (1·4) is trivially implied by the condition

$$\sum_{x^\varepsilon<p\leqslant x}\frac{\{r(p)-\Re e\,f(p)\}^{\mathfrak{h}}}{p}\ll\varepsilon^{\delta_1\mathfrak{h}},\tag{1·7}$$

and, of course, also by the uniform bound

$$\max_{x^\varepsilon<p\leqslant x}\{r(p)-\Re e\,f(p)\}\ll\varepsilon^{\delta_1}.\tag{1·8}$$

We also quote from [9] the remark that, while the assumptions of Theorem 1.1 imply

$$\prod_p\sum_{p^\nu\leqslant x}\frac{f(p^\nu)}{p^\nu}\ll\mathrm{e}^{Z(x;f)},$$

the two sides of (1·6) have the same order of magnitude if

$$\min_{p,x}\Big|\sum_{0\leqslant\nu\leqslant(\log x)/\log p}\frac{f(p^\nu)}{p^\nu}\Big|\gg1.\tag{1·9}$$

Under this condition generically satisfied, formula (1·6) becomes

$$M(x;f)=\big\{1+O(\varepsilon^\delta)\big\}\frac{\mathrm{e}^{-\gamma\varrho}x}{\Gamma(\varrho)\log x}\prod_p\sum_{p^\nu\leqslant x}\frac{f(p^\nu)}{p^\nu}.\tag{1·10}$$

For the next statement, we introduce the notation

$$\beta_0=\beta_0(\mathfrak{b},A):=1-\frac{\sin(2\pi\mathfrak{b}/A)}{2\pi\mathfrak{b}/A},\qquad\delta_0(\mathfrak{b})=\delta_0(\mathfrak{b},A):=\tfrac13\mathfrak{b}\beta_0.\tag{1·11}$$

Theorem 1.2 ([9]). *Let*

$$\mathfrak{a}\in]0,\tfrac14],\quad\mathfrak{b}\in[\mathfrak{a},\tfrac12[,\quad A\geqslant2\mathfrak{b},\quad B>0,\quad x\geqslant\mathrm{e}^4,\quad 1/\sqrt{\log x}<\varepsilon\leqslant\tfrac12.$$

Assume that the multiplicative functions f, r, *such that* $r\in\mathcal{M}(2A,B)$, $|f|\leqslant r$, *satisfy conditions* (1·3), (1·4) *with* $\mathfrak{h}:=(1-\mathfrak{b})/\mathfrak{b}$, (1·7) *with* $\mathfrak{h}=1$, *and*

$$\sum_{y<p\leqslant y^{1+\varepsilon_1}}\frac{r(p)\log p}{p}\geqslant4\mathfrak{b}\varepsilon_1\log y\qquad\big(\mathrm{e}^{1/\varepsilon_1}\leqslant y\leqslant x^{1/(1+\varepsilon_1)}\big)\tag{1·12}$$

where $\varepsilon_1:=\sqrt{\varepsilon}$. *Assume furthermore that* $\delta_1\in]0,\delta_0(\mathfrak{b})]$. *Then we have*

$$M(x;f)=M(x;r)\prod_p\frac{\sum_{p^\nu\leqslant x}f(p^\nu)/p^\nu}{\sum_{p^\nu\leqslant x}r(p^\nu)/p^\nu}+O\bigg(\frac{x\,\varepsilon^\delta\mathrm{e}^{Z(x;r)-\mathfrak{c}Z(x;|f|-f)}}{\log x}\bigg)\tag{1·13}$$

where $\delta:=w_f\delta_1$, $\mathfrak{c}:=\mathfrak{b}/A$. *Moreover, the above formula persists without requiring* (1·7) *to hold with* $\mathfrak{h}=1$ *provided* $\min_{x^\varepsilon\leqslant p\leqslant x}r(p)\geqslant4\mathfrak{b}$. *The implicit constant in* (1·13) *depends at most on* A, B, $\mathfrak{a}$, $\mathfrak{b}$, *as well as on the implicit constants in* (1·4) *and if need be in* (1·7).

The main novelty in the statements of Theorems 1.1 and 1.2 is effectivity in hypothesis (1·4), which finds its counterpart in the estimates for the remainder terms. The introduction of parameters A, B, $\mathfrak{a}$, $\mathfrak{b}$, is purely contingent and is designed to facilitate applicability.

2. An effective comparison theorem of Wirsing type

Theorem 1.2 provides an estimate for $M(x;f)/M(x;r)$ when $r(p)-\Re e\, f(p)$ is small in suitable respects. However, under the assumption that, for suitable $\varrho>0$, we have

$$\sum_{p\leqslant x}\frac{\{r(p)-\varrho\}\log p}{p}=o\big(\log x\big), \tag{2·1}$$

Wirsing's theorem [10] also provides, via partial summation, an asymptotic formula for $M(x;f)$ whenever the condition

$$\sum_{p}\frac{r(p)-\Re e\,\{f(p)/p^{i\tau}\}}{p}<\infty \tag{2·2}$$

holds for some real number τ, necessarily unique. A further extension has recently been established under condition (2·2) by Indlekofer & Kaya [6], assuming moreover that $r=r_1*r_2$, $r_1\geqslant 0$, $r_2\geqslant 0$, with r_1 satisfying (2·1), and the r_j are multiplicative functions with disjoint supports on the set of prime powers.

While, both in the classical framework and in the setting considered in [6], the transition from $\tau=0$ to general τ is obtained through simple partial integration, the deduction is not straightforward when an effective estimate is aimed at. The following result, which is a consequence of Theorem 1.2, furnishes the desired extension under the much weaker hypothesis (1·12). For given $\tau\in\mathbb{R}$, we write $f_\tau(n):=f(n)/n^{i\tau}$ $(n\geqslant 1)$ and recall notation (1·11) for $\delta_0(\mathfrak{b})$.

Theorem 2.1. *Let*

$$\mathfrak{a}\in]0,\tfrac14],\quad \mathfrak{b}\in[\mathfrak{a},\tfrac12[,\quad A\geqslant 2\mathfrak{b},\quad B>0,\quad x\geqslant \mathrm{e}^4,\quad 1/\sqrt{\log x}<\varepsilon\leqslant\tfrac12,\quad \tau\in\mathbb{R}.$$

Assume that the multiplicative functions f, r, with $r\in\mathcal{M}(2A,B)$, $|f|\leqslant r$, are such that conditions (1·3), (1·4) *with $\mathfrak{h}:=(1-\mathfrak{b})/\mathfrak{b}$, and* (1·7) *with $\mathfrak{h}=1$ are satisfied with f_τ in place of f. Suppose furthermore that* (1·12) *holds and that $\delta_1\in]0,\delta_0(\mathfrak{b})]$. Then we have*

$$M(x;f)=\frac{x^{i\tau}M(x;r)}{1+i\tau}\prod_p\frac{\sum_{p^\nu\leqslant x}f(p^\nu)/p^{\nu(1+i\tau)}}{\sum_{p^\nu\leqslant x}r(p^\nu)/p^\nu}+O\Big(\varepsilon^\delta M(x;r)\Big), \tag{2·3}$$

where $\delta:=w_f\delta_1$. Moreover, the above formula persists without requiring (1·7) *to hold with $\mathfrak{h}=1$ provided $\min_{x^\varepsilon\leqslant p\leqslant x}r(p)\geqslant 4\mathfrak{b}$. The implicit constant in* (2·3) *depends at most on A, B, $\mathfrak{a}$, $\mathfrak{b}$, τ, as well as on the implicit constants in* (1·4) *and if need be in* (1·7).

For simplicity, we have omitted in the above statement the potential sharpening of the error term involving $Z(x;|f|-f)$ and appearing in (1·13). With some extra care, it could be reinserted.

We start with a lemma potentially useful in other contexts.

Lemma 2.2. *Assume $r\in\mathcal{M}(2A,B)$, $r\geqslant 0$, and that* (1·12) *holds with ε sufficiently small. Then*

$$M(z;r)\asymp\frac{z\mathrm{e}^{Z(z;r)}}{\log z}\qquad (x^{2\varepsilon_1}<z\leqslant x). \tag{2·4}$$

Proof. The corresponding upper bound is standard and follows from [5] or [7]. To establish the lower bound, define

$$\varepsilon_2:=\varepsilon\varepsilon_1=\varepsilon^{3/2},\quad \mathcal{K}:=\left[\frac{\log(\varepsilon_2\log x)}{\log(1+\varepsilon_1)},\frac{\log_2 x}{\log(1+\varepsilon_1)}-1\right]\cap\mathbb{N},$$

and apply (1·12) with $y=y_k:=\exp\{(1+\varepsilon_1)^k\}$ for $k\in\mathcal{K}$ to get

$$\sum_{y_k<p\leqslant y_{k+1}}\frac{r(p)\log p}{p}=\mathfrak{b}_k\varepsilon_1\log y_k\qquad \big(k\in\mathcal{K}\big), \tag{2·5}$$

with

$$4\mathfrak{b}\leqslant\mathfrak{b}_k\leqslant 2A+O\Big(\mathrm{e}^{-\sqrt{\log y_k}}\Big).$$

Then define

$$(2{\cdot}6)\qquad s(p) := 2\mathfrak{b}r(p)/\mathfrak{b}_k \quad \big(y_k < p \leqslant y_{k+1},\, k \in \mathcal{K}\big), \quad s(p) := 0 \quad \Big(p \in [2, x] \smallsetminus \cup_{k\in\mathcal{K}}]y_k, y_{k+1}]\Big).$$

Thus $0 \leqslant s(p) \leqslant \frac{1}{2}r(p)$ for all $p \leqslant x$. By summation, it follows that

$$(2{\cdot}7)\qquad \sum_{p\leqslant y} \frac{\{s(p) - 2\mathfrak{b}\}\log p}{p} \ll \varepsilon_1 \log y \qquad (x^{\varepsilon} < y \leqslant x).$$

Next, define two multiplicative functions s_1 and t_1, supported on squarefree integers, by the formulae $s_1(p) := s(p)$, $t_1(p) := r(p) - s_1(p)$, and put $r_1 := s_1 * t_1$.

For $x^{\varepsilon_1} < \xi \leqslant x$, and hence $\xi^{\varepsilon_1} > x^{\varepsilon}$, we may apply Theorem 1.1 to $(\xi, s_1, s_1, 2\mathfrak{b}, \varepsilon_1, 2\delta_1)$ in place of $(x, f, r, \varrho, \varepsilon, \delta)$: indeed, we check that $2\delta_1 \leqslant 2\delta_0(\mathfrak{b}) = \frac{2}{3}\beta\mathfrak{b}$ for β defined by (1·2) with $\varrho = 2\mathfrak{b}$, and $\varepsilon_1 = \sqrt{\varepsilon} \geqslant 1/\sqrt{\log \xi}$. This yields

$$M(\xi; s_1) \asymp \frac{\xi \mathrm{e}^{Z(\xi; s_1)}}{\log \xi} \qquad (x^{\varepsilon_1} < \xi \leqslant x).$$

Moreover, Theorem 1.2 immediately yields $M(\xi; r) \asymp M(\xi; r_1)$ for the same values of ξ. Therefore, for $x^{2\varepsilon_1} < z \leqslant x$, we have

$$\begin{aligned} M(z; r) \gg M(z; r_1) &= \sum_{m\leqslant z} t_1(m) M\Big(\frac{z}{m}; s_1\Big) \\ &\gg \sum_{m\leqslant\sqrt{z}} \frac{z t_1(m) \mathrm{e}^{Z(z/m; s_1)}}{m \log(2z/m)} \asymp \frac{z \mathrm{e}^{Z(z; s_1)}}{\log z} \sum_{m\leqslant\sqrt{z}} \frac{t_1(m)}{m}\cdot \end{aligned}$$

Now a standard manipulation resting on Rankin's method—see e.g. [8; (1·4)]—yields

$$\sum_{m\leqslant\sqrt{z}} \frac{t_1(m)}{m} \asymp \mathrm{e}^{Z(z; t_1)}.$$

Appealing to the identity $Z(z; s_1) + Z(z; t_1) = Z(z; r)$ completes the proof of (2·4). □

We now embark on the proof of (2·3). Without loss of generality, we may assume ε arbitrarily small. Indeed, in the opposite circumstance the required estimate reduces to $M(x; f) \ll M(x; r)$ and hence follows from the inequality $|f| \leqslant r$.

For $x^{2\varepsilon_1} < z \leqslant x$, and hence $z^{\varepsilon_1} > x^{\varepsilon}$, we have, by the assumptions of Theorem 2.1,

$$(1{\cdot}4)'\qquad \sum_{z^{\varepsilon_1} < p \leqslant y} \frac{\{r(p) - \Re e\, f_\tau(p)\}^{\mathfrak{h}} \log p}{p} \ll \varepsilon_1^{2\delta_1\mathfrak{h}} \log y \qquad (z^{\varepsilon_1} < y \leqslant z),$$

$$(1{\cdot}7)'\qquad \sum_{z^{\varepsilon_1} < p \leqslant z} \frac{r(p) - \Re e\, f_\tau(p)}{p} \ll \varepsilon_1^{2\delta_1},$$

the latter being only necessary if the extra hypothesis on $\min_{x^{\varepsilon} < p \leqslant x} r(p)$ is not assumed. Therefore, we may apply Theorem 1.2 with $(z, r, f_\tau, \varepsilon_1, 2\delta_1)$ in place of $(x, r, f, \varepsilon, \delta_1)$. Let $L_r(x; f)$ denote the product appearing on the right-hand side of (1·13). Applying this formula for f_τ while taking (2·4) into account, we see that

$$M(z; f_\tau) = M(z; r) L_r(z; f_\tau) + O\big(\varepsilon^{\delta} M(z; r)\big) \qquad (x^{2\varepsilon_1} < z \leqslant x).$$

Hence

$$(2{\cdot}8)\qquad \begin{aligned} M(x; f) &= \int_1^x z^{i\tau}\, \mathrm{d}M(z; f_\tau) = x^{i\tau} M(x; f_\tau) - i\tau \int_1^x z^{i\tau - 1} M(z; f_\tau)\, \mathrm{d}z \\ &= x^{i\tau} M(x; r) L_r(x; f_\tau) - i\tau \int_{\varepsilon x}^x z^{i\tau-1} M(z; r) L_r(z; f_\tau)\, \mathrm{d}z + O\big(\varepsilon^{\delta} M(x; r)\big), \end{aligned}$$

where we used, for both $\xi = \varepsilon x$ and $\xi = x$, the bound

$$\int_1^\xi \frac{M(z;r)}{z}\,\mathrm{d}z \ll \int_1^\xi \frac{\mathrm{e}^{Z(z;r)}}{\log 2z}\,\mathrm{d}z \ll \frac{\xi\mathrm{e}^{Z(\xi;r)}}{\log 2\xi} \ll M(\xi;r).$$

Now, observe that, by $(1{\cdot}7)'$, we have

$$L_r(z;f_\tau) = L_r(x;f_\tau)\{1+O(\varepsilon^{\delta_1})\} \qquad (x^{2\varepsilon_1} < z \leqslant x).$$

Since $\delta = w_f\delta_1 \leqslant \delta_1$, we infer that

$$\int_{\varepsilon x}^x z^{i\tau-1}M(z;r)L_r(z;f_\tau)\,\mathrm{d}z = L_r(x;f_\tau)I(x) + O(\varepsilon^\delta M(x;r)). \tag{2·9}$$

with

$$I(x) := \int_{\varepsilon x}^x z^{i\tau-1}M(z;r)\,\mathrm{d}z. \tag{2·10}$$

At this stage, we extend s as defined on the primes in (2·6) to an exponentially multiplicative function by setting $s(p^\nu) := s(p)^\nu/\nu!$ $(\nu \geqslant 0)$ for all primes p, and define the multiplicative function t by the convolution formula $s * t = r$. For $\varepsilon x \leqslant z \leqslant x$, we write

$$M(z;r) = \sum_{m\leqslant z} t(m)M\Big(\frac{z}{m};s\Big) = S + R, \tag{2·11}$$

where S corresponds to the contribution of $m \leqslant \varepsilon z/x^{\varepsilon_1}$ and R denotes the complementary sum.

Now, observing that $M(z/m;s) = 1$ for $z/x^{\varepsilon_2} < m \leqslant z$, we have

$$\begin{aligned} R &\ll \sum_{\varepsilon z/x^{\varepsilon_1}<m\leqslant z/x^{\varepsilon_2}} \frac{z|t(m)|\mathrm{e}^{Z(z/m;s)}}{m\log(z/m)} + \sum_{z/x^{\varepsilon_2}<m\leqslant z} |t(m)| \\ &\ll \sum_{\varepsilon z/x^{\varepsilon_1}<m\leqslant z/x^{\varepsilon_2}} \frac{z|t(m)|\{\log(z/m)\}^{2\mathfrak{b}-1}}{m(\varepsilon_2\log x)^{2\mathfrak{b}}} + \frac{z\mathrm{e}^{Z(z;t)}}{\log z} \\ &\ll \int_{\varepsilon z/x^{\varepsilon_1}}^{z/x^{\varepsilon_2}} \frac{z\{\log(z/v)\}^{2\mathfrak{b}-1}}{v(\varepsilon_2\log x)^{2\mathfrak{b}}}\,\mathrm{d}O\Big(\frac{v\mathrm{e}^{Z(z;t)}}{\log z}\Big) + \frac{z\mathrm{e}^{Z(z;t)}}{\log z} \ll \frac{z(\varepsilon_1/\varepsilon_2)^{2\mathfrak{b}}\mathrm{e}^{Z(z;t)}}{\log z} \\ &\ll \frac{z(\varepsilon_1/\varepsilon_2)^{2\mathfrak{b}}\mathrm{e}^{Z(x;r)-Z(z;s)}}{\log z} \ll M(z;r)\varepsilon_1^{2\mathfrak{b}} \ll \varepsilon^\delta M(z;r), \end{aligned} \tag{2·12}$$

where we used (2·6) in the form

$$Z(z;s) = 2\mathfrak{b}\log(1/\varepsilon_2) + O(1) \qquad (\sqrt{x} \leqslant z \leqslant x), \tag{2·13}$$

together with the inequalities $\varepsilon_2^{2\mathfrak{b}} = \varepsilon^{3\mathfrak{b}} \leqslant \varepsilon^{9\delta_1} \leqslant \varepsilon^\delta$.

In the range $m \leqslant \varepsilon z/x^{\varepsilon_1}$, the bound (2·7) is a sufficient hypothesis to evaluate $M(z/m;s)$ by Theorem 1.1 with $(z/m, s, s, \varepsilon_1, 2\delta_1)$ in place of $(x, r, f, \varepsilon, \delta)$. This furnishes the estimate

$$M\Big(\frac{z}{m};s\Big) = \frac{\{1+O(\varepsilon^\delta)\}\mathrm{e}^{-2\mathfrak{b}\gamma}z}{\Gamma(2\mathfrak{b})m(\varepsilon_2\log x)^{2\mathfrak{b}}}\Big(\log\frac{z}{m}\Big)^{2\mathfrak{b}-1} \qquad \big(m \leqslant \varepsilon z/x^{\varepsilon_1},\ \varepsilon x \leqslant z \leqslant x\big), \tag{2·14}$$

where we took into account the fact that, by (2·6), we have $Z(x^{\varepsilon_2};s) = 0$.

Therefore, in view of (2·10), (2·11) and (2·12), we get

$$I(x) = \frac{\mathrm{e}^{-2\mathfrak{b}\gamma}}{\Gamma(2\mathfrak{b})}J(x) + O(\varepsilon^\delta M(x;r)),$$

with

$$\begin{aligned} J(x) &:= \int_{\varepsilon x}^x \sum_{m\leqslant \varepsilon z^{1-\varepsilon_1}} \frac{t(m)z^{i\tau}\{\log(z/m)\}^{2\mathfrak{b}-1}}{m(\varepsilon_2\log x)^{2\mathfrak{b}}}\{1+O(\varepsilon^\delta)\}\,\mathrm{d}z \\ &= \sum_{m\leqslant \varepsilon x^{1-\varepsilon_1}} \int_{\varepsilon x}^x \frac{t(m)z^{i\tau}\{\log(z/m)\}^{2\mathfrak{b}-1}}{m(\varepsilon_2\log x)^{2\mathfrak{b}}}\,\mathrm{d}z + R_1 + R_2 + R_3, \end{aligned} \tag{2·15}$$

and

$$\begin{aligned} R_1 &\ll \int_{\varepsilon x}^{x} \sum_{\varepsilon z^{1-\varepsilon_1}<m\leqslant \varepsilon x^{1-\varepsilon_1}} \frac{|t(m)|(\log z/m)^{2\mathfrak{b}-1}}{m(\varepsilon_2\log x)^{2\mathfrak{b}}}\,\mathrm{d}z \\ &\ll \frac{(\varepsilon_1\log x)^{2\mathfrak{b}-1}}{(\varepsilon_2\log x)^{2\mathfrak{b}}}\int_{\varepsilon x}^{x}\sum_{\varepsilon z^{1-\varepsilon_1}<m\leqslant \varepsilon x^{1-\varepsilon_1}}\frac{|t(m)|}{m}\,\mathrm{d}z \ll \sum_{\varepsilon^2 x^{1-\varepsilon_1}<m\leqslant \varepsilon x^{1-\varepsilon_1}}\frac{x|t(m)|(\varepsilon_1/\varepsilon_2)^{2\mathfrak{b}}}{m\varepsilon_1\log x} \\ &\ll \frac{x(\varepsilon_1/\varepsilon_2)^{2\mathfrak{b}}}{\varepsilon_1\log x}\int_{\varepsilon^2x^{1-\varepsilon_1}}^{\varepsilon x^{1-\varepsilon_1}}\frac{1}{v}\,\mathrm{d}O\Big(\frac{v\mathrm{e}^{Z(x;t)}}{\log x}\Big) \ll \frac{x(\varepsilon_1/\varepsilon_2)^{2\mathfrak{b}}\mathrm{e}^{Z(x;t)}\log(1/\varepsilon)}{\varepsilon_1(\log x)^2} \\ &\ll \frac{\varepsilon_1^{2\mathfrak{b}}\log(1/\varepsilon)x\mathrm{e}^{Z(x;r)}}{\varepsilon_1(\log x)^2} \ll \frac{\varepsilon_1^{2\mathfrak{b}}\log(1/\varepsilon)x\mathrm{e}^{Z(x;r)}}{\log x} \ll \varepsilon^{\delta}M(x;r), \\ R_2 &:= \sum_{\sqrt{x}<m\leqslant \varepsilon x^{1-\varepsilon_1}} \frac{\varepsilon^{\delta}x|t(m)|\{\log(x/m)\}^{2\mathfrak{b}-1}}{m(\varepsilon_2\log x)^{2\mathfrak{b}}} \ll \frac{\varepsilon^{\delta}x}{(\varepsilon_2\log x)^{2\mathfrak{b}}}\int_{\sqrt{x}}^{x^{1-\varepsilon_2}}\frac{(\log x/v)^{2\mathfrak{b}-1}}{v}\,\mathrm{d}O\Big(\frac{v\mathrm{e}^{Z(x;t)}}{\log x}\Big) \\ &\ll \frac{\varepsilon^{\delta}x\mathrm{e}^{Z(x;t)}}{\varepsilon_2^{2\mathfrak{b}}\log x} \ll \varepsilon^{\delta}M(x;r), \\ R_3 &\ll \sum_{m\leqslant\sqrt{x}}\frac{\varepsilon^{\delta}x|t(m)|}{m\varepsilon_2^{2\mathfrak{b}}\log x} \ll \frac{\varepsilon^{\delta}x\mathrm{e}^{Z(x;t)}}{\varepsilon_2^{2\mathfrak{b}}\log x} = \frac{\varepsilon^{\delta}x\mathrm{e}^{Z(x;r)-Z(x;s)}}{\varepsilon_2^{2\mathfrak{b}}\log x} \asymp \varepsilon^{\delta}M(x;r). \end{aligned}$$

To evaluate the main term appearing in the right-hand side of (2·15), it is sufficient to observe that, by partial summation, we have, for $m\leqslant \varepsilon x^{1-\varepsilon_1}$,

$$\begin{aligned} \int_{\varepsilon x}^{x} z^{i\tau}\Big(\log\frac{z}{m}\Big)^{2\mathfrak{b}-1}\mathrm{d}z &= \{1+O(\varepsilon)\}\frac{x^{1+i\tau}}{1+i\tau}\Big(\log\frac{x}{m}\Big)^{2\mathfrak{b}-1} + O\Big(x\Big(\log\frac{x}{m}\Big)^{2\mathfrak{b}-2}\Big) \\ &= \big\{1+O\big(\sqrt{\varepsilon}\big)\big\}\frac{x^{1+i\tau}}{1+i\tau}\Big(\log\frac{x}{m}\Big)^{2\mathfrak{b}-1}. \end{aligned}$$

Thus we can summarise our estimates as

$$\begin{aligned} I(x) &= \sum_{m\leqslant \varepsilon x^{1-\varepsilon_1}}\frac{\mathrm{e}^{-2\mathfrak{b}\gamma}t(m)x^{1+i\tau}\{\log(x/m)\}^{2\mathfrak{b}-1}}{\Gamma(2\mathfrak{b})(1+i\tau)m(\varepsilon_2\log x)^{2\mathfrak{b}}} + O\Big(\varepsilon^{\delta}M(x;r)\Big) \\ &= \sum_{m\leqslant \varepsilon x^{1-\varepsilon_1}}\frac{x^{i\tau}t(m)}{1+i\tau}M\Big(\frac{x}{m};s\Big) + O\Big(\varepsilon^{\delta}M(x;r)\Big) = \frac{x^{i\tau}M(x;r)}{1+i\tau} + O\Big(\varepsilon^{\delta}M(x;r)\Big) \end{aligned}$$

Carrying this back into (2·9) and (2·8), we obtain

$$M(x;f) = \frac{x^{i\tau}L_r(x;f_\tau)M(x;r)}{1+i\tau} + O\Big(\varepsilon^{\delta}M(x;r)\Big),$$

which coincides with (2·3).

3. Moments

Given a non-negative multiplicative function $r\in\mathcal{M}(A,B)$ and a real additive function h, let us consider the distribution function $z\mapsto F_x(z;h,r)$ of the random variable $h(n)$ on the set of integers not exceeding x equipped with the measure attributing to each integer $n\leqslant x$ the weight $r(n)/M(x;r)$, viz.

$$F_x(z;h,r) := \frac{1}{M(x;r)}\sum_{\substack{n\leqslant x\\ h(n)\leqslant z}} r(n) \qquad (z\in\mathbb{R}).$$

Put

$$E_h(x;r) := \sum_{p\leqslant x}\frac{r(p)h(p)}{p},\quad D_h(x;r)^2 := \sum_{p\leqslant x}\frac{r(p)h(p)^2}{p},$$

and denote by $\Phi(z) := \int_{-\infty}^{z}\mathrm{e}^{-u^2/2}\,\mathrm{d}u/\sqrt{2\pi}$ the distribution function of the normal law.

The following theorem is established in [9] as a corollary to Theorem 1.2. We write

$$\mu_x = \mu(x;h,r) := \max_{p\leqslant x}\frac{|h(p)|}{D_h(x;r)},\qquad \vartheta_x = \vartheta(x;h,r) := \mu_x + 1/D_h(x;r),$$

and note that $\vartheta_x\asymp\mu_x$ if $\max_{p\leqslant x}|h(p)|\gg 1$ and that $\vartheta_x = o(1)$ if $D_h(x;r)\to\infty$.

Theorem 3.1 ([9]). *Let A, B, denote positive constants. Let $x \geqslant 2$, $r \in \mathcal{M}(A, B)$, and let h be a real additive function. Assume that:*

(i) $\min_{\exp\sqrt{\log x} < p \leqslant x} r(p) \gg 1$; (ii) $D_h(x; r) \gg 1$;

(iii) $\mu_x \leqslant 1$; (iv) $\sum_{p^\nu \leqslant x} \sum_{\nu \geqslant 2} \frac{r(p^\nu)|h(p^\nu)| \log p^\nu}{p^\nu} \ll 1.$[(1)]

Then

$$F_x\Big(E_h(x; r) + zD_h(x; r); h, r\Big) = \Phi(z) + O\big(\vartheta_x\big). \tag{3·1}$$

In this section, we apply the above result to evaluate, as $x \to \infty$, the weighted moments

$$G_m(x; r, h) := \frac{1}{M(x; r)} \sum_{n \leqslant x} r(n)\big\{h(n) - E_h(x; r)\big\}^m \qquad (m \geqslant 1).$$

We denote by ν_m the mth integral moment of the normal law.

Theorem 3.2. *Let $m \geqslant 1$ and let h be a real strongly additive function. Under the hypotheses of Theorem 3.1 and assuming $\vartheta_x \leqslant \frac{1}{2}$, we have*

$$G_m(x; r, h) = \Big\{\nu_m + O\Big(\vartheta_x (\log 1/\vartheta_x)^{m/2}\Big)\Big\} D_h(x; r)^m. \tag{3·2}$$

Remark. The assumption that h is strongly additive is not essential here but it simplifies the analysis. It could be relaxed by writing $h = h_1 + h_2$, where h_1 (resp. h_2) is supported on the set of squarefree (resp. squareful) integers and making *ad hoc* hypotheses on the values $h(p^\nu)$ for $\nu \geqslant 2$.

Proof. To lighten the writing, put $E := E_h(x; r)$, $D := D_h(x; r)$. With the aim of applying (3·1), we need an upper bound for the contribution to the left-hand side of (3·2) of large values of $|h(n) - E|$. For $\sigma \in \mathbb{R}$, $|\sigma| \leqslant 1/\mu_x$, Shiu's bound [7] furnishes, in view of (2·4),

$$\begin{aligned}\sum_{n \leqslant x} r(n)\mathrm{e}^{\sigma h(n)/D} &\ll x \prod_{p \leqslant x} \Big(1 + \frac{r(p)\mathrm{e}^{\sigma h(p)/D} - 1}{p}\Big) \ll M(x; r) \exp\Big\{\sum_{p \leqslant x} \frac{r(p)\{\mathrm{e}^{\sigma h(p)/D} - 1\}}{p}\Big\} \\ &\ll M(x; r) \exp\Big\{\sum_{p \leqslant x} \frac{\sigma r(p)h(p)}{pD} + \sum_{p \leqslant x} \frac{\sigma^2 r(p)h(p)^2}{pD^2}\Big\} = M(x; r)\mathrm{e}^{\sigma E/D + \sigma^2},\end{aligned}$$

where we used the inequality $\mathrm{e}^v - 1 \leqslant v + v^2$ ($|v| \leqslant 1$). Applying this for $\sigma = \pm t$, $0 \leqslant t \leqslant 1/\mu_x$, we get

$$\sum_{n \leqslant x} r(n)\mathrm{e}^{t|h(n) - E|/D} \ll \mathrm{e}^{t^2} M(x; r),$$

whence, selecting $t := \sqrt{\log 1/\vartheta_x}$ and applying the above for $2t$,

$$\begin{aligned}\sum_{\substack{n \leqslant x \\ |h(n) - E| > 5tD}} r(n)|h(n) - E|^m &\leqslant \sum_{n \leqslant x} r(n) \frac{m! D^m}{t^m} \mathrm{e}^{2t|h(n) - E|/D - 5t^2} \\ &\ll M(x; r) D^m \mathrm{e}^{-t^2} = \vartheta_x D^m M(x; r).\end{aligned} \tag{3·3}$$

Now combining (3·3) and (3·1) yields

$$\begin{aligned}G_m(x; r, h) &= D^m \int_{-5t}^{5t} z^m \,\mathrm{d}\Big\{\Phi(z) + O\big(\vartheta_x\big)\Big\} + O\big(\vartheta_x D^m\big) \\ &= \Big\{\nu_m - \frac{1}{\sqrt{2\pi}} \int_{|z| > 5t} \mathrm{e}^{-z^2/2} \,\mathrm{d}z + O\big(t^m \vartheta_x\big)\Big\} D^m = \Big\{\nu_m + O\big(t^m \vartheta_x\big)\Big\} D^m,\end{aligned}$$

as required. □

1. Due to a misprint, the factor $\log p^\nu$ is missing in the corresponding hypothesis of [9; cor. 2.5].

4. Sifted mean-values

Given an arithmetic function f and an integer D, put $f_D(n) := \mathbf{1}_{\{(n,D)=1\}}f(n)$. Moreover, if f satisfies $\sum_{\nu\geqslant 0}|f(p^\nu)|/p^\nu < \infty$ for all primes p, define

$$W_f(n) := \prod_{p|n}\sum_{\nu\geqslant 0}\frac{f(p^\nu)}{p^\nu} \qquad (n\geqslant 1),$$

and denote by $P^+(n)$ the largest prime factor of an integer $n>1$, with the convention that $P^+(1)=1$.

In his paper [1], Elliott indicates that the following effective estimate is « provided by combining the argument of Elliott & Kish [3], [1; th. 2], with that of the taxonomy section of Elliott & Kish [2] »: *Let r denote a non-negative, exponentially multiplicative function, satisfying, for suitable positive constants A, $\mathfrak{b}$, and c_1,*

$$\sup_p r(p)\leqslant A, \qquad \sum_{w<p\leqslant v}\frac{r(p)-\mathfrak{b}}{p}\geqslant -c_1 \qquad (\tfrac32\leqslant w\leqslant v). \tag{4·1}$$

Then, uniformly for $x\geqslant 2$, $D\geqslant 1$, $P^+(D)\leqslant x$, we have

$$M(x;r_D)=M(x;r)\bigg\{\frac{1}{W_r(D)}+O\bigg(\frac{(\log_2 2D)^{1+A}}{(\log x)^{\mathfrak{c}}}\bigg)\bigg\} \tag{4·2}$$

where $\mathfrak{c} := \mathfrak{b}^3/\{\mathfrak{b}^2+3456A^2\}$ provided $\mathfrak{b}\leqslant 12\sqrt{2A}$.[2]

The above statement can be directly compared with an almost immediate consequence of Theorem 1.2. Put

$$\beta=\beta(\mathfrak{b},A):=1-\frac{\sin(\pi\mathfrak{b}/A)}{\pi\mathfrak{b}/A}, \qquad \delta=\delta(\mathfrak{b},A):=\tfrac{1}{12}\mathfrak{b}\beta, \tag{4·3}$$

and recall definition (1·1) of the class $\mathcal{M}(A,B)$. Note that, if $\mathfrak{b}\leqslant A$, we have $\beta\geqslant\mathfrak{b}^2/A^2$ by the product formula for $\sin(\pi z)/\pi z$, and hence $\delta\geqslant\mathfrak{b}^3/12A^2$.

Theorem 4.1. *Let $A>0$, $B>0$, $r\in\mathcal{M}(A,B)$, $r\geqslant 0$, $x\geqslant e^4$. Assume that, for a suitable constant $\mathfrak{b}$, $0<\mathfrak{b}\leqslant\min(1,A)$, and $\eta_x := (\log x)^{-1/4}$, we have*

$$\sum_{y<p\leqslant y^{1+\eta_x}}\frac{r(p)\log p}{p}\geqslant\mathfrak{b}\eta_x\log y \qquad \big(\mathrm{e}^{1/\eta_x}\leqslant y\leqslant x^{1/(1+\eta_x)}\big). \tag{4·4}$$

Then, uniformly for $x\geqslant 2$, $D\geqslant 1$, $P^+(D)\leqslant x$, we have

$$M(x;r_D)=M(x;r)\bigg\{\frac{1+O(\chi)}{W_r(D)}+O\Big(\frac{1}{(\log x)^{\delta/2}}\Big)\bigg\}, \tag{4·5}$$

where $\chi := \mathbf{1}_{\{\log_2 3D>(\log x)^{\mathfrak{b}^3/17A^3}\}}$.

Remarks. (i) The restriction to exponentially multiplicative functions has been dropped.

(ii) Condition (4·4) is significantly less restrictive than (4·1).

(iii) The error term of (4·5) is always smaller than that of (4·2) by a power of $\log x$.

Proof. We shall apply Theorem 1.2 to the pair $(r,f)=(r,r_D)$, replacing $(A,\mathfrak{b})$ by $(A/2,\mathfrak{b}/4)$, and selecting $\varepsilon := 1/\sqrt{\log x}$, so that $\varepsilon_1=\eta_x$.

First consider the case when $\log_2 3D\leqslant(\log x)^{\mathfrak{b}^3/17A^3}$. We then have, for large x,

$$\sum_{p|D}\frac{r(p)}{p}\leqslant A\log_3 3D+O(1)\leqslant\frac{\mathfrak{b}^3}{16A^2}\log_2 x\leqslant\tfrac18\beta\mathfrak{b}\log(1/\varepsilon).$$

2. The condition $P^+(D)\leqslant x$ is omitted in [1]. We reinserted it since it does not involve any loss of generality.

Therefore, condition (1·3) holds for our modified parameters. To check that conditions (1·4) with $\mathfrak{h} := (4-\mathfrak{b})/\mathfrak{b}$, and (1·7) with $\mathfrak{h}=1$ are also fulfilled, we note that a well known estimate provides

$$\sum_{p|D} \frac{\log p}{p} \ll \log_2 3D.$$

We hence have

$$\sum_{\substack{x^\varepsilon<p\leqslant y\\ p|D}} \frac{\log p}{p} \ll (\log x)^{\mathfrak{b}^3/16A^3} \leqslant \frac{\log y}{(\log x)^{\frac12(1-\mathfrak{b}^3/8A^3)}} \leqslant \frac{\log y}{(\log x)^{7/16}} \qquad (x^\varepsilon<y\leqslant x).$$

Since, for $\mathfrak{h} := (4-\mathfrak{b})/\mathfrak{b}$, we have $\frac{7}{16} > \frac12\delta\mathfrak{h} = \frac{1}{24}\beta(4-\mathfrak{b})$, we see that (1·4) holds for this value of $\mathfrak{h}$ and $\delta_1=\delta$.

Next,

$$\sum_{\substack{x^\varepsilon<p\leqslant x\\ p|D}} \frac{1}{p} \leqslant \sum_{p|D} \frac{\log p}{p\sqrt{\log x}} \ll \frac{1}{(\log x)^{\frac12(1-\mathfrak{b}^3/8A^3)}} \ll \varepsilon^\delta,$$

with a lot to spare, since $\delta \leqslant \frac{1}{12}$, $1-\mathfrak{b}^3/8A^3 \geqslant \frac78$. This shows that (1·7) with $\mathfrak{h}=1$ is satisfied.

Applying Theorem 1.2 with the parameters defined above, we get in the case under consideration

$$M(x;r_D) = M(x;r)\Big\{\frac{1}{W_r(D)} + O\Big(\frac{1}{(\log x)^{\delta/2}}\Big)\Big\},$$

which is compatible with (4·5).

If $\log_2 3D > (\log x)^{\mathfrak{b}^3/17A^3}$, estimate (4·5) reduces to the Halberstam-Richert upper bound [5] since, by Lemma 2.2, $M(x;r) \asymp x\mathrm{e}^{Z(x;r)}/\log x$. □

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