On simply normal numbers
with digit dependencies

Verónica Becher, Agustín Marchionna, Gérald Tenenbaum

Abstract. Given an integer $b \geqslant 2$ and a set $\mathcal{P}$ of prime numbers, the set $\mathcal{T}_{\mathcal{P}}$ of Toeplitz numbers comprises all elements of $[0, b[$ whose digits $(a_n)_{n\geqslant 1}$ in the base-b expansion satisfy $a_n = a_{pn}$ for all $p \in \mathcal{P}$ and $n \geqslant 1$. Using a completely additive arithmetical function, we construct a number in $\mathcal{T}_{\mathcal{P}}$ that is simply Borel normal if, and only if, $\sum_{p\notin\mathcal{P}} 1/p = \infty$. We then provide an effective bound for the discrepancy.

Let $\mathbb{P}$ denote the set of prime numbers, and let $\mathcal{P} \subset \mathbb{P}$. Following Jacobs and Keane's definition of Toeplitz sequences in [4], we define the set $\mathcal{T}_{\mathcal{P}}$ of *Toeplitz numbers* as the set of all real numbers $\xi \in [0, b[$ whose base-b expansion $\xi = \sum_{n\geqslant 1} a_n/b^n$ satisfies

$$a_n = a_{np} \qquad (n \geqslant 1,\, p \in \mathcal{P}).$$

For example, $0.a_1a_2a_3\dots$ is a Toeplitz number for $\mathcal{P} = \{2, 3\}$ if, for every $n \geqslant 1$, we have

$$a_n = a_{2n} = a_{3n}.$$

Then, $a_1, a_5, a_7, a_{11}, \dots$ are independent while $a_2, a_3, a_4, a_6, \dots$ are completely determined by earlier digits.

As defined by Émile Borel, a real number is called *simply normal* to the integer base $b \geqslant 2$ if every possible digit in $\mathbb{Z}/b\mathbb{Z}$ occurs in its b-ary expansion with the same asymptotic frequency $1/b$. A real number is said to be *normal* to the base b if it is simply normal to all the bases b^j, $j \geqslant 1$. Borel proved that, with respect to the Lebesgue measure, almost all numbers are normal to all integer bases at least equal to 2. For a presentation of the theory of normal numbers see for example [3, 5].

In [1], Aistleitner, Becher and Carton considered the notion of Borel normality under the assumption of dependencies between the digits of the expansion. Thus [1, th. 1] states that, given any integer base $b \geqslant 2$ and any finite subset $\mathcal{P}$ of the primes, almost all numbers, with respect to the uniform probability measure on $\mathcal{T}_{\mathcal{P}}$, are normal to the base b. In the particular case $\mathcal{P} = \{2\}$, they show [1, th. 2] that almost all elements in $\mathcal{T}_{\mathcal{P}}$ (still with respect to the uniform measure on $\mathcal{T}_{\mathcal{P}}$) are normal to all integer bases greater than or equal to 2. For $\mathcal{P} = \{2\}$, a construction of an explicit number in $\mathcal{T}_{\mathcal{P}}$ that is normal to the base 2 appears in [2]. This construction can be generalized to any integer base b and any singleton $\mathcal{P}$.

Let $\Omega_{\mathcal{P}}$ denote the completely additive arithmetical function defined by $\Omega_{\mathcal{P}}(p) = \mathbb{1}_{(\mathbb{P}\smallsetminus\mathcal{P})}(p)$. Then, $\Omega_{\mathcal{P}}(n)$ is the sum of the exponents in the canonical factorization of n of those prime

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Date: April 12, 2023.

2020 *Mathematics Subject Classification.* Primary 11K16, 11N60; Secondary 11N56.

Key words and phrases. normal numbers, Toeplitz sequences, discrepancy, additive and multiplicative functions.

factors that do *not* belong to $\mathcal{P}$. For $n \geqslant 1$ and $b \geqslant 2$, let $a_n = a_{n,b}$ denote the representative of the congruence class $\Omega_{\mathcal{P}}(n) + b\mathbb{Z}$ lying in $[0, b[$. Thus, given $b \geqslant 2$, the real number

$$\xi_{\mathcal{P}} = \sum_{n \geqslant 1} a_n / b^n \tag{1}$$

is a an element of $\mathcal{T}_{\mathcal{P}}$.

Motivated by the question posed in [1] on how to exhibit a normal number in $\mathcal{T}_{\mathcal{P}}$ for any set $\mathcal{P}$ of primes, we construct in this note simply normal numbers for arbitrary bases and a large family of sets $\mathcal{P}$.

Theorem. *Let $\mathcal{P} \subset \mathbb{P}$, $\mathcal{Q} := \mathbb{P} \smallsetminus \mathcal{P}$, and let b be an integer $\geqslant 2$. The number $\xi_{\mathcal{P}}$ is simply normal to the base b if, and only if,*

$$\sum_{p \in \mathcal{Q}} 1/p = \infty. \tag{2}$$

Moreover, defining, for $0 \leqslant k < b$,

$$\varepsilon_{N,k} := \left| \frac{1}{N} |\{1 \leqslant n \leqslant N : a_n = k\}| - \frac{1}{b} \right|, \quad E(N) := \sum_{p \leqslant N,\, p \in \mathcal{Q}} \frac{1}{p} \quad (N \geqslant 1),$$

we have

$$\varepsilon_{N,k} \ll \mathrm{e}^{-2E(N)/9b^2}. \tag{3}$$

Our proof rests on the following auxiliary result where we use the traditional notation $\mathrm{e}(u) := \mathrm{e}^{2\pi i u}$ $(u \in \mathbb{R})$.

Lemma. *Let $\mathcal{P} \subset \mathbb{P}$ and let b be an integer $\geqslant 2$. The number $\xi_{\mathcal{P}}$ is simply normal to the base b if, and only if,*

$$\frac{1}{N} \sum_{n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = o(1) \quad (a = 1, 2, \ldots b-1,\ N \to \infty). \tag{4}$$

Proof. The necessity of the criterion is clear. We show the sufficiency. Define

$$b_{k,N} = \frac{1}{N} \big|\{1 \leqslant n \leqslant N : a_n = k\}\big| \quad (0 \leqslant k < b,\ N \geqslant 1).$$

Then

$$b_{k,N} = \frac{1}{bN} \sum_{0 \leqslant a < b} \mathrm{e}(-ak/b) \sum_{1 \leqslant n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) = \frac{1}{b} + o(1) \tag{5}$$

since by (4) all inner sums with $a \neq 0$ contribute $o(N)$. □

We may now embark on the proof of the Theorem. Let

$$S(N; a/b) := \sum_{n \leqslant N} \mathrm{e}\big(a\Omega_{\mathcal{P}}(n)/b\big) \quad (a \in \mathbb{Z},\ b \geqslant 2,\ N \geqslant 1).$$

We aim at necessary and sufficient conditions that ensure $S(N, a/b) = o(N)$ as $N \to +\infty$, and seek effective upper bounds for $S(N; a/b)$ when such conditions are met.

Whenever a and b are coprime, $b \geqslant 2$ and $|a| \leqslant b/2$, we may apply [7, cor. 2.4(i)] with $r = 1$, $z = \mathrm{e}(a/b)$, $\vartheta = 2\pi a/b$ and $\kappa = 1$. Using [7, (7.4)], from which the bound [7, (2.19)] is actually derived, this yields

$$S(N; a/b) \ll N\mathrm{e}^{-2a^2E(N)/(9b^2)}.$$

So, if (2) holds, then the above lemma implies that $\xi_{\mathcal{P}}$ is simply normal to the base b. Notice that $\{a \in \mathbb{Z} : |a| \leqslant \frac{1}{2}b\}$ describes a complete set of residues $\pmod b$. The effective bound (3) is then provided by (5).

If, on the contrary, condition (4) fails, we apply [7, cor. 2.2], which is an effective version of a result of Delange—see [6, th. III.4.4]. We have

$$\sum_{p\in\mathcal{Q},\ p\leqslant N} \frac{\log p}{p} \ll \eta_N \log N \tag{6}$$

for some $\eta_N \to 0$. A possible choice is

$$\eta_N := \min_{1\leqslant z\leqslant N}\left(\frac{\log z}{\log N} + \sum_{p\in\mathcal{Q},\, p>z}\frac{1}{p}\right).$$

The validity of (6) is then obtained by bounding $\log p$ by $\log z$ if $p \leqslant z$ and by $\log N$ otherwise. That $\eta_N = o(1)$ follows by noticing that the last sum tends to 0 as $z \to \infty$. Then we get

$$S(N; a/b) = \frac{N}{\log N}\left(\prod_p \sum_{p^\nu\leqslant N} \frac{\mathrm{e}(\nu a\Omega_{\mathcal{P}}(p)/b)}{p^\nu} + O\left(\eta_N^{1/8}\mathrm{e}^{E(N)} + \frac{\mathrm{e}^{E(N)}}{(\log N)^{1/12}}\right)\right),$$

where we are picking the corresponding values from [7, cor. 2.2] as $a = 1/8$, $b = 1/12$, and $\varrho = 1$.

To prove that

$$S(N, a/b) \gg N, \tag{7}$$

it hence suffices to show that

$$\log N \ll \prod_p \sum_{p^\nu\leqslant N} \frac{\mathrm{e}(\nu a\Omega_{\mathcal{P}}(p)/b)}{p^\nu} = \prod_{p\in\mathcal{Q}} \frac{1 - \mathrm{e}(\nu_p a/b)/p^{\nu_p}}{1 - \mathrm{e}(a/b)/p} \prod_{p\in\mathcal{P}} \frac{1 - 1/p^{\nu_p}}{1 - 1/p},$$

where we have put $\nu_p := 1 + \lfloor(\log N)/\log p\rfloor$, so that $p^{\nu_p} \geqslant N$. Now the double product above is clearly

$$\sim \sigma_N := \prod_{p\leqslant N}\frac{1}{1 - 1/p}\prod_{p\in\mathcal{Q}}\frac{1 - 1/p}{1 - \mathrm{e}(a/b)/p}.$$

Since the general factor of the last product equals $1 + \{\mathrm{e}(a/b) - 1\}/p + O(1/p^2)$, we deduce from the convergence of $\sum_{p\in\mathcal{Q}} 1/p$ and Mertens' formula that $\sigma_N \sim c\log N$ for some $c \neq 0$. This yields (7) as required.

References

[1] Christoph Aistleitner, Verónica Becher, and Olivier Carton. Normal numbers with digit dependencies. *Trans. Amer. Math. Soc.*, 372(6):4425–4446, 2019.

[2] Verónica Becher, Olivier Carton, and Pablo Ariel Heiber. Finite-state independence. *Theory Comput. Syst.*, 62(7):1555–1572, 2018.

[3] Yann Bugeaud. *Distribution Modulo One and Diophantine Approximation.* Series: Cambridge Tracts in Mathematics 193. Cambridge University Press, 2012.

[4] Konrad Jacobs and Michael Keane. $0-1$-sequences of Toeplitz type. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 13:123–131, 1969.

[5] Lauwerens Kuipers and Harald Niederreiter. *Uniform distribution of sequences.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.

[6] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, third edition, 2015.

[7] Gérald Tenenbaum. Moyennes effectives de fonctions multiplicatives complexes. *The Ramanujan Journal*, 44(3):641–701, 2017. Correction in: *The Ramanujan Journal* 53:1:243–244, 2020.

Verónica Becher
Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires e ICC CONICET
Pabellón 0, Ciudad Universitaria, C1428EGA Buenos Aires, Argentina
`vbecher@dc.uba.ar`

Agustín Marchionna
Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires
Pabellón 0, Ciudad Universitaria, C1428EGA Buenos Aires, Argentina
`agusmarchionna1998@gmail.com`

Gérald Tenenbaum
Institut Élie Cartan, Université de Lorraine
BP 70239
54506 Vandœuvre-lès-Nancy Cedex France
`gerald.tenenbaum@univ-lorraine.fr`