POWER PARTITIONS AND SADDLE-POINT METHOD*

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Abstract. For $k \geqslant 1$, denote by $p_k(n)$ the number of partitions of an integer n into k-th powers. In this note, we apply the saddle-point method to provide a new proof for the well-known asymptotic expansion of $p_k(n)$. This approach turns out to significantly simplify those of Wright (1934), Vaughan (2015) and Gafni (2016).

1. Introduction

Let $p(n)$ denote, as usual, the number of unrestricted partitions of an integer n, i.e. the number of solutions to the equation

$$n = a_1 + a_2 + \cdots + a_d,$$

where $d \geqslant 1$ and the a_j are positive integers such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d \geqslant 1$. In 1918, Hardy and Ramanujan [4] proved the asymptotic formula

$$p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}\,n} \qquad (n \to \infty). \tag{1.1}$$

by using modular properties of Jacobi's Δ-function.

More generally, given an integer $k \geqslant 1$, let $p_k(n)$ denote the number of partitions of the integer n into k-th powers, i.e. the number of solutions to the equation

$$n = a_1^k + \cdots + a_d^k$$

where, as before, $d \geqslant 1$ and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d \geqslant 1$. Thus, $p_1(n) = p(n)$.

In 1918 too, Hardy and Ramanujan [4] stated without proof the asymptotic formula

$$p_k(n) \sim \frac{\mathfrak{b}_k \exp\big\{\mathfrak{c}_k n^{1/(k+1)}\big\}}{n^{(3k+1)/(2k+2)}} \qquad (n \to \infty), \tag{1.2}$$

where the constants $\mathfrak{b}_k$ and $\mathfrak{c}_k$ are defined by

$$\mathfrak{a}_k := \{k^{-1}\zeta(1+k^{-1})\Gamma(1+k^{-1})\}^{k/(k+1)}, \tag{1.3}$$

$$\mathfrak{b}_k := \frac{\mathfrak{a}_k}{(2\pi)^{(k+1)/2}\sqrt{(1+1/k)}}, \tag{1.4}$$

$$\mathfrak{c}_k := (k+1)\mathfrak{a}_k, \tag{1.5}$$

and ζ is the Riemann zeta-function. In 1934, introducing a number of complicated objects including generalised Bessel functions, Wright [9] obtained an asymptotic expansion of $p_k(n)$: for any integer $k \geqslant 1$, there is a real sequence $\{\alpha_{kj}\}_{j\geqslant 1}$ such that, for any $J \geqslant 1$, we have

$$p_k(n) = \frac{\mathfrak{b}_k \exp(\mathfrak{c}_k(n+h_k)^{1/(k+1)})}{(n+h_k)^{(3k+1)/(2k+2)}}\bigg\{1 + \sum_{1\leqslant j<J} \frac{(-1)^j\alpha_{kj}}{(n+h_k)^{j/(k+1)}} + O\bigg(\frac{1}{n^{J/(k+1)}}\bigg)\bigg\}, \tag{1.6}$$

where

$$h_k := \begin{cases} 0 & \text{if } k \text{ is even,} \\ (-1)^{(k+1)/2}(2\pi)^{-(k+1)}k!\zeta(k+1) & \text{if } k \text{ is odd,} \end{cases}$$

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and the implied constant depends at most on J and k. Apart from an explicit formula for α_{k1}, no further information was given about the α_{kj} beyond the statement that they depend only on k and j and that they "may be calculated with sufficient labour for any given values of k, j".

Of course, taking $J = 1$ in (1.6) yields an effective form of (1.2).

More recently, appealing to a relatively simple implementation of the Hardy-Littlewood circle method, Vaughan [7] obtained an explicit version of (1.6) in the case $k = 2$ and Gafni [3] generalised the argument to arbitrary, fixed k.

Gafni states her result in the following way. Let $X = X_k(n)$ denote the real solution to the equation

$$n = (\mathfrak{a}_k X)^{1+1/k} - \tfrac{1}{2}X - \tfrac{1}{2}\zeta(-k), \tag{1.7}$$

and write

$$Y = Y_k(n) := (1 + 1/k)\mathfrak{a}_k^{1+1/k} X^{1/k} - \tfrac{1}{4}. \tag{1.8}$$

Then, given any $k \geqslant 1$, there is a real sequence $\{\beta_{kj}\}_{j\geqslant 1}$ such that for any fixed $J \geqslant 1$, we have

$$p_k(n) = \frac{\exp\left\{(k+1)\mathfrak{a}_k^{1+1/k} X^{1/k} - \frac{1}{2}\right\}}{(2\pi)^{(k+1)/2} X^{3/2} Y^{1/2}} \left\{1 + \sum_{1\leqslant j<J} \frac{\beta_{kj}}{Y^j} + O\left(\frac{1}{Y^J}\right)\right\}. \tag{1.9}$$

It may be checked that the asymptotic formulae (1.6) and (1.9) match each other.

In this note, our aim is to provide a new proof of (1.2), and indeed also of (1.6) and (1.9), by applying the saddle-point method along lines very similar to those employed in [5] in the case of $p(n)$.

Our approach appears to be significantly simpler than those of the quoted previous works.

Indeed, as mentioned above, Wright's method, which provides a sharper error term, rests on the introduction of generalised Bessel functions and so may be regarded as far more difficult. Note that Vaughan [7] does motivate his study by the search of a 'relatively simple argument'.

Our method simplifies the matter further. Actually, the saddle-point method may be seen as a crude version of the circle method in which the major arcs are reduced to a single neighbourhood of one point. In the present context, it actually provides the same accuracy.

We thank the referee for pointing out to us that the Vaughan-Gafni method has been very recently generalised to enumerate other classes of partitions—see [1] and [2]. We believe that the saddle-point method could still be used in these contexts, with expected substantial simplifications in the analysis.

The constants $\mathfrak{b}_k$ and $\mathfrak{c}_k$ being defined as in (1.4) and (1.5), we can state the following.

Theorem 1. *Let $k \geqslant 1$ be a fixed integer. There is a real sequence $\{\gamma_{kj}\}_{j\geqslant 1}$ such that, for any given integer $J \geqslant 1$, we have*

$$p_k(n) = \frac{\mathfrak{b}_k \exp(\mathfrak{c}_k n^{1/(k+1)})}{n^{(3k+1)/(2k+2)}} \left\{1 + \sum_{1\leqslant j<J} \frac{\gamma_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{J/(k+1)}}\right)\right\} \tag{1.10}$$

uniformly for $n \geqslant 1$. The implied constant depends at most on J and k.

The coefficients γ_{kj} can be made explicit directly from the computations in our proof. For instance, we find that $\gamma_{k1} = -(11k^2 + 11k + 2)/(24k\mathfrak{c}_k)$ when $k \geqslant 2$, in accordance with the expression given by Wright. (It can be checked, after some computations, that it matches Gafni's formula too.) We also have

$$\gamma_{11} = -\tfrac{1}{48}\mathfrak{c}_1 - 1/\mathfrak{c}_1 = -\sqrt{\frac{2}{3}}\left(\frac{\pi}{48} + \frac{3}{2\pi}\right).$$

It may be seen that $|\gamma_{kj}|$ grows like $\Gamma(j)\mathrm{e}^{O(j)}$ and thus that the series $\sum_{j\geqslant 1} \gamma_{kj} z^j$ has radius of convergence 0.

2. Technical preparation

Define

$$F_k(s) := \sum_{n\geqslant 0} p_k(n)\mathrm{e}^{-ns} \qquad (\Re e\, s > 0), \tag{2.1}$$

so that

$$p_k(n) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\pi}^{\sigma+\mathrm{i}\pi} F_k(s)\mathrm{e}^{ns}\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(\sigma+\mathrm{i}\tau)\mathrm{e}^{n\sigma+\mathrm{i}n\tau}\,\mathrm{d}\tau. \tag{2.2}$$

According to the principles of the saddle-point method, we aim at selecting the integration abscissa σ as a solution σ_n of $-F_k'(\sigma)/F_k(\sigma) = n$. We plainly have

$$F_k(s) = \prod_{m\geqslant 1}\big(1-\mathrm{e}^{-m^k s}\big)^{-1} \qquad (\Re e\, s > 0). \tag{2.3}$$

Thus, in the same half-plane, we may define a determination of $\log F_k(s)$ by the formula

$$\Phi_k(s) := \sum_{m\geqslant 1}\log\Big(\frac{1}{1-\mathrm{e}^{-m^k s}}\Big)$$

where the complex logarithms are taken in principal branch. Expanding throughout and inverting summations, we get

$$\Phi_k(s) = \sum_{m\geqslant 1}\sum_{n\geqslant 1}\frac{\mathrm{e}^{-m^k ns}}{n} = \sum_{r\geqslant 1}\frac{w_k(r)}{r}\mathrm{e}^{-rs}, \quad -\Phi_k'(s) = \sum_{r\geqslant 1} w_k(r)\mathrm{e}^{-rs} \quad (\Re e\, s>0), \tag{2.4}$$

where

$$w_k(r) := \sum_{m^k\,|\,r} m^k \qquad (r\geqslant 1).$$

Hence $-\Phi_k'(\sigma)$ decreases from $+\infty$ to $0+$ on $(0,\infty)$, and so the equation $-\Phi_k'(\sigma) = n$ has for each integer $n\geqslant 1$ a unique real solution $\sigma_n = \sigma_n(k)$. Moreover, the sequence $\{\sigma_n\}_{n\geqslant 1}$ is decreasing and the trivial estimates $1\leqslant w_k(r)\leqslant r^2$ yield $1/n \ll \sigma_n \ll 1/\sqrt[3]{n}$.

We start with an asymptotic expansion for the derivatives $\Phi_k^{(m)}(\sigma_n)$ in terms of powers of σ_n. It turns out that all coefficients but a finite number vanish.

Lemma 2.1. *Let $J\geqslant 1$, $k\geqslant 1$. As $n\to\infty$, we have*

$$\Phi_k(\sigma_n) = \frac{k\mathfrak{a}_k^{1+1/k}}{\sigma_n^{1/k}} + \tfrac12\log\Big(\frac{\sigma_n}{(2\pi)^k}\Big) + \tfrac12\zeta(-k)\sigma_n + O\big(\sigma_n^J\big), \tag{2.5}$$

Moreover, for fixed $m\geqslant 1$,

$$(-1)^m\Phi_k^{(m)}(\sigma_n) = \frac{\mathfrak{a}_k^{1+1/k}}{\sigma_n^{m+1/k}}\prod_{1\leqslant \ell<m}\Big(\ell+\frac1k\Big) - \frac{(m-1)!}{2\sigma_n^m} - \tfrac12\delta_{1m}\zeta(-k) + O(\sigma_n^J), \tag{2.6}$$

where δ_{1m} is Kronecker's symbol.

Proof. Considering Mellin's inversion formula

$$\mathrm{e}^{-s} = \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\Gamma(z)s^{-z}\,\mathrm{d}z \qquad (\Re e\, s>0)$$

and the convolution identity

$$\sum_{r\geqslant 1}\frac{w_k(r)}{r^{1+z}} = \zeta(z+1)\zeta(kz) \qquad (\Re e\, z > 1/k), \tag{2.7}$$

we derive from the series representation (2.4) the integral formula

$$\Phi_k(s) = \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\zeta(z+1)\zeta(kz)\Gamma(z)\frac{\mathrm{d}z}{s^z}, \tag{2.8}$$

and in turn

$$(-1)^m\Phi_k^{(m)}(\sigma_n) = \frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\zeta(z+1)\zeta(kz)\Gamma(z+m)\frac{\mathrm{d}z}{\sigma_n^{z+m}} \qquad (m\geqslant 0). \tag{2.9}$$

Using the classical fact that $\zeta(z)$ has finite order in any vertical strip $a \leqslant \Re e\, z \leqslant b$ ($a, b \in \mathbb{R}$ with $a < b$), or, in other words, satisfies

$$\zeta(x + \mathrm{i}y) \ll_{a,b} 1 + |y|^A \qquad (a \leqslant x \leqslant b,\ |y| \geqslant 1),$$

for suitable $A = A(a, b)$, and invoking Stirling's formula in the form

$$|\Gamma(x + \mathrm{i}y)| = \sqrt{2\pi}|y|^{x-1/2}\mathrm{e}^{-\pi|y|/2}\big\{1 + O_{a,b}\big(1/y\big)\big\} \qquad (a \leqslant x \leqslant b,\ |y| \geqslant 1)$$

we may move the line of integration to $\Re e\, z = -J - m - \frac{1}{2}$.

The shifted integral is clearly $\ll \sigma_n^J$.

Let us first consider the case $m = 0$ in (2.9). Then the crossed singularities are a pole of order 2 at $z = 0$, and two simple poles at $z = 1/k$ and $z = -1$. Indeed, $\zeta(z + 1)\zeta(kz)$ vanishes at all negative integers $\leqslant -2$, so the corresponding zeros compensate the poles of $\Gamma(z)$ at negative integers $\leqslant -2$.

The residue at $z = 1/k$ is equal to

$$k^{-1}\zeta(1 + k^{-1})\Gamma(k^{-1})\sigma_n^{-1/k} = k\mathfrak{a}_k^{1+1/k}\sigma_n^{-1/k}.$$

The residue at $z = 0$ is the coefficient of z in the Taylor expansion of

$$\begin{aligned} z^2\zeta(z + 1)\zeta(kz)\Gamma(z)\sigma_n^{-z} &= z\zeta(z + 1)\zeta(kz)\Gamma(z + 1)\sigma_n^{-z} \\ &= (1 - \gamma z)\{\zeta(0) + k\zeta'(0)z\}(1 + \gamma z)(1 - z\log\sigma_n) + O(z^2) \\ &= \zeta(0) + \{-\zeta(0)\log\sigma_n + k\zeta'(0)\}z + O(z^2). \end{aligned}$$

Since $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, this residue equals $\frac{1}{2}\log\{\sigma_n/(2\pi)^k\}$.

The residue at $z = -1$ equals $\frac{1}{2}\zeta(-k)\sigma_n$.

This completes the proof of (2.5).

When $m = 1$, the three crossed singularities are simple poles. The residues at $z = 1/k$, $z = 0$ and $z = -1$ are respectively $(1/k)\Gamma(1 + 1/k)\zeta(1 + 1/k)\sigma_n^{-1-1/k}$, $-1/2\sigma_n$ and $-\frac{1}{2}\zeta(-k)$.

When $m \geqslant 2$, the only crossed singularities are two simple poles, at $z = 1/k$ and $z = 0$, with respective residues $(1/k)\Gamma(m + 1/k)\zeta(1 + 1/k)\sigma_n^{-m-1/k}$ and $-\frac{1}{2}(m - 1)!\sigma_n^{-m}$. This proves (2.6). □

Lemma 2.2. *Let $J \geqslant 1$, $k \geqslant 1$, $m \geqslant 1$.*

(i) *There is a real sequence $\{a_{kj}\}_{j\geqslant 1}$ with $a_{k1} = -k/(2\mathfrak{c}_k)$, $a_{k2} = k/(8\mathfrak{c}_k^2)$, such that*

$$\sigma_n = \frac{\mathfrak{a}_k}{n^{k/(k+1)}}\bigg\{1 + \sum_{1\leqslant j<J}\frac{a_{kj}}{n^{j/(k+1)}} + O\Big(\frac{1}{n^{J/(k+1)}}\Big)\bigg\} \qquad (n \to \infty). \tag{2.10}$$

(ii) *There is a real sequence $\{b_{kj}\}_{j\geqslant 1}$ with $b_{k1} = -a_{k1}/k$ such that, as $n \to \infty$, we have*

$$\Phi_k(\sigma_n) = k\mathfrak{a}_k n^{1/(k+1)}\bigg\{1 + \sum_{1\leqslant j<J}\frac{b_{kj}}{n^{j/(k+1)}} + O\Big(\frac{1}{n^{J/(k+1)}}\Big)\bigg\} + \tfrac{1}{2}\log\Big(\frac{\sigma_n}{(2\pi)^k}\Big). \tag{2.11}$$

(iii) *There is a real sequence $\{b_{kmj}\}_{j\geqslant 1}$ such that, as $n \to \infty$, we have*

$$\begin{aligned} &(-1)^m\Phi_k^{(m)}(\sigma_n) + \tfrac{1}{2}\delta_{1m}\zeta(-k) \\ &\qquad = \frac{n^{(mk+1)/(k+1)}}{\mathfrak{a}_k^{m-1}}\prod_{1\leqslant \ell<m}\Big(\ell + \frac{1}{k}\Big)\bigg\{1 + \sum_{1\leqslant j<J}\frac{b_{kmj}}{n^{j/(k+1)}} + O\Big(\frac{1}{n^{J/(k+1)}}\Big)\bigg\}. \end{aligned} \tag{2.12}$$

Proof. We infer from (2.6) that

$$n = \frac{\mathfrak{a}_k^{1+1/k}}{\sigma_n^{1+1/k}} - \frac{1}{2\sigma_n} - \tfrac{1}{2}\zeta(-k) + O(\sigma_n^J). \tag{2.13}$$

This immediately implies (2.10) by Lagrange's inversion formula — see, e.g. [8, §7.32]. We may obtain an explicit expression for the a_{kj} from the formula

$$\sigma_n = \frac{\mathfrak{a}_k}{2\pi\mathrm{i}\, n^{k/(k+1)}}\oint_{|z-1|=\varrho}\frac{zG'(z)}{G(z)}\,\mathrm{d}z + O\Big(\frac{1}{n^{(k+J)/(k+1)}}\Big) \tag{2.14}$$

where ϱ is a fixed, small positive constant and

$$G(z) := z^{-1-1/k} - 1 - \frac{1}{2\mathfrak{a}_k z n^{1/(k+1)}} - \frac{\zeta(-k)}{2n}\cdot$$

This is classically derived from Rouché's theorem and we omit the details. The values of a_{k1} and a_{k2} may be retrieved from the above or by formally inserting (2.10) into (2.13).

Inserting (2.14) back into (2.5) and (2.6) immediately yields (2.11) and (2.12). □

With the aim of applying Laplace's method to evaluate the integral on the right-hand side of (2.2), we need to show that it is dominated by a small neighbourhood of the saddle-point σ_n. The next result meets this requirement. Here and in the sequel, all constants c_j ($j \geqslant 0$) are assumed, unless otherwise stated, to depend at most upon k.

Lemma 2.3. *We have*

$$\frac{|F_k(\sigma_n + \mathrm{i}\tau)|}{|F_k(\sigma_n)|} \leqslant \begin{cases} \mathrm{e}^{-c_1\tau^2\sigma_n^{-(2+1/k)}} & \text{if } |\tau| \leqslant 2\pi\sigma_n, \\ \mathrm{e}^{-c_2\sigma_n^{-1/k}} & \text{if } 2\pi\sigma_n < |\tau| \leqslant \pi. \end{cases} \tag{2.15}$$

Proof. Noticing that

$$\begin{aligned}\big|1 - \mathrm{e}^{-m^k(\sigma_n+\mathrm{i}\tau)}\big|^2 &= \big|1-\mathrm{e}^{-m^k\sigma_n}\big|^2 + 4\mathrm{e}^{-m^k\sigma_n}\sin^2(\tfrac12 m^k\tau)\\ &\geqslant \big|1-\mathrm{e}^{-m^k\sigma_n}\big|^2 + 16\mathrm{e}^{-m^k\sigma_n}\|m^k\tau/(2\pi)\|^2,\end{aligned}$$

we can write

$$\begin{aligned}\frac{|F_k(\sigma_n+\mathrm{i}\tau)|^2}{|F_k(\sigma_n)|^2} &\leqslant \prod_{m\geqslant 1}\left(1 + \frac{16\|m^k\tau/(2\pi)\|^2}{\mathrm{e}^{m^k\sigma_n}(1-\mathrm{e}^{-m^k\sigma_n})^2}\right)^{-1}\\ &\leqslant \prod_{(4\sigma_n)^{-1/k}<m\leqslant(2\sigma_n)^{-1/k}}\left(1 + \frac{16\|m^k\tau/(2\pi)\|^2}{\mathrm{e}^{m^k\sigma_n}(1-\mathrm{e}^{-m^k\sigma_n})^2}\right)^{-1}.\end{aligned}$$

Thus, there is an absolute positive constant c_3 such that

$$\frac{|F_k(\sigma_n+\mathrm{i}\tau)|}{|F_k(\sigma_n)|} \leqslant \mathrm{e}^{-c_3 S(\tau;\sigma_n)} \tag{2.16}$$

with

$$S(\tau;\sigma_n) := \sum_{m\in I}\|m^k\tau/(2\pi)\|^2.$$

where we have put $I := \big](4\sigma_n)^{-1/k}, (2\sigma_n)^{-1/k}\big]$.

If $|\tau| \leqslant 2\pi\sigma_n$ and $m \in I$, we have $|m^k\tau/(2\pi)| \leqslant \frac12$. Thus

$$S(\tau;\sigma_n) = \sum_{(4\sigma_n)^{-1/k}<m\leqslant(2\sigma_n)^{-1/k}} m^{2k}\tau^2/4\pi^2 \asymp \tau^2\sigma_n^{-(2+1/k)}. \tag{2.17}$$

When $2\pi\sigma_n < |\tau| \leqslant 2\pi\sigma_n^{1-1/3k}$, we proceed similarly, noting that for any integer h with $|\tau|/4\sigma_n < 2\pi h \leqslant |\tau|/2\sigma_n$ there are $\gg (1/\sigma_n)^{1/k}/h$ integers m in I such that

$$\tfrac14 \leqslant |m^k\tau/(2\pi) - h| \leqslant \tfrac12.$$

This yields the required estimate

$$S(\tau;\sigma_n) \gg \sigma_n^{-1/k}. \tag{2.18}$$

When $2\pi(\sigma_n)^{1-1/3k} < |\tau| \leqslant \pi$, we note that (2.18) follows, via the Cauchy-Schwarz inequality, from the inequalities

$$\sum_{m\in I}\|\tau m^k/(2\pi)\| \geqslant \sum_{m\in I}|1-\mathrm{e}^{\mathrm{i}\tau m^k}| \geqslant |I| - \Big|\sum_{m\in I}\mathrm{e}^{\mathrm{i}\tau m^k}\Big|$$

provided we can show the modulus of the last exponential sum is, say, $\leqslant (1-c)|I|$ for some positive constant c depending at most upon k. Now Dirichlet's approximation lemma guarantees that there exist integers $a \in \mathbb{Z}^*$ and q, with $1 < q \leqslant (1/\sigma_n)^{1-1/3k}$, $(a,q)=1$,

$$|\tau/(2\pi) - a/q| \leqslant (\sigma_n)^{1-1/3k}/q.$$

If $q \leqslant (1/\sigma_n)^{1/3k}$, we readily deduce the required estimate from [6, Lemma 2.7 & Theorem 4.2]. If $(1/\sigma_n)^{1/3k} < q \leqslant (1/\sigma_n)^{1-1/3k}$, we may apply Weyl's inequality, as stated for instance in [6, Lemma 2.4], to get that the exponential sum under consideration is $\ll |I|^{1-\varepsilon_k}$ for some positive ε_k depending only on k. Hence, (2.18) holds in all circumstances. □

3. Completion of the proof

Proposition 3.1. *Let $k \geqslant 1$, $J \geqslant 1$. Then there is a real sequence $\{e_{kj}\}_{j\geqslant 1}$ such that for any integer $J \geqslant 1$ we have*

$$p_k(n) = \frac{\exp(n\sigma_n + \Phi_k(\sigma_n))}{\sqrt{2\pi\Phi_k''(\sigma_n)}}\bigg\{1 + \sum_{2\leqslant j<J} \frac{e_{kj}}{n^{j/(k+1)}} + O\bigg(\frac{1}{n^{J/(k+1)}}\bigg)\bigg\} \qquad (n\to\infty). \tag{3.1}$$

Proof. By (2.2), we have

$$p_k(n) = \frac{\mathrm{e}^{n\sigma_n}}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\Phi_k(\sigma_n+\mathrm{i}\tau)+\mathrm{i}n\tau}\,\mathrm{d}\tau. \tag{3.2}$$

From (2.15), we deduce that

$$\begin{aligned} \int_{2\pi\sigma_n<|\tau|\leqslant\pi} \mathrm{e}^{\Phi_k(\sigma_n+\mathrm{i}\tau)+\mathrm{i}n\tau}\,\mathrm{d}\tau &\ll \mathrm{e}^{\Phi_k(\sigma_n)-c_4\sigma_n^{-1/k}} \\ \int_{\sigma_n^{1+1/3k}<|\tau|\leqslant 2\pi\sigma_n} \mathrm{e}^{\Phi_k(\sigma_n+\mathrm{i}\tau)+\mathrm{i}n\tau}\,\mathrm{d}\tau &\ll \mathrm{e}^{\Phi_k(\sigma_n)-c_4\sigma_n^{-1/3k}}. \end{aligned} \tag{3.3}$$

Since these bounds are exponentially small with respect to the expected main term, it only remains to estimate the contribution of the interval $\mathfrak{I} :=]-\sigma_n^{1+1/(3k)}, \sigma_n^{1+1/(3k)}[$, corresponding to a small neighbourhood of the saddle-point.

In this range, we have

$$\Phi_k(\sigma_n+\mathrm{i}\tau) = \sum_{0\leqslant m\leqslant 2J+1} \frac{\Phi_k^{(m)}(\sigma_n)}{m!}(\mathrm{i}\tau)^m + O\bigg(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\bigg),$$

where the estimate for the error term follows from (2.6). The same formula ensures that $|\Phi_k^{(m)}(\sigma_n)\tau^m| \ll 1$ for $m \geqslant 3$. Thus for $\tau\in\mathfrak{I}$, we can write

$$\begin{aligned} &\mathrm{e}^{\Phi_k(\sigma_n+\mathrm{i}\tau)+\mathrm{i}n\tau} \\ &= \mathrm{e}^{\Phi_k(\sigma_n)-\frac12\Phi_k''(\sigma_n)\tau^2}\bigg\{1 + \sum_{1\leqslant\ell\leqslant 2J}\frac{1}{\ell!}\bigg(\sum_{3\leqslant m\leqslant 2J+1}\frac{\Phi_k^{(m)}(\sigma_n)}{m!}(\mathrm{i}\tau)^m\bigg)^\ell + O\bigg(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\bigg)\bigg\} \\ &= \mathrm{e}^{\Phi_k(\sigma_n)-\frac12\Phi_k''(\sigma_n)\tau^2}\bigg\{1 + \sum_{1\leqslant\ell\leqslant 2J}\frac{1}{\ell!}\sum_{3\ell\leqslant m\leqslant(2J+1)\ell}\lambda_{k,\ell,m}(n)\tau^m + O\bigg(\frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}}\bigg)\bigg\}, \end{aligned}$$

where

$$\lambda_{k,\ell,m}(n) := \mathrm{i}^m \sum_{\substack{3\leqslant m_1,\dots,m_\ell\leqslant 2J+1\\ m_1+\cdots+m_\ell=m}} \prod_{1\leqslant r\leqslant\ell}\frac{\Phi_k^{(m_r)}(\sigma_n)}{m_r!}. \tag{3.4}$$

Since the contributions from odd powers of τ vanish, we get

$$\int_{\mathfrak{I}} \mathrm{e}^{\Phi_k(\sigma_n+\mathrm{i}\tau)+\mathrm{i}n\tau}\,\mathrm{d}\tau = \mathrm{e}^{\Phi_k(\sigma_n)}\bigg\{I_0 + \sum_{1\leqslant\ell\leqslant 2J}\frac{1}{\ell!}\sum_{3\ell\leqslant 2m\leqslant(2J+1)\ell}\lambda_{k,\ell,2m}(n)I_m + O(R)\bigg\}, \tag{3.5}$$

with

$$I_m := \int_{\mathfrak{I}} \mathrm{e}^{-\frac{1}{2}\Phi_k''(\sigma_n)\tau^2}\tau^{2m}\,\mathrm{d}\tau, \qquad R := \sigma_n^{-1/k-2J-2}\int_{\mathfrak{I}} \mathrm{e}^{-\frac{1}{2}\Phi_k''(\sigma_n)\tau^2}\tau^{2J+2}\,\mathrm{d}\tau.$$

Extending the range of integration in I_m involves an exponentially small error, so we get from the classical formula for Laplace integrals

$$I_m = \frac{\sqrt{2\pi}(2m)!}{m!2^m\Phi_k''(\sigma_n)^{m+1/2}} + O\Big(\mathrm{e}^{-c_5 n^{1/(3k+3)}}\Big), \quad R \asymp \sigma_n^{1+(J+1/2)/k} \asymp \frac{1}{\sqrt{\Phi_k''(\sigma_n)}n^{J/(k+1)}}.$$

Inserting these estimates back into (3.5) and expanding all arising factors $\Phi_k^{(m)}(\sigma_n)$ by (2.12), we obtain (3.1). □

Remark. From (3.3) and (3.5) we see that, when $k \geqslant 2$,

$$p_k(n) = \frac{\mathrm{e}^{n\sigma_n+\Phi_k(\sigma_n)}}{\sqrt{2\pi\Phi_k''(\sigma_n)}}\Big\{1 - \frac{2k^2+5k+2}{24k\mathfrak{c}_k}\Big(\frac{\sigma_n}{\mathfrak{a}_k}\Big)^{1/k} + O\Big(\sigma_n^{2/k}\Big)\Big\} \tag{3.6}$$

where, in view of (2.6), the quantity inside curly brackets may be replaced by an asymptotic series in powers of $\sigma_n^{1/k}$. Inserting (2.5) and (2.10) in the main term, we thus get a formula which is very close to, but simpler than (1.9), since it follows from (2.13) that X and $1/\sigma_n$ agree to any power of σ_n.

We are now in a position to complete the proof of Theorem 1.

We infer from (2.10) and (2.11) that

$$n\sigma_n + \Phi_k(\sigma_n) = \mathfrak{c}_k n^{1/(k+1)} + \sum_{1\leqslant j<J}\frac{a_{kj}^*}{n^{j/(k+1)}} + O\Big(\frac{1}{n^{J/(k+1)}}\Big) + \tfrac{1}{2}\log\Big(\frac{\sigma_n}{(2\pi)^k}\Big)$$

with $a_{kj}^* := \mathfrak{a}_k(ka_{k,j+1} + b_{k,j+1})$. Exponentiating and expanding, we get

$$\begin{aligned}\exp(n\sigma_n + \Phi(\sigma_n)) &\\ &= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}}\exp\big(\mathfrak{c}_k n^{1/(k+1)}\big)\Big\{1 + \sum_{1\leqslant\ell<J}\frac{1}{\ell!}\Big(\sum_{1\leqslant j<J}\frac{a_{kj}^*}{n^{j/(k+1)}}\Big)^{\ell} + O\Big(\frac{1}{n^{J/(k+1)}}\Big)\Big\}\\ &= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}}\exp\big(\mathfrak{c}_k n^{1/(k+1)}\big)\Big\{1 + \sum_{1\leqslant j<J}\frac{f_{kj}}{n^{j/(k+1)}} + O\Big(\frac{1}{n^{J/(k+1)}}\Big)\Big\}\end{aligned} \tag{3.7}$$

with

$$f_{kj} := \sum_{1\leqslant\ell<J}\frac{1}{\ell!}\sum_{\substack{1\leqslant j_1,\dots,j_\ell<J\\ j_1+\cdots+j_\ell=j}} a_{kj_1}^*\cdots a_{kj_\ell}^*.$$

It remains to insert back into (3.1) and expand $\sqrt{\sigma_n/\Phi_k''(\sigma_n)}$ according to (2.10) and (2.12) with $m=2$ to obtain the required asymptotic formula.

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