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LOW MOMENTS OF DIRICHLET SERIES

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Abstract. We determine the maximum possible size of the q^{th} moment of a Dirichlet series, for $1 \leqslant q \leqslant 2$.

1. Introduction

In order to bound the mean value of multiplicative functions, Halász [5] introduced a majorant principle which (after a little refining) asserts that if $\lambda_1, \lambda_2, \ldots$ are real numbers, if $|a_n| \leqslant A_n$ for all n, and $\sum_{n\geqslant 1} A_n < \infty$, then

$$\int_{-T}^{T} \Big| \sum_{n\geqslant 1} a_n \mathrm{e}^{i\lambda_n t} \Big|^2 \,\mathrm{d}t \leqslant 3 \int_{-T}^{T} \Big| \sum_{n\geqslant 1} A_n \mathrm{e}^{i\lambda_n t} \Big|^2 \,\mathrm{d}t. \tag{1.1}$$

For a proof of the principle in this form, see Montgomery [8, §7.3]. In Halász's theory, one needs bounds for integrals of the shape

$$I(q) = \int_{-1}^{1} \Big| \sum_{p} \frac{a_p \log p}{p^{\sigma+it}} \Big|^q \,\mathrm{d}t, \qquad J(q) = \int_{-1}^{1} \Big| \sum_{n\geqslant 1} \frac{b_n}{n^{\sigma+it}} \Big|^q \,\mathrm{d}t, \tag{1.2}$$

when $|a_p| \leqslant 1$ for all p, and $|b_n| \leqslant 1$ for all n. From (1.1) it is immediate that

$$I(2) \leqslant 3 \int_{-1}^{1} \Big| \sum_{n\geqslant 2} \frac{\Lambda(n)}{n^{\sigma+it}} \Big|^2 \,\mathrm{d}t = 3 \int_{-1}^{1} \Big| \frac{\zeta'}{\zeta}(\sigma+it) \Big|^2 \,\mathrm{d}t \ll \int_{-1}^{1} \frac{\mathrm{d}t}{|\sigma+it-1|^2} \ll \frac{1}{\sigma-1}$$

uniformly for $1 < \sigma \leqslant 2$. Similarly, $J(2) \ll 1/(\sigma-1)$ for σ in this range. By applying the majorant principle to the squares of these Dirichlet series we find that $I(4) \ll (\sigma-1)^{-3}$ and $J(4) \ll (\sigma-1)^{-3}$. Hence by Hölder's inequality,

$$I(q) \ll (\sigma-1)^{1-q}, \qquad J(q) \ll (\sigma-1)^{1-q} \qquad (2 \leqslant q \leqslant 4), \tag{1.3}$$

$$I(q) \ll (\sigma-1)^{-q/2}, \qquad J(q) \ll (\sigma-1)^{-q/2} \qquad (1 \leqslant q \leqslant 2) \tag{1.4}$$

uniformly for $1 < \sigma \leqslant 2$. The estimate (1.3) is best possible, as we see by taking $a_p = 1$ for all p and $b_n = 1$ for all n. For purposes of Halász's theory, it would be helpful if (1.3) would hold also when $1 < q \leqslant 2$. However, we construct examples that show that the weaker estimate (1.4) is best possible.

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Theorem 1.1. *Let* $I(q)$ *and* $J(q)$ *be defined as in* (1.2). *There exist numbers* a_p *with* $a_p = \pm 1$ *for all* p, *and* b_n *with* $b_n = \pm 1$ *for all* n, *such that*

$$I(q) \asymp (\sigma - 1)^{-q/2}, \qquad J(q) \asymp (\sigma - 1)^{-q/2}$$

uniformly for $1 < \sigma \leqslant 2$, $1 \leqslant q \leqslant 2$.

This is analogous to the situation for Fourier series. For example, if $|b_n| \leqslant 1$ for $-N \leqslant n \leqslant N$ and $\mathrm{e}(\vartheta) = \mathrm{e}^{2\pi i\vartheta}$, then

$$\int_0^1 \Big| \sum_{|n| \leqslant N} b_n \mathrm{e}(nx) \Big|^q \, \mathrm{d}x \ll N^{q-1} \tag{1.5}$$

uniformly for $2 \leqslant q \leqslant 4$, and

$$\int_0^1 \Big| \sum_{|n| \leqslant N} b_n \mathrm{e}(nx) \Big|^q \, \mathrm{d}x \ll N^{q/2} \tag{1.6}$$

for $1 \leqslant q \leqslant 2$, but there exists a choice of the b_n with $b_n = \pm 1$ for all n such that

$$\int_0^1 \Big| \sum_{|n| \leqslant N} b_n \mathrm{e}(nx) \Big|^q \, \mathrm{d}x \asymp N^{q/2} \tag{1.7}$$

uniformly for $1 \leqslant q \leqslant 2$. Indeed, we use such b_n in our construction.

Antecedents of Halász's majorant principle (1.1) are found in Wiener & Wintner [15] and in Erdős & Fuchs [3]. Logan [6] showed that the constant 3 in (1.1) is best-possible.

2. LEMMAS

We begin with a generalization of a result of H. S. Shapiro [12].

Let the sequence $\{r_n\}_{n=0}^{\infty}$ be defined by the relations $r_0 = 1$, $r_{2n} = r_n$ and $r_{2n+1} = (-1)^n r_n$. The sequence $\{r_n\}_{n=0}^{\infty}$ is the classical Rudin–Shapiro sequence. Suppose that the binary expansion of n is $n = \sum_{j \geqslant 0} e_j(n) 2^j$ where $e_j(n) = 0$ or 1. A well-known alternative definition is

$$r_n = (-1)^{H(n)}, \text{ where } H(n) := \sum_{j \geqslant 0} e_j(n) e_{j+1}(n) \qquad (n \geqslant 0).$$

Let $p_m(z)$, $q_m(z)$ denote polynomials defined recursively by the relations $p_0(z) = 1$, $q_0(z) = 1$ and

$$\begin{aligned} p_{m+1}(z) &= p_m(z) + z^{2^m} q_m(z) \\ q_{m+1}(z) &= p_m(z) - z^{2^m} q_m(z). \end{aligned} \tag{2.1}$$

One can easily check that

$$p_m(z) = \sum_{0 \leqslant n \leqslant 2^m - 1} r_n z^n \qquad (m \geqslant 0).$$

The Rudin–Shapiro sequence may be generalized by the so-called paper-folding twist. This amounts to introducing a sequence $\{\varepsilon_m\}_{m=0}^{\infty} \in \{\pm 1\}^{\mathbb{N}}$ and replacing (2.1) by

$$\begin{aligned} p_{m+1}(z) &= p_m(z) + \varepsilon_m z^{2^m} q_m(z) \\ q_{m+1}(z) &= p_m(z) - \varepsilon_m z^{2^m} q_m(z). \end{aligned} \tag{2.2}$$

We then obtain

$$p_m(z) = \sum_{0\leqslant n\leqslant 2^m-1} c_n z^n \qquad (m = 0, 1, \ldots)$$

with

$$c_n = r_n \prod_{j\geqslant 0} \varepsilon_j^{e_j(n)} \qquad (n \geqslant 0), \tag{2.3}$$

a formula for which we did not find a reference in the literature and which B. Saffari kindly pointed out to us.

Lemma 2.1. *Let the sequence* c_m *be defined as above. Put*

$$P_M(\vartheta) = \sum_{0\leqslant m<M} c_m \mathrm{e}(m\vartheta)\,. \tag{2.4}$$

Then

$$|P_M(\vartheta)| \leqslant \big(2+\sqrt{2}\big)\sqrt{M}$$

for all positive integers M *and all real* ϑ.

Shapiro proved this in the case $c_m = r_m$ $(m \geqslant 0)$ but never published his work. The coefficients r_m were independently discovered by Golay [4]. Rudin [11] published an account of Shapiro's argument in the case $M = 2^k$, but obtained an inferior constant in the general case. The above lemma is proved in [7, théorème 2].

We note in passing that it follows from the proof of theorem 2 of [7] that, given an arbitrary sequence $\{\eta_j\}_{j=0}^{\infty} \in \{\pm 1\}^{\mathbb{N}}$, a generalized Rudin–Shapiro sequence may alternatively be written as

$$c_m = (-1)^{v_m} \tag{2.5}$$

where v_m equals 0 or 1 according to whether $\sum_{j\geqslant 0} \eta_j |e_j(m) - e_{j+1}(m)|$ belongs to $\{0,1\}$ or to $\{2,3\}$ modulo 4. In this setting, we recover r_m by selecting $\eta_j = (-1)^j$ $(j \geqslant 0)$. Also, this easily enables retrieving (2.3).

Lemma 2.2. *For* $|z| < 1$, *let* $f(z) = \sum_{m\geqslant 0} c_m z^m$ *where* c_m *is defined as in* (2.3). *Then*

$$|f(r\mathrm{e}(\vartheta))| \leqslant \frac{2+\sqrt{2}}{\sqrt{1-r}}\cdot$$

Numerical studies suggest that, at least in the case $c_j = r_j$ $(j \geqslant 0)$,

$$\max_{\vartheta} |f(r\mathrm{e}(\vartheta))| = f(r)\,. \tag{2.6}$$

Moreover, it is easy to show that, in the above circumstance, $f(r)\sqrt{1-r}$ does not tend to a limit as $r \to 1^-$. Indeed, it is proved in Brillhart, Erdős and Morton [1] that

$$\sum_{m<n} r_m = \sqrt{n}\, G\Big(\frac{\log n}{\log 4}\Big) \qquad (n \geqslant 0) \tag{2.7}$$

where G is 1-periodic and continuous. Moreover, Dumont and Thomas [2] showed that G is nowhere differentiable and Tenenbaum [13] obtained the oscillation result

$$G(x+h) - G(x) = \Omega\big(\sqrt{h}\big) \qquad (h \geqslant 0)$$

for any given real number x.

By partial summation it is readily derived from (2.7) that $f(r)\sqrt{1-r}$ oscillates as $r \to 1-$: indeed, as $y \to \infty$,

$$2^y f\big(\exp(-4^{-y})\big) \tag{2.8}$$

tends to a 1-periodic, nowhere differentiable function of y.

It is noteworthy that

$$\begin{aligned} f(z) &= f(z^2) + f(-z^2), \\ f(-z) &= f(z^2) - f(-z^2)\,. \end{aligned} \tag{2.9}$$

Kumiko Nishioka [10] showed that $f(z)$ and $f(-z)$ are algebraically independent, and then used these recurrences and Mahler's method to show that if α is algebraic with $0 < |\alpha| < 1$, then $f(\alpha)$ and $f(-\alpha)$ are algebraically independent.

Proof. Clearly

$$\frac{f(re(\vartheta))}{1-r} = \sum_{m\geqslant 0} P_{m+1}(\vartheta) r^m\,.$$

Hence by Lemma 2.1 and the triangle inequality it follows that

$$\Big|\frac{f(re(\vartheta))}{1-r}\Big| \leqslant (2+\sqrt{2}) \sum_{m\geqslant 0} \sqrt{m+1}\, r^m\,. \tag{2.10}$$

But

$$\sqrt{m+1} \leqslant \binom{m+1/2}{m} \tag{2.11}$$

for all non-negative integers m. Hence the right hand side of (2.10) is

$$\leqslant (2+\sqrt{2}) \sum_{m\geqslant 0} \binom{m+1/2}{m} r^m = \frac{2+\sqrt{2}}{(1-r)^{3/2}},$$

which gives the stated result.

To prove (2.11), let $a_m = \binom{m+1/2}{m}/\sqrt{m+1}$. To show that $a_m \geqslant 1$, it suffices to note that $a_0 = 1$, and to show that the a_m are increasing. As to this latter point, we observe that

$$\frac{a_m}{a_{m-1}} = \frac{m+1/2}{\sqrt{m(m+1)}} = \frac{2m+1}{\sqrt{(2m+1)^2-1}} > 1.$$

□

Lemma 2.3. *Let f be defined as in Lemma 2.2. For each r, $0 < r < 1$, there is a measurable set $A_r \subseteq \mathbb{T}$ with Lebesgue measure $\lambda\big(A_r\big) \geqslant 1/50$, such that*

$$|f(re(\vartheta))| \geqslant \frac{1}{2\sqrt{1-r}} \tag{2.12}$$

for all $\vartheta \in A_r$.

Proof. Let $B_r = \mathbb{T} \smallsetminus A_r$ be the complementary set of those ϑ on which $|f|$ is small; precisely $|f(re(\vartheta))| < 1/(2\sqrt{1-r})$. By Parseval's identity we know that

$$\int_0^1 |f(re(\vartheta))|^2 \, \mathrm{d}\vartheta = \sum_{m\geqslant 0} r^{2m} = \frac{1}{1-r^2} > \frac{1}{2(1-r)}\cdot \tag{2.13}$$

By Lemma 2.2, the left hand side above equals

$$\int_{A_r} |f(re(\vartheta))|^2 \,\mathrm{d}\vartheta + \int_{B_r} |f(re(\vartheta))|^2 \,\mathrm{d}\vartheta \leqslant \frac{(2+\sqrt{2})^2}{1-r}\lambda(A_r) + \frac{1-\lambda(A_r)}{4(1-r)}\cdot$$

On combining this with (2.13), we find that

$$\tfrac14 \leqslant \big((2+\sqrt{2})^2 - \tfrac14\big)\lambda(A_r),$$

which gives the stated result. □

Lemma 2.4. *Write* $s = \sigma + it$. *Then*

$$\sum_{n\leqslant x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} + O\Big(\frac{x^{1-\sigma}}{\exp(\sqrt{\log x})}\Big)$$

for $x \geqslant 2$, $1 < \sigma \leqslant 2$, $-1 \leqslant t \leqslant 1$.

This is included in equation (III.5.72) of [14] and is proved by Perron's summation formula (see Montgomery–Vaughan [9, Theorem 5.2] or Tenenbaum [14, corollary II.2.4]) appealing to the classical zero-free region and estimates for $\zeta'(s)/\zeta(s)$ in the zero-free region.

Lemma 2.5. *For* $x \geqslant 2$, $1 < \sigma \leqslant 2$, *and* $-1 \leqslant t \leqslant 1$, *we have*

$$\sum_{n\leqslant x} \frac{1}{n^s} = \zeta(s) + \frac{x^{1-s}}{1-s} + O\Big(\frac{1}{x^\sigma}\Big).$$

This is immediate by partial summation; see Montgomery–Vaughan [9, Theorem 1.12] or Tenenbaum [14, theorem II.3.5] for the details.

3. Proof of the Theorem

In view of the upper bounds of (1.4), it suffices to establish lower bounds. For $I(q)$ we let c_m be defined as in (2.3), and take $a_p := c_m$ for $\mathrm{e}^{\pi m} < p < \mathrm{e}^{\pi(m+1)}$. Then, for $s = \sigma + it$, $1 < \sigma \leqslant 2$, $|t| \leqslant 1$,

$$\begin{aligned}\sum_p \frac{a_p \log p}{p^s} &= \sum_{m\geqslant 0} c_m \sum_{\mathrm{e}^{\pi m} < p < \mathrm{e}^{\pi(m+1)}} \frac{\log p}{p^s}\\ &= \sum_{m\geqslant 0} c_m \Big(\sum_{\mathrm{e}^{\pi m} < n < \mathrm{e}^{\pi(m+1)}} \frac{\Lambda(n)}{n^s} + O\big(\mathrm{e}^{\pi m(1-2\sigma)}\big)\Big). \qquad (3.1)\end{aligned}$$

By Lemma 2.4 this is

$$= \sum_{m\geqslant 0} c_m \Big(\frac{\mathrm{e}^{\pi(m+1)(1-s)}}{1-s} - \frac{\mathrm{e}^{\pi m(1-s)}}{1-s}\Big) + O\Big(\sum_{m\geqslant 0} \frac{\mathrm{e}^{\pi m(1-\sigma)}}{\exp(c\sqrt{m})}\Big) + O(1).$$

Here the first error term is also $O(1)$, uniformly for $\sigma \geqslant 1$. The main term is

$$F(s) f\big(e^{\pi(1-s)}\big) \qquad (3.2)$$

where f is defined in Lemma 2.2, and

$$F(s) = \frac{e^{\pi(1-s)} - 1}{1-s}\cdot \qquad (3.3)$$

The zeros of this entire function are the numbers $1+2im$ where m runs over non-zero integers. Thus $|F(s)|$ is bounded away from 0 uniformly on the rectangle $1 \leqslant \sigma \leqslant 2$,

$-1 \leqslant t \leqslant 1$. With a little computation one can in fact show that the minimum of $|F(s)|$ in this rectangle is $|F(2 \pm i)| \approx 0.73766$. By Lemma 2.3 it follows that if σ is fixed, $1 < \sigma \leqslant 2$, then

$$\Big| \sum_p \frac{a_p \log p}{p^{\sigma+it}} \Big| \gg \frac{1}{\sqrt{1-\mathrm{e}^{\pi(\sigma-1)}}} \asymp \frac{1}{\sqrt{\sigma-1}}$$

when $t/2 \in A_{\mathrm{e}^{\pi(\sigma-1)}}$, i.e. on a subset of $-1 \leqslant t \leqslant 1$ of measure $> 1/25$. Hence $I(q) \gg (\sigma-1)^{-q/2}$.

The proof for $J(q)$ is the same, except that now the passage from $\log p$ to $\Lambda(n)$ in (3.1) is unnecessary, and instead of Lemma 2.4 we use Lemma 2.5, in which the error term is smaller.

References

[1] J. Brillhart, P. Erdős & P. Morton, On sums of Rudin–Shapiro coefficients II, *Pacific J. Math.* **107** (1983), 39–69.

[2] J.-M. Dumont & A. Thomas, Systèmes de numération et fonctions fractales relatifs aux substitutions, *Theoret. Comput. Sci.* **65** (1989), 153–169.

[3] P. Erdős & W. H. J. Fuchs, On a problem in additive number theory, *J. London Math. Soc.* **31** (1956), 67–73.

[4] M. J. E. Golay, Multi-slit spectroscopy, *J. Opt. Soc. Amer.* **39** (1949), 437–444.

[5] G. Halász, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hungar.* **19** (1968), 365–403.

[6] B. F. Logan, An interference problem for exponentials,*Michigan Math. J.* **35** (1988), 369–393.

[7] M. Mendès France & G. Tenenbaum, Dimension des courbes planes, papiers pliés, et suites de Rudin–Shapiro, *Bull. Soc. Math. France* **109** (1981), 207–215.

[8] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, CBMS No. 84, Amer. Math. Soc. (Providence), 1994, xiii+220 pp.

[9] H. L. Montgomery & R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge University Press (Cambridge), 2007, xvii+552 pp.

[10] K. Nishioka, New approach in Mahler's method, *J. Reine Angew. Math.* **407** (1990), 202–219.

[11] W. Rudin, Some theorems on Fourier coefficients, *Proc. Amer. Math. Soc.* **10** (1959), 855–859.

[12] H. S. Shapiro, *Extremal problems for polynomials and power series*, M.S. Thesis, Massachusetts Institute of Technology, 1951.

[13] G. Tenenbaum, Sur la non-dérivabilité de fonctions périodiques associées à certaines fonctions sommatoires, in: R.L. Graham & J. Nešetřil (eds), *The Mathematics of Paul Erdős*, Algorithms and combinatorics 13 (Springer Verlag, 1997), 117–128.

[14] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, troisième édition, coll. Échelles, Belin, 2008, 592 pp.

[15] N. Wiener & A. Wintner, On a local L^2-variant of Ikehara's theorem, *Rev. Math. Cuyana* **2** (1956), 53–59; *Norbert Wiener: Collected Works*, vol. II, MIT Press, Cambridge, 1979, pp. 758–764; 788–790.

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