

# On the gap distribution of prime factors\*

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**Abstract.** Let  $\{p_j(n)\}_{j=1}^{\omega(n)}$  denote the increasing sequence of distinct prime factors of an integer  $n$ . For  $z \geq 0$ , let  $G(n; z)$  denote the number of those indexes  $j$  such that  $p_{j+1}(n) > p_j(n)^{\exp z}$ . We show uniform convergence, with almost optimal effective estimate of the speed, of the distribution of  $G(n; z)$  on  $\{n : 1 \leq n \leq N\}$  to a Gaussian limit law with mean  $e^{-z} \log_2 n$ , variance  $\{e^{-z} - 2ze^{-2z}\} \log_2 n$ , and we establish an asymptotic formula with remainder for all centered moments.

**Keywords:** Distribution of prime factors, normal order, gaps between prime factors, moments of arithmetical functions, Fourier inversion, Laplace transform, Berry-Esseen inequality.

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## 1. Introduction

Let  $\{p_j(n)\}_{j=1}^{\omega(n)}$  denote the increasing sequence of distinct prime factors of an integer  $n$ . For  $z \geq 0$ , define

$$G(n; z) := |\{1 \leq j < \omega(n) : p_{j+1}(n) > p_j(n)^{\exp z}\}| \quad (n \geq 1, z \geq 0).$$

Let  $\lambda = \lambda_z := e^{-z}$ . By the sieve, it is easy to see that, for fixed  $z > 0$ ,  $N \rightarrow \infty$ , and  $j = j_N \rightarrow \infty$ , we have  $p_{j+1}(n) > p_j(n)^{\exp z}$  for  $\lambda N + o(N)$  integers  $n \leq N$ , and so that the average order of  $G(n; z)$  is  $\{\lambda + o(1)\} \log_2 n$ . Here and in the sequel, we denote by  $\log_k$  the  $k$ th fold iterated logarithm, defined for sufficiently large value of the argument.

Write  $G_N^*(n; z) := G(n, z) - \lambda \log_2 2N$  ( $1 \leq n \leq N$ ). Erdős [2] showed that  $G_N^*(n; z)^2$  has mean-value  $o((\log_2 N)^2)$  over the set of integers  $n \leq N$ , thus deducing that  $\lambda \log_2 n$  is a normal order for  $G(n; z)$ . In the same spirit, he stated later [3] that, for any given  $\lambda > 0$ , the inequality  $\max_{1 \leq j < \omega(n)} \{\log p_{j+1}(n)\} / \log p_j(n) > (\log_2 n)^\lambda$  holds on a sequence of integers  $n$  with asymptotic density  $1 - \lambda$ . The second author [9] provided the necessary details for the proof of this statement.

In a recent preprint, Sofos [5] studied the centered moments of  $\{G(n; z)\}_{1 \leq n \leq N}$  and established, for each fixed  $z \geq 0$ , the asymptotic estimate

$$\sum_{n \leq N} G_N^*(n; z)^r = \{\mu_r + o(1)\} N(\gamma \log_2 N)^{r/2} \quad (r \in \mathbb{N}, N \rightarrow \infty)$$

with  $\gamma = \gamma_z := \lambda(1 - 2\lambda z)$ , where  $\mu_r$  denotes the  $r$ th moment of the normal law.

Sofos' method rests on a probabilistic theorem of Stein [6], which provides a quantitative estimate for the speed of weak convergence of a sum of weakly dependent random variables to the Gaussian law. A second, crucial part of the analysis involves studying the convergence of moments of the model. The proof is then completed by showing sufficient agreement between the model and the arithmetical setting. According to the terminology introduced by Ruzsa [4], this proof thus pertains to the class of *indirect* approaches, in which arithmetic results are derived from genuine probabilistic statements via comparison theorems between the set  $\{n : 1 \leq n \leq N\}$  ( $N \in \mathbb{N}$ ) equipped with uniform probability and a suitable probabilistic model.

Here, we propose a *direct* approach in which the Fourier transform

$$(1) \quad \frac{1}{N} \sum_{n \leq N} e^{iy G_N^*(n; z)} \quad (y \in \mathbb{R}, N \geq 1)$$

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\* A typo in the printed version has been corrected here.

is evaluated by purely arithmetic means, with sufficient accuracy to apply an effective inversion result such as the Berry-Esseen inequality—see, e.g., [8; th. II.7.16]<sup>(1)</sup>. An upper bound for the corresponding bilateral Laplace transform then readily solves, in a comparable effective way, the question of convergence of moments. This method has three advantages: (i) the proof turns out to be very short; (ii) the speed of convergence is explicitly bounded; (iii) the effective asymptotic formula for moments is a straight corollary of the Fourier inversion theorem and the Laplace transform upper bound.

Writing  $\Phi(t) := (1/\sqrt{2\pi}) \int_{-\infty}^t e^{-y^2/2} dy$  ( $t \in \mathbb{R}$ ), we prove the following result.

**Theorem 1.** *Let  $z \geq 0$ ,  $\lambda_z := e^{-z}$ ,  $\gamma_z := \lambda_z(1 - 2z\lambda_z)$ . Uniformly for  $t \in \mathbb{R}$ ,  $N \geq 16$ , we have*

$$(2) \quad \frac{1}{N} \sum_{\substack{n \leq N \\ G_N^*(n; z) \leq t\sqrt{\gamma_z \log_2 N}}} 1 = \Phi(t) + O\left(\frac{\log_3 N}{\sqrt{\log_2 N}}\right).$$

Moreover, for any given non-negative integer  $r$ , we have

$$(3) \quad \frac{1}{N} \sum_{n \leq N} G_N^*(n; z)^r = (\gamma_z \log_2 N)^{r/2} \left\{ \mu_r + O\left(\frac{(\log_3 N)^{1+r/2}}{\sqrt{\log_2 N}}\right) \right\},$$

where  $\mu_r$  is the  $r$ th moment of the normal law. This formula persists for all real  $r \geq 0$  provided  $G_N^*(n; z)^r$  is replaced by  $|G_N^*(n; z)|^r$  and  $\mu_r$  by  $2^{r/2} \Gamma(\frac{1}{2}r + \frac{1}{2})/\sqrt{\pi}$ .

As follows from a sharp version [1] of the Erdős-Kac theorem, which corresponds to  $z = 0$ , the error terms in (2) and (3) are optimal up to the factor involving  $\log_3 N$ .

## 2. Proof

Assuming throughout that  $z > 0$ , we first prove (2). Let  $\chi_p(n)$  denote the indicator of the set of those integers  $n \leq N$  that are divisible by the prime  $p$  but by no prime  $q$  such that  $p < q \leq p^{1/\lambda}$ .

Let  $T = T_N$  to be precisely defined later, with  $T_N = o((\log_2 N)/\log_4 N)$ . Put

$$u := \exp\{(\log N)^{1/T}\}, \quad v := \exp\{(\log N)^{1-1/T}\}, \quad w := \log\left(\frac{\log v}{\log u}\right),$$

and select  $T$  in such a way that  $w/z$  is an integer. We define

$$g(n) := \sum_{u < p \leq v} \chi_p(n) - \lambda w,$$

and aim to show that the distribution of  $g(n)/\sqrt{w}$  is approximately Gaussian over the set of integers  $n \in [1, N]$ . With an effective Fourier inversion in mind, we consequently introduce the quantity

$$L(N; \vartheta) := \sum_{n \leq N} e^{i\vartheta g(n)} \quad (\vartheta \in \mathbb{R}),$$

with  $\vartheta := y/\sqrt{w}$ . We shall only need to consider  $|y| \ll \sqrt{w}$ , hence  $|\vartheta| \ll 1$ .

We have

$$L(N; \vartheta) = e^{-i\vartheta \lambda w} \sum_{n \leq N} \prod_{\substack{p|n \\ u < p \leq v}} \{1 + \chi_p(n)(e^{i\vartheta} - 1)\}.$$

Let  $\mathcal{M} = \mathcal{M}(u, v)$  denote the set of those squarefree integers  $m \leq N$  all of whose prime factors belong to the interval  $]u, v]$  and such that  $\chi_p(m) = 1$  whenever  $p|m$ . For each  $m \in \mathcal{M}$ , define  $P_m := \prod_{p|m} \prod_{p < q \leq p^{1/\lambda}} q$ . Thus

$$(4) \quad L(N; \vartheta) = e^{-i\vartheta \lambda w} \sum_{m \in \mathcal{M}} (e^{i\vartheta} - 1)^{\omega(m)} \sum_{\substack{d \leq N/m \\ (d, P_m)=1}} 1.$$

1. This inequality is crucial in some of the proofs of the Erdős-Kac theorem with optimal remainder, see e.g., [8; th. III.4.15].

A standard application of Rankin's method (see, e.g., [8; th. III.5.1]) yields that the contribution of those  $m > \sqrt{N}$  is  $\ll N/u$ . By a sieve result, e.g. [7; lemma], that of the remaining integers equals

$$\frac{N\varphi(P_m)}{mP_m} \left\{ 1 + O\left(\frac{1}{u}\right) \right\} = \frac{N\lambda^{\omega(m)}}{m} \exp \left\{ O\left(\frac{1 + \omega(m)}{\exp \sqrt{\log u}}\right) \right\},$$

where the second estimate follows from a strong form of the prime number theorem. Observing that, by definition of  $\mathcal{M}$  we must have  $z\omega(m) \leq w$ , we obtain

$$L(N; \vartheta) = Ne^{-i\vartheta\lambda w} \sum_{k \leq w/z} \lambda^k (e^{i\vartheta} - 1)^k \sum_{\substack{m \in \mathcal{M} \\ \omega(m)=k}} \frac{1}{m} + O_z \left( Ne^{-\frac{1}{2}\sqrt{\log u}} \right).$$

In the inner sum, we may write  $m = p_1 \cdots p_k$ , with

$$\log u < \log p_1 \leq \lambda \log p_2 \leq \lambda^2 \log p_3 \leq \cdots \leq \lambda^{k-1} \log p_k \leq \lambda^{k-1} \log v.$$

The sum over  $p_k$  is

$$\log_2 v - z - \log_2 p_{k-1} + O\left(\frac{1}{\log p_{k-1}}\right).$$

Summing this over  $p_{k-1}$  furnishes in turn

$$\frac{1}{2}(\log_2 v - 2z - \log_2 p_{k-2})^2 + O\left(\frac{1}{\log p_{k-2}}\right).$$

Iterating, the sum over  $p_{k-j-1}$  is

$$\frac{1}{j!}(\log_2 v - jz - \log_2 p_{k-j})^j + O\left(\frac{1}{\log p_{k-j}} + \frac{j}{(\log p_{k-j})^2}\right)$$

and finally

$$\sum_{\substack{m \in \mathcal{M} \\ \omega(m)=k}} \frac{1}{m} = \frac{(w - kz)^k}{k!} + O\left(\frac{1}{\log u}\right).$$

Therefore

$$(5) \quad L(N; \vartheta) = NF(\vartheta, w) + O\left(\frac{N}{(\log N)^{1/T}}\right),$$

with

$$(6) \quad F(\vartheta, w) := e^{-i\lambda\vartheta w} \sum_{k \leq w/z} \frac{\lambda^k (e^{i\vartheta} - 1)^k (w - kz)^k}{k!}.$$

Write  $\Delta(\vartheta, w) := F(\vartheta, w) - e^{-\frac{1}{2}\vartheta^2\gamma w}$ . We now aim to show that, still with  $\gamma = \gamma_z$  and for suitable positive constants  $c_0, c_1$  depending at most on  $z$ , we have

$$(7) \quad \Delta(\vartheta, w) \ll \vartheta \quad (|\vartheta| \leq w^{-2}),$$

$$(8) \quad \Delta(\vartheta, w) \ll \vartheta^2 w \quad (w^{-2} < |\vartheta| \leq w^{-3/4}),$$

$$(9) \quad \Delta(\vartheta, w) \ll \vartheta^3 w e^{-\frac{1}{2}\gamma\vartheta^2 w} \quad (w^{-3/4} < |\vartheta| \leq w^{-1/3}),$$

$$(10) \quad \Delta(\vartheta, w) \ll e^{-c_1\vartheta^2 w} \quad (w^{-1/3} < |\vartheta| \leq c_0).$$

We first note that (7) readily follows from the estimates

$$e^{-i\lambda\vartheta w} = 1 - i\lambda\vartheta w + O(\vartheta), \quad \sum_{k \leq w/z} \frac{\lambda^k (e^{i\vartheta} - 1)^k (w - kz)^k}{k!} = 1 + i\lambda\vartheta w + O(\vartheta) \quad (|\vartheta| \leq w^{-2}).$$

We next prove (8). We introduce the notation

$$(11) \quad \mu := \lambda(e^{i\vartheta} - 1), \quad \varrho := |\mu| = 2\lambda|\sin(\frac{1}{2}\vartheta)|.$$

Put  $m := \lfloor \varrho w + \sqrt{w} \rfloor$ . Then

$$\sum_{k \geq m+1} \frac{\varrho^k w^k}{k!} \ll \frac{\varrho^m w^m}{m!} \ll \left(\frac{\varrho e w}{m}\right)^m \ll e^{-\sqrt{w}} \ll \vartheta^2 w.$$

Therefore, using the fact that  $(w - kz)^k = w^k \{1 + O(k^2/w)\}$  for  $k \leq m$ ,

$$F(\vartheta, w) = e^{-i\lambda\vartheta w} \left\{ e^{\mu w} + O(\vartheta^2 w) \right\} = e^{(\mu - i\lambda\vartheta)w} + O(\vartheta^2 w) = 1 + O(\vartheta^2 w).$$

This is plainly sufficient.

We now embark to proving (9) and (10). Define  $K := w/z$  (an integer by construction), so that, for any  $R > 0$ ,

$$\begin{aligned} e^{i\lambda\vartheta w} F(\vartheta, w) &= \sum_{0 \leq k \leq K} \frac{\mu^k (w - kz)^k}{k!} = \frac{1}{2\pi i} \oint_{|\zeta|=R} \sum_{0 \leq k \leq K} \frac{e^{\mu(w-kz)\zeta}}{\zeta^{k+1}} d\zeta \\ &= \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{1 - e^{-(K+1)\mu z \zeta} / \zeta^{K+1}}{\zeta - e^{-\mu z \zeta}} e^{\mu w \zeta} d\zeta. \end{aligned}$$

Selecting  $R := 1/|\vartheta|$ , we see that the contribution of the term  $e^{-(K+1)\mu z \zeta} / \zeta^{K+1}$  is

$$\ll e^{2w} |\vartheta|^{w/z} \ll e^{-w}$$

in the considered range. In the remaining integral, we observe that  $-\Re(\mu z \zeta) \ll 1$ , so, by Rouché's theorem, the equation  $\zeta - e^{-\mu z \zeta} = 0$  has exactly one solution, say  $\beta$ , inside the circle. Moreover, we have  $\beta = 1 - \mu z + O(\vartheta^2)$ . The residue theorem thus yields

$$\begin{aligned} e^{i\lambda\vartheta w} F(\vartheta, w) &= \frac{e^{\mu w \beta}}{1 + \mu z e^{-\mu z \beta}} + O(e^{-w}) \quad (w^{-3/4} < |\vartheta| \leq w^{-1/3}). \\ &= e^{\mu w - \mu^2 z w} \{1 + O(\vartheta)\} \end{aligned}$$

Since  $-i\lambda\vartheta + \mu - \mu^2 z = -\frac{1}{2}\gamma\vartheta^2 + O(\vartheta^3)$ , this completes the proof of (9). When  $x^{-1/3} < |\vartheta| \leq c_0$  with sufficiently small  $c_0 = c_0(z)$ , we note that  $\Re\{\mu w \beta - i\lambda\vartheta w\} \leq -c_1 \vartheta^2 w$ : this confirms (10).

Selecting with  $T_N := \{1 + O(1/\log_3 N)\}(\log_2 N)/\log_3 N$ , we may now apply the Berry-Esseen inequality to obtain, uniformly for real  $t$ ,

$$\frac{1}{N} \sum_{\substack{n \leq N \\ g(n) \leq \lambda w + t\sqrt{\gamma w}}} 1 - \Phi(t) \ll \frac{1}{\sqrt{w}} + \int_{-c_0}^{c_0} \left| \frac{\Delta(\vartheta, w)}{\vartheta} \right| d\vartheta \ll \frac{1}{\sqrt{\log_2 N}}.$$

However, a standard estimate (see, e.g., [8; th. III.3.8]) furnishes that

$$g(n) - \lambda w = G_N^*(n; z) + O(\log_3 N)$$

for all but at most  $\ll N/\sqrt{\log_2 N}$  integers  $n \leq N$ . This plainly establishes (2).

In order to deduce (3) from (2), we need an argument enabling us to discard very large values of  $G^*(n; z)$ . To this end, we employ the Laplace transform

$$\mathcal{L}(N; \vartheta) := L(N; -i\vartheta) = \sum_{n \leq N} e^{\vartheta g(n)} \quad (\vartheta \in \mathbb{R}).$$

Arguing as previously, we get, for  $\vartheta \ll 1$ ,

$$(12) \quad \mathcal{L}(N; \vartheta) = N\mathcal{F}(\vartheta, w) + O\left(\frac{N}{(\log N)^{1/T}}\right),$$

with now

$$(13) \quad \mathcal{F}(\vartheta, w) := e^{-\vartheta\lambda w} \sum_{k \leq w/z} \frac{\lambda^k (e^\vartheta - 1)^k (w - kz)^k}{k!}.$$

We aim to show that

$$(14) \quad \mathcal{F}(\vartheta, w) \ll e^{\frac{1}{2}\lambda\vartheta^2 w} \quad (w^{-1/2} \leq |\vartheta| \leq w^{-1/3}).$$

For  $\vartheta \geq 0$ , this follows trivially from bounding  $w - kz$  by  $w$  in (6). The complementary case turns out to be more subtle. Writing  $\nu := \lambda(1 - e^{-\vartheta})$ , we have, for  $\alpha > 0$ ,  $K = w/z$ ,

$$e^{-\vartheta\lambda w} \mathcal{F}(-\vartheta, w) = \sum_{0 \leq k \leq w/z} \frac{(-1)^k \nu^k (w - kz)^k}{k!} = \frac{I(\alpha) - I(-K - \frac{1}{2})}{2\pi i},$$

where

$$I(a) := \int_{a-i\infty}^{a+i\infty} H(s) ds \quad (a \geq -K - \frac{1}{2}), \quad H(s) := \frac{\Gamma(s)}{\{\nu(w + sz)\}^s}.$$

This stems from the fact that the residue of  $\Gamma(s)$  at  $s = -k$  is  $(-1)^k/k!$  and from a classical upper bound for  $\Gamma(s)$  in vertical strips.

We select  $\alpha := w\nu/(1 - z\nu)$ , so that  $\nu(w + \alpha z) = \alpha \gg \sqrt{w}$ . By the complex Stirling's formula (see, e.g., [8; th. II.0.12]), we have, for  $s = \alpha + i\tau$ ,

$$H(s) \ll \frac{e^{-\alpha}}{\sqrt{\alpha}} \left| \left( \frac{1 + i\tau/\alpha}{1 + i\tau\nu z/\alpha} \right)^s \right|.$$

Now

$$\begin{aligned} \Re \{(\alpha + i\tau) \log(1 + i\tau/\alpha)\} &= \frac{1}{2}\alpha \log\left(1 + \frac{\tau^2}{\alpha^2}\right) - \tau \arctan\left(\frac{\tau}{\alpha}\right) \\ &= \int_0^{|\tau|/\alpha} \frac{\alpha t - |\tau|}{1 + t^2} dt, \end{aligned}$$

and similarly

$$-\Re \{(\alpha + i\tau) \log(1 + i\tau\nu z/\alpha)\} \leq \int_0^{|\tau\nu z/\alpha} \frac{|\tau| - \alpha t}{1 + t^2} dt.$$

Therefore  $H(s) \ll e^{-\alpha}/\sqrt{\alpha}$ , and we infer that the contribution to  $I(\alpha)$  of the range  $|\tau| \leq w^2$  is

$$(15) \quad \ll e^{-\alpha} w^2 / \sqrt{\alpha} \ll e^{-\alpha} w^2.$$

However, in the case  $|\tau| > w^2$ , we have, by a further appeal to Stirling's complex formula,

$$H(s) = \sqrt{\frac{2\pi}{s}} (\nu z e)^{-s} \left(1 + \frac{w}{zs}\right)^{-s} \left\{1 + O\left(\frac{1}{s}\right)\right\} = \sqrt{\frac{2\pi}{s}} (\nu z e)^{-s} e^{-w/z} \left\{1 + O\left(\frac{w^2}{s}\right)\right\}.$$

The main term may be estimated by partial integration, while the remainder furnishes an absolutely convergent contribution. The overall upper bound is  $\ll e^{-w/z + \alpha \log(1/\nu z e)}$ , which is negligible in front of (15).

Thus we have proved that

$$(16) \quad I(\alpha) \ll e^{-w\nu/(1-z\nu)} w^2.$$

To bound  $I(-K - \frac{1}{2})$ , we observe that, for  $\Re s = -\sigma = -K - \frac{1}{2}$  and suitable  $c = c_z > 0$ , we have

$$\left| \frac{\Gamma(s)}{\nu^s (w + sz)^s} \right| = \left| \frac{\nu^\sigma \Gamma(s + K + 1)}{(w + sz)^s \prod_{0 \leq j \leq K} (j + \frac{1}{2} + i\tau)} \right| \ll \nu^K e^{-c|\tau|},$$

using Stirling's formula on the line  $\Re s = \frac{1}{2}$  and the estimates

$$|(w + sz)^{-s}| \ll (z^2 + \tau^2)^{\sigma/2} \leq (z^2 + \tau^2)^{K/2+1/4}, \quad \prod_{1 \leq j \leq K} \left\{ \frac{z^2 + \tau^2}{(j + \frac{1}{2})^2 + \tau^2} \right\}^{1/2} \ll e^{-c|\tau|}.$$

This furnishes the bound  $I(-K - \frac{1}{2}) \ll \nu^K$ , which, in the considered range, is negligible in front of that of (16).

Gathering our estimates, we obtain

$$\mathcal{F}(-\vartheta, w) \ll e^{\vartheta \lambda w - w\nu/(1-z\nu)} w^2 \ll e^{\frac{1}{2}\vartheta^2 \gamma w} w^2 \ll e^{\frac{1}{2}\lambda \vartheta^2 w} \quad (0 \leq \vartheta \leq w^{-1/3}),$$

which finishes the proof of (14).

Since (14) implies that the contribution to the  $r$ th moment of those  $n$  with  $|G_N^*(n; z)| > T\sqrt{\log_2 N}$  is bounded by the claimed error term provided  $T > c_r\sqrt{\log_3 N}$  for suitable constant  $c_r > 0$ , we obtain (3) as an immediate consequence. The case of absolute moments is similar.

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