On integral boxes of minimal surface

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1 Introduction and statements of results

Given an integer $k \geqslant 2$, consider for each integer n a k-dimensional box with integral edges and volume n. How should we select the lengths $d_1, \ldots, d_k$ of the edges so that the surface of the box is minimal? Since the surface of a k-dimensional box with edges $d_1, \ldots, d_k$ is given by

$$\sigma(d_1, \ldots, d_k) := 2 \sum_{1 \leqslant h \leqslant k} \prod_{\substack{1 \leqslant m \leqslant k \\ m \neq h}} d_m \quad (k \geqslant 2),$$

it is clear that any sequence $\{\varrho_h(n)\}_{1 \leqslant h \leqslant k} = \{\varrho_{k,h}(n)\}_{1 \leqslant h \leqslant k}$ realizing this minimum and arranged in increasing order is a solution of the optimisation problem

$$(P_{n,k}) \quad \begin{cases} \displaystyle\prod_{1 \leqslant h \leqslant k} \varrho_h(n) = n, \\ \varrho_1(n) \leqslant \varrho_2(n) \leqslant \cdots \leqslant \varrho_k(n), \\ \displaystyle\sum_{1 \leqslant h \leqslant k} \frac{1}{\varrho_h(n)} = \min_{d_1 \ldots d_k = n} \sum_{1 \leqslant h \leqslant k} \frac{1}{d_h} \cdot \end{cases}$$

This setting generalizes the case $k = 2$ introduced by the second author in [6]. In this case we have

$$\varrho_1(n) = \max\{d | n : d \leqslant \sqrt{n}\}, \quad \varrho_2(n) = \min\{d | n : d \geqslant \sqrt{n}\}, \tag{1.1}$$

so the pair $(\varrho_1(n), \varrho_2(n))$ is unique. We note in passing that uniqueness of the solution to problem $(P_{n,k})$ is not necessarily granted in the general case. We leave this question open for the time being and merely indicate that numerical tests show that uniqueness holds for $k \in \{3, 4, 5\}$ and $n \leqslant 10^8$.

Estimates for the mean-values of $\varrho_{2,1}(n)$ and $\varrho_{2,2}(n)$ were given in [6]. The purpose of this note is to provide corresponding estimates in the general case of dimension k. This last parameter will be fixed throughout the paper, consequently we shall omit to indicate dependence upon it.

We shall see that evaluating the average of ϱ_j is significantly more delicate for $j = 1$ than for $j \geqslant 2$. By [6, th. 2], we have

$$\sum_{n \leqslant x} \varrho_{2,2}(n) = \frac{\pi^2 x^2}{12 \log x} \left\{ 1 + O\left(\frac{1}{\log x} \right) \right\} \quad (x \geqslant 2). \tag{1.2}$$

Improving upon estimates established in [6, th. 1] and (implicitly) in [7], Ford [1, cor. 6] showed that

$$\sum_{n \leqslant x} \varrho_{2,1}(n) \asymp \frac{x^{3/2}}{(\log x)^{\delta} (\log_2 x)^{3/2}} \quad (x \geqslant 3), \tag{1.3}$$

with

$$\delta := 1 - \frac{1 + \log_2 2}{\log 2} \approx 0.086071. \tag{1.4}$$

Here and throughout, we write $\log_j$ for the jth iterated logarithm.

De Koninck and Razafindrasoanaivolala ([2],[3]) estimated the mean-values of the ratios $\varrho_{2,1}(n)/\varrho_{2,2}(n)$ and $\varrho_{2,2}(n)/\varrho_{2,1}(n)^r$, $r > -1$, and also of ratios involving the logarithms of these functions.

A very recent result of Haddad [4] provides the formula

$$\sum_{n\leqslant x} \log \varrho_{2,1}(n) = cx\log x + O(x)$$

for a suitable constant $c \in]0, \frac{1}{2}]$.

We state our results as the two following theorems. We denote Riemann's zeta function by $\zeta(s)$. We also put

$$Q(v) := v\log v - v + 1 \quad (v > 0), \qquad \delta_k := Q\Big(\frac{k-1}{\log k}\Big) \quad (k \geqslant 2). \tag{1.5}$$

Theorem 1.1. *If the sequence* $\{\varrho_h(n)\}_{h=1}^k$ *satisfies* $(P_{n,k})$, *we have*

$$\sum_{n\leqslant x} \varrho_1(n) \asymp_k \frac{x^{1+1/k}}{(\log x)^{\delta_k}(\log_2 x)^{3/2}} \quad (x \geqslant 3). \tag{1.6}$$

Theorem 1.2. *Let* $2 \leqslant j \leqslant k$ *and* $\gamma_j := k+1-j$. *If the sequence* $\{\varrho_h(n)\}_{h=1}^k$ *satisfies* $(P_{n,k})$, *we have*

$$\sum_{n\leqslant x} \varrho_j(n) = \frac{x^{1+1/\gamma_j}}{(\log x)^{\gamma_j}}\bigg\{\frac{\gamma_j^{2\gamma_j}}{(\gamma_j+1)!}\zeta\Big(1+\frac{1}{\gamma_j}\Big) + O\Big(\frac{1}{\log x}\Big)\bigg\} \quad (x \geqslant 2). \tag{1.7}$$

Remarks. (i) The upper bound in (1.6) is actually valid if $\varrho_1(n)$ is replaced by the smallest divisor d_1 in any representation $n = d_1 \cdots d_k$.

(ii) Formulae (1.6) and (1.7) hold for any solution to $(P_{n,k})$ and hence are unaffected by the fact that uniqueness of solution has not been established for $k \geqslant 3$.

(iii) The trivial bound $\varrho_j(n)^{k+1-j} \leqslant n$ readily implies

$$\sum_{n\leqslant x} \varrho_j(n) \leqslant \frac{2x^{1+1/\gamma_j}}{1+\gamma_j} \quad (1 \leqslant j \leqslant k).$$

(iv) Generalising an observation made in [6] for the case $k = 2$, we shall see that the average of $\varrho_j(n)$ is dominated by integers with exactly $k+1-j$ "large" prime factors, in a sense to be made precise later.

(v) For $k = j = 2$, (1.7) coincides with (1.2).

(vi) For $2 \leqslant j \leqslant k$ and $0 \leqslant h \leqslant j-2$, we have

$$\sum_{n\leqslant x} \varrho_{k-h,j-h}(n) \sim \sum_{n\leqslant x} \varrho_{k,j}(n) \qquad (x \to \infty).$$

2 A lemma

For any $\boldsymbol{v} = (v_1, \dots, v_k) \in (\mathbb{N}^*)^k$, set

$$S(\boldsymbol{v}) := \sum_{1\leqslant j\leqslant k} \frac{1}{v_j},$$

and let us equip

$$E_n := \{\boldsymbol{v} \in (\mathbb{N}^*)^k : v_1 \leqslant \cdots \leqslant v_k,\ v_1v_2\cdots v_k = n\},$$

with the total preorder relation $\preccurlyeq$ defined by

$$\boldsymbol{v} \preccurlyeq \boldsymbol{w} \Leftrightarrow S(\boldsymbol{v}) \leqslant S(\boldsymbol{w}) \quad (\boldsymbol{v}, \boldsymbol{w} \in E_n).$$

The following result provides a necessary condition for optimality of a k-tuple. We denote by $P^-(n)$ the smallest prime factor of an integer $n > 1$ and make the standard convention $P^-(1) = +\infty$.

Lemma 2.1. *Let* $n \geqslant 1$ *and* $\boldsymbol{v} \in E_n$. *If* $v_j P^-(v_h) < v_h$ *for some* $1 \leqslant j < h \leqslant k$, *then the ordered* k*-tuple* $\boldsymbol{w}$ *obtained from* $\boldsymbol{v}$ *on replacing* v_j *by* $v_j P^-(v_h)$ *and* v_h *by* $v_h/P^-(v_h)$ *is an element of* E_n *satisfying* $\boldsymbol{w} \prec \boldsymbol{v}$.

Proof. Since $\boldsymbol{v} \in E_n$ and $\boldsymbol{w}$ is ordered, we have $\boldsymbol{w} \in E_n$. Moreover,

$$S(\boldsymbol{w}) - S(\boldsymbol{v}) = \frac{\{P^-(v_h) - 1\}\{v_j P^-(v_h) - v_h\}}{v_j v_h P^-(v_h)} < 0. \qquad \square$$

For purpose of further reference, we note as an immediate consequence of Lemma 2.1 that, for each $n \geqslant 1$, we have

$$\varrho_j(n) P^-(\varrho_h(n)) \geqslant \varrho_h(n) \quad (1 \leqslant j < h \leqslant k). \tag{2.1}$$

3 Proof of Theorem 1.1

Set

$$N_{j,\ell}(x) := \left\{ n \leqslant x : \frac{x^{1/k}}{2^{\ell+1}} < \varrho_j(n) \leqslant \frac{x^{1/k}}{2^\ell} \right\} \quad (x \geqslant 1,\ 1 \leqslant j \leqslant k,\ \ell \in \mathbb{Z}).$$

Lemma 3.1. *Let* $x \geqslant 1$. *For each* $n \leqslant x$, *there exists a unique* $\ell \geqslant 0$ *such that* $n \in N_{1,\ell}(x)$ *and*

$$\frac{x^{1/k}}{2^{\ell+1}} < \varrho_j(n) \leqslant 2^{(\ell+1)(k-1)} x^{1/k} \quad (2 \leqslant j \leqslant k). \tag{3.1}$$

Proof. The requirement that $n \in N_{1,\ell}(x)$ implies that ℓ is the greatest integer such that $\varrho_1(n) \leqslant x^{1/k}/2^\ell$. Moreover, since $\varrho_1(n) \leqslant x^{1/k}$ by definition, we must have $\ell \geqslant 0$.

When $2 \leqslant j \leqslant k$, we have $\varrho_j(n) \geqslant \varrho_1(n) > x^{1/k}/2^{\ell+1}$ and

$$\varrho_j(n) \leqslant \varrho_k(n) \leqslant \frac{x}{\{\varrho_1(n)\}^{k-1}} \leqslant 2^{(\ell+1)(k-1)} x^{1/k}. \qquad \square$$

We can now embark on the proof of Theorem 1.1. For $\boldsymbol{v} = (v_1, \ldots, v_m) \in [0, \infty[^m$ $(m \geqslant 1)$, define

$$\tau_{m+1}(n, \boldsymbol{v}) := |\{(d_1, \ldots, d_m) \in \mathbb{N}^m : d_1 \ldots d_m \mid n,\ v_j < d_j \leqslant 2v_j\ (1 \leqslant j \leqslant m)\}| \quad (n \geqslant 1),$$
$$H^{(m+1)}(x, \boldsymbol{v}) := |\{n \leqslant x : \tau_{m+1}(n, \boldsymbol{v}) \geqslant 1\}| \quad (x \geqslant 1,\ m \geqslant 1).$$

We know from [5, th. 1] that

$$H^{(m+1)}(x, \boldsymbol{v}) \asymp \frac{x}{(\log x)^{Q(1/\log r)} (\log_2 x)^{3/2}}, \tag{3.2}$$

with $r := (m+1)^{1/m}$, provided $2^{m+1} \prod_{1 \leqslant h \leqslant m} v_h \leqslant x/v_1^{c_1}$ and $v_m \leqslant v_1^{c_2}$ for suitable constants c_1, c_2, $0 < c_1 \leqslant 1 \leqslant c_2$. Note right-away that $Q(1/\log r) = \delta_k$ as defined in (1.5) when $m = k-1$.

By Lemma 3.1, for every $n \leqslant x$, there exists a unique k-tuple $\boldsymbol{\ell}_n = (\ell_{n,1}, \ldots, \ell_{n,k}) \in \mathbb{Z}^k$ such that $n \in N_{j,\ell_{n,j}}(x)$ $(1 \leqslant j \leqslant k)$, $\ell_{n,1} \geqslant 0$, and $-(\ell_{n,1}+1)(k-1) \leqslant \ell_{n,k} \leqslant \cdots \leqslant \ell_{n,1}$. Therefore

$$|N_{1,\ell}(x)| \leqslant \sum_{\substack{(\ell+1)(1-k) \leqslant \ell_k \leqslant \cdots \leqslant \ell_2 \leqslant \ell \\ \ell + \ell_2 + \cdots + \ell_k \geqslant 0}} H^{(k)}\Big(\frac{x}{2^{\ell + \ell_2 + \cdots + \ell_k}}, \Big\{ \frac{x^{1/k}}{2^{\ell_j + 1}} \Big\}_{2 \leqslant j \leqslant k} \Big) \quad (\ell \geqslant 0). \tag{3.3}$$

Now, setting $L_x := \lceil 2\delta_k (\log_2 x)/\log 2 \rceil$ $(x \geqslant 3)$, we can write

$$\sum_{n \leqslant x} \varrho_1(n) = \sum_{\substack{n \leqslant x \\ \varrho_1(n) \geqslant x^{1/k}/(\log x)^{2\delta_k}}} \varrho_1(n) + O\Big(\frac{x^{1+1/k}}{(\log x)^{2\delta_k}} \Big) \leqslant x^{1/k} \sum_{0 \leqslant \ell \leqslant L_x} \frac{|N_{1,\ell}(x)|}{2^\ell} + O\Big(\frac{x^{1+1/k}}{(\log x)^{2\delta_k}} \Big).$$

Since the required hypotheses for (3.2) are satisfied for $v_j := x^{1/k}/2^{\ell_j+1}$ $(2 \leqslant j \leqslant k)$, $0 \leqslant \ell \leqslant L_x$, $\ell_j \asymp \ell+1$, and sufficiently large x, we deduce from (3.3) that

$$\begin{aligned}\sum_{n\leqslant x} \varrho_1(n) &\ll_k \frac{x^{1+1/k}}{(\log x)^{\delta_k}(\log_2 x)^{3/2}} \sum_{0\leqslant \ell\leqslant L_x} \frac{1}{2^\ell} \sum_{\substack{(\ell+1)(1-k)\leqslant \ell_k\leqslant\cdots\leqslant\ell_2\leqslant\ell\\ \ell+\ell_2+\cdots+\ell_k\geqslant 0}} \frac{1}{2^{\ell+\ell_2+\cdots+\ell_k}}\\ &\ll_k \frac{x^{1+1/k}}{(\log x)^{\delta_k}(\log_2 x)^{3/2}} \sum_{0\leqslant \ell\leqslant L_x} \frac{\{k(\ell+1)\}^{k-1}}{2^\ell} \ll_k \frac{x^{1+1/k}}{(\log x)^{\delta_k}(\log_2 x)^{3/2}}.\end{aligned}$$

It remains to establish the lower bound included in (1.6). From [5, th. 2], we know that

$$\mathcal{A}(x) := \Big|\Big\{\tfrac12 x < n \leqslant x : \mu(n)^2 = 1,\ \tau_k\big(n, \{\tfrac12 x^{1/k}\}_{1\leqslant j\leqslant k-1}\big) \geqslant 1\Big\}\Big| \gg_k \frac{x}{(\log x)^{\delta_k}(\log_2 x)^{3/2}}, \tag{3.4}$$

where μ refers to the Möbius function. Observe that

$$\frac{1}{\varrho_1(n)} < \sum_{1\leqslant j\leqslant k} \frac{1}{\varrho_j(n)} < \frac{2k}{x^{1/k}} \qquad \big(n \in \mathcal{A}(x)\big),$$

hence $x^{1/k}/2k < \varrho_1(n) \leqslant x^{1/k}$ by construction. In view of (3.4), we finally have

$$\begin{aligned}\sum_{n\leqslant x} \varrho_1(n) &\geqslant \sum_{\substack{n\leqslant x\\ x^{1/k}/2k<\varrho_1(n)\leqslant x^{1/k}}} \varrho_1(n) \geqslant \frac{x^{1/k}}{2k}\Big|\Big\{n \leqslant x : \frac{x^{1/k}}{2k} < \varrho_1(n) \leqslant x^{1/k}\Big\}\Big|\\ &\geqslant \frac{x^{1/k}\mathcal{A}(x)}{2k} \gg_k \frac{x^{1+1/k}}{(\log x)^{\delta_k}(\log_2 x)^{3/2}}.\end{aligned}$$

This completes the proof.

4 Proof of Theorem 1.2

Recall definition $\gamma_j := k + 1 - j$, and put

$$\alpha_j := \frac{1}{\gamma_j + 1/2} = \frac{1}{k - j + 3/2} \quad (1 \leqslant j \leqslant k). \tag{4.1}$$

Our proof of Theorem 1.2 is based on the observation that the sums (1.7) are dominated by large values of $\varrho_j(n)$. More precisely, we shall see that the structure of the integers $n \leqslant x$ such that $\varrho_j(n) > x^{\alpha_j}$ is very constrained: in that case the $\varrho_h(n)$ $(j \leqslant h \leqslant k)$ are all prime. To lighten notation, we write

$$\pi_j(n) := \prod_{1\leqslant h\leqslant j} \varrho_h(n) \quad (n \geqslant 1,\ 1 \leqslant j \leqslant k). \tag{4.2}$$

Lemma 4.1. *Let $1 \leqslant n \leqslant x$ and $2 \leqslant j \leqslant k$. If $\varrho_j(n) > x^{\alpha_j}$, then $\varrho_h(n)$ is a prime number for $j \leqslant h \leqslant k$.*

Proof. Under the assumption $\varrho_j(n) > x^{\alpha_j}$, we have

$$\pi_{j-1}(n) \leqslant \frac{x}{\{\varrho_j(n)\}^{\gamma_j}} < x^{1-\gamma_j\alpha_j} = x^{1/(2\gamma_j+1)}. \tag{4.3}$$

If $\varrho_h(n)$ is composite for some $h \in [j,k]$, then $\varrho_h(n) \geqslant p^2$, with $p := P^-(\varrho_h(n))$. However, in view of Lemma 2.1 we have

$$p \geqslant \frac{\varrho_h(n)}{\varrho_{j-1}(n)} \geqslant \frac{\varrho_j(n)}{\varrho_{j-1}(n)} \geqslant \frac{\varrho_j(n)}{\pi_{j-1}(n)} > x^{1/(2\gamma_j+1)}, \tag{4.4}$$

by (4.3), while, since $\varrho_{j-1}(n)p \geqslant \varrho_h(n) \geqslant p^2$, we also have

$$p \leqslant \varrho_{j-1}(n) \leqslant \pi_{j-1}(n) < x^{1/(2\gamma_j+1)}, \tag{4.5}$$

by another appeal to (4.3). Since (4.4) and (4.5) are incompatible, $\varrho_h(n)$ must be prime. □

Observe that, trivially, $\varrho_j(n) \leqslant x^{1/\gamma_j}$ $(1 \leqslant j \leqslant k)$ and that the contribution to the left-hand side of (1.7) of those integers n such that $\varrho_j(n) \leqslant x^{\alpha_j}$ is clearly $\ll x^{1+\alpha_j}$, a quantity exceeded by the error term of (1.7). Therefore, we can focus on those n such that $\varrho_j(n) \in]x^{\alpha_j}, x^{1/\gamma_j}]$. By Lemma 4.1, these integers admit a representation of the form $n = mpq_1 \dots q_{k-j}$ with $p \leqslant q_1 \leqslant \dots \leqslant q_{k-j}$, where p and the q_h are all prime. It follows that

$$\sum_{n \leqslant x} \varrho_j(n) = S_j(x) + O\big(x^{1+\alpha_j}\big), \tag{4.6}$$

with

$$S_j(x) := \sum_{x^{\alpha_j} < p \leqslant x^{1/\gamma_j}} p \sum_{\substack{p \leqslant q_1 \leqslant \dots \leqslant q_{k-j} \\ q_1 \cdots q_{k-j} \leqslant x/p}} \left\lfloor \frac{x}{pq_1 \dots q_{k-j}} \right\rfloor \quad (x \geqslant 1,\ 2 \leqslant j \leqslant k). \tag{4.7}$$

We now evaluate $S_j(x)$. Let $\Omega(n)$ denote the total number of prime factors of an integer n. We have

$$S_j(x) = \sum_{x^{\alpha_j} < p \leqslant x^{1/\gamma_j}} p \sum_{\substack{n \leqslant x/p \\ P^-(n) \geqslant p \\ \Omega(n) = k-j}} \sum_{m \leqslant x/np} 1 = \sum_{m \leqslant x^{1-\gamma_j \alpha_j}} \sum_{x^{\alpha_j} < p \leqslant (x/m)^{1/\gamma_j}} p \sum_{\substack{n \leqslant x/pm \\ P^-(n) \geqslant p \\ \Omega(n) = k-j}} 1. \tag{4.8}$$

Now from the trivial estimate

$$\sum_{p \leqslant x^{\alpha_j}} p \sum_{\substack{n \leqslant x/pm \\ P^-(n) \geqslant p \\ \Omega(n) = k-j}} 1 \leqslant \sum_{p \leqslant x^{\alpha_j}} p \sum_{\substack{n \leqslant x/pm \\ P^-(n) \geqslant p \\ \Omega(n) = k-j}} \frac{x}{pmn} \ll \frac{x\pi(x^{\alpha_j})(\log_2 x)^{k-j}}{m} \ll \frac{x^{1+\alpha_j}(\log_2 x)^{k-j}}{m \log x},$$

we see that removing the lower bound for p in the summation conditions of (4.8) introduces a global error $\ll x^{1+\alpha_j}(\log_2 x)^{k-j}$. Observing that $1 - \gamma_j \alpha_j = \alpha_j/2$ and defining

$$T_j(y) := \sum_{p \leqslant y^{1/\gamma_j}} p \sum_{\substack{n \leqslant y/p \\ P^-(n) \geqslant p \\ \Omega(n) = k-j}} 1 \leqslant \sum_{p \leqslant y^{1/\gamma_j}} \sum_{\substack{n \leqslant y/p \\ P^-(n) \geqslant p \\ \Omega(n) = k-j}} \frac{y}{n} \ll y^{1+1/\gamma_j}, \tag{4.9}$$

we arrive at

$$S_j(x) = \sum_{m \leqslant x^{\alpha_j/3}} T_j\Big(\frac{x}{m}\Big) + O\Big(x^{1+(1-\alpha_j/3)/\gamma_j}\Big), \tag{4.10}$$

in view of the inequality $\alpha_j < (1 - \alpha_j/3)/\gamma_j$.

The next lemma furnishes an asymptotic formula for $T_j(y)$ as defined in (4.9).

Lemma 4.2. *Let $2 \leqslant j \leqslant k$. We have*

$$T_j(y) = \frac{\gamma_j^{2\gamma_j} y^{1+1/\gamma_j}}{(\gamma_j + 1)!(\log y)^{\gamma_j}} \left\{1 + O\Big(\frac{1}{\log y}\Big)\right\} \quad (y \geqslant 2). \tag{4.11}$$

Proof. Set

$$A_h(v,t) := \sum_{\substack{q_1 \geqslant \dots \geqslant q_h \geqslant t \\ q_1 \dots q_h \leqslant v}} 1 \quad (h \geqslant 1,\ 2 \leqslant t \leqslant v),$$

so that

$$T_j(y) = \int_2^{y^{1/\gamma_j}} tA_{k-j}\Big(\frac{y}{t}, t\Big)\,\mathrm{d}\pi(t). \tag{4.12}$$

An iterated application of a strong form of the prime number theorem yields, for any fixed $h \geqslant 1$,

$$A_h(v,t) = \int_{\substack{u_1 \geqslant \dots \geqslant u_h \geqslant t \\ u_1 \cdots u_h \leqslant v}} \mathrm{d}\pi(u_1) \cdots \mathrm{d}\pi(u_h) = \int_{\substack{u_1 \geqslant \dots \geqslant u_h \geqslant t \\ u_1 \cdots u_h \leqslant v}} \frac{\mathrm{d}u_1 \cdots \mathrm{d}u_h}{(\log u_1) \cdots (\log u_h)} + O\Big(\frac{v}{\mathrm{e}^{\sqrt{\log v}}}\Big).$$

Therefore

$$T_j(y)=\frac{1}{(k-j)!}\int_2^{y^{1/\gamma_j}}\int_{\substack{u_1,\dots,u_{k-j}\geqslant t\\ u_1\cdots u_{k-j}\leqslant y/t}}\frac{t\,\mathrm{d}u_1\cdots\mathrm{d}u_{k-j}\mathrm{d}t}{(\log t)(\log u_1)\dots(\log u_{k-j})}+O\Big(\frac{y^{1+1/\gamma_j}}{\mathrm{e}^{\sqrt{\log y}}}\Big).$$

Performing the change of variables $t=y^{v_0}$, $u_h=y^{v_h}$ $(1\leqslant h\leqslant k-j)$, we obtain

$$T_j(y)=\frac{1}{(k-j)!}\int_{(\log 2)/\log y}^{1/\gamma_j}\int_{\substack{v_1,\dots,v_{k-j}\geqslant v_0\\ v_1+\cdots+v_{k-j}\leqslant 1-v_0}}\frac{y^{2v_0+v_1+\cdots+v_{k-j}}\,\mathrm{d}v_0\,\mathrm{d}v_1\cdots\mathrm{d}v_{k-j}}{v_0v_1\cdots v_{k-j}}+O\Big(\frac{y^{1+1/\gamma_j}}{\mathrm{e}^{\sqrt{\log y}}}\Big).$$

For $2\leqslant h\leqslant k-j$, write

$$W_h:=\sum_{1\leqslant m\leqslant h}w_m\quad(w_1,\dots,w_h\in\mathbb{R}).$$

The further change of variables

$$v_0=\frac{1}{\gamma_j}-\frac{s}{\log y},\quad v_h=v_0+\frac{w_h}{\log y}=\frac{1}{\gamma_j}+\frac{w_h-s}{\log y}\quad(1\leqslant h\leqslant k-j)$$

yields

$$T_j(y)=\frac{U_j(y)}{(k-j)!}+O\bigg(\frac{y^{1+1/\gamma_j}}{\mathrm{e}^{\sqrt{\log y}}}\bigg),\tag{4.13}$$

with

$$\begin{aligned}U_j(y)&:=\frac{y^{1+1/\gamma_j}}{(\log y)^{\gamma_j}}\int_0^{\log(y^{1/\gamma_j}/2)}\frac{\mathrm{d}s}{\mathrm{e}^{(\gamma_j+1)s}}\int_{\substack{w_1,\dots,w_{k-j}\geqslant 0\\ W_{k-j}\leqslant\gamma_j s}}\frac{\mathrm{e}^{W_{k-j}}\,\mathrm{d}w_1\cdots\mathrm{d}w_{k-j}}{D_j(s,y)},\\ D_j(s,y)&:=\Big(\frac{1}{\gamma_j}-\frac{s}{\log y}\Big)\prod_{1\leqslant h\leqslant k-j}\Big(\frac{1}{\gamma_j}+\frac{w_h-s}{\log y}\Big).\end{aligned}$$

For $s\leqslant\log(\frac12 y^{1/\gamma_j})$, we have

$$\frac{1}{\gamma_j}+\frac{w_h-s}{\log y}\geqslant\frac{1}{\gamma_j}-\frac{s}{\log y}\geqslant\frac{\log 2}{\log y},$$

hence the contribution of $s\geqslant Y:=(\gamma_j+2)\log_2 y$ to $U_j(y)$ is

$$\begin{aligned}&\ll y^{1+1/\gamma_j}\int_Y^\infty\mathrm{e}^{-s}\,\mathrm{d}s\int_{\substack{w_1,\dots,w_{k-j}\geqslant 0\\ W_{k-j}\leqslant\gamma_j s}}\mathrm{d}w_1\cdots\mathrm{d}w_{k-j}\\ &\ll y^{1+1/\gamma_j}\int_Y^\infty s^{k-j}\,\mathrm{e}^{-s}\,\mathrm{d}s\ll y^{1+1/\gamma_j}Y^{k-j}\,\mathrm{e}^{-Y}\ll\frac{y^{1+1/\gamma_j}}{(\log y)^{\gamma_j+1}}.\end{aligned}\tag{4.14}$$

Moreover, for $s\leqslant Y$, we have

$$D_j(s,y)=\gamma_j^{-\gamma_j}+\gamma_j^{1-\gamma_j}\frac{W_{k-j}-\gamma_j s+O(1)}{\log y},$$

and so

$$\frac{1}{D_j(s,y)}=\gamma_j^{\gamma_j}\bigg\{1-\gamma_j\frac{W_{k-j}-\gamma_j s+O(1)}{\log y}\bigg\}.$$

It follows that

$$U_j(y)=\frac{\gamma_j^{\gamma_j}y^{1+1/\gamma_j}}{(\log y)^{\gamma_j}}\int_0^Y\frac{\mathrm{d}s}{\mathrm{e}^{(\gamma_j+1)s}}\int_{\substack{w_1,\dots,w_{k-j}\geqslant 0\\ W_{k-j}\leqslant\gamma_j s}}\mathrm{e}^{W_{k-j}}\,\Big\{1+O\Big(\frac{s+1}{\log y}\Big)\Big\}\,\mathrm{d}w_1\cdots\mathrm{d}w_{k-1}.$$

Since the contribution of the error term is clearly

$$\ll \frac{y^{1+1/\gamma_j}}{(\log y)^{\gamma_j+1}}\int_0^\infty (s+1)^{\gamma_j}\,\mathrm{e}^{-s}\,\mathrm{d}s \ll \frac{y^{1+1/\gamma_j}}{(\log y)^{\gamma_j+1}},$$

we finally get

$$U_j(y)=\frac{\gamma_j^{\gamma_j}y^{1+1/\gamma_j}}{(\log y)^{\gamma_j}}\int_0^Y\frac{\mathrm{d}s}{\mathrm{e}^{(\gamma_j+1)s}}\int_{\substack{w_1,\cdots w_{k-j}\geqslant 0\\ W_{k-j}\leqslant\gamma_j s}}\mathrm{e}^{W_{k-j}}\,\mathrm{d}w_1\cdots\,\mathrm{d}w_{k-j}+O\Big(\frac{y^{1+1/\gamma_j}}{(\log y)^{\gamma_j+1}}\Big).\tag{4.15}$$

The range of the outer integral may now be extended to $[0,\infty[$ for the involved error is

$$\ll\int_Y^\infty s^{k-j}\,\mathrm{e}^{-s}\,\mathrm{d}s\ll Y^{k-j}\,\mathrm{e}^{-Y}\ll\frac{1}{(\log y)^{\gamma_j+1}}.$$

Inverting the order of integrations in the multiple integral thus modified, we get that it is

$$\int_{[0,\infty[^{k-j}}\mathrm{e}^{W_{k-j}}\,\mathrm{d}w_1\cdots\mathrm{d}w_{k-j}\int_{W_{k-j}/\gamma_j}^\infty\frac{\mathrm{d}s}{\mathrm{e}^{(\gamma_j+1)s}}=\frac{1}{\gamma_j+1}\int_{[0,\infty[^{k-j}}\frac{\mathrm{d}w_1\cdots\mathrm{d}w_{k-j}}{\mathrm{e}^{W_{k-j}/\gamma_j}}=\frac{\gamma_j^{\gamma_j-1}}{\gamma_j+1}.\tag{4.16}$$

Formula (4.11) then follows from (4.13) and (4.15). □

We are now in a position to complete the proof of Theorem 1.2. By (4.6), (4.10) and (4.11), we have

$$\begin{aligned}\sum_{n\leqslant x}\varrho_j(n)&=\frac{\gamma_j^{2\gamma_j}x^{1+1/\gamma_j}}{(\gamma_j+1)!(\log x)^{\gamma_j}}\sum_{1\leqslant m\leqslant x^{\alpha_j/3}}\frac{1}{m^{1+1/\gamma_j}}\Big\{1+O\Big(\frac{1+\log m}{\log x}\Big)\Big\}+O\Big(\frac{x^{1+1/\gamma_j}}{(\log x)^{\gamma_j+1}}\Big)\\&=\frac{\gamma_j^{2\gamma_j}\zeta(1+1/\gamma_j)x^{1+1/\gamma_j}}{(\gamma_j+1)!(\log x)^{\gamma_j}}+O\Big(\frac{x^{1+1/\gamma_j}}{(\log x)^{\gamma_j+1}}\Big),\end{aligned}$$

as required.

References

[1] K. Ford, The distribution of integers with a divisor in a given interval, *Ann. of Math.* 168 (2008), 367–433.

[2] J.-M. De Koninck & A. A. B. Razafindrasoanaivolala, On the middle divisors of an integer, *Ann. Univ. Sci. Budapest, Sect. Comput.* 50 (2020), 113–126.

[3] J.-M. De Koninck & A. A. B. Razafindrasoanaivolala, On the quotient of the logarithms of the middle divisors of an integer, *Afrika Matematika* 34 (2023), no. 2, 1–15.

[4] T. Haddad, Arratia's coupling for prime factors and its applications to divisors, Ph.D thesis, Montreal Univ., 2025.

[5] D. Koukoulopoulos, Localized factorizations of integers, *Proc. London Math. Soc.* (3) 101 (2010), no. 2, 392–426.

[6] G. Tenenbaum, Sur deux fonctions de diviseurs, *J. London Math. Soc.* (2) 14 (1976), 521–526; corrigendum *ibid.* (2) 17 (1978), p. 212.

[7] G. Tenenbaum, Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné, *Compositio Math.* **51** (1984), 243–263.

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