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On the densities of sets of multiples

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1. Introduction and statement of results

Let $\mathcal{A}$ denote a strictly increasing sequence of integers exceeding 1 and let

$$\mathcal{M}(\mathcal{A}) := \{ma : m \geqslant 1,\, a \in \mathcal{A}\}$$

denote its set of multiples. It was conjectured in the early thirties that $\mathcal{M}(\mathcal{A})$ has a density for any $\mathcal{A}$, but this was disproved by Besicovitch [Bes] in 1934. However, Davenport and Erdős showed in 1937 [DE1] (see also [DE2]) that a slightly weaker result did hold, namely that any set of multiples has a logarithmic density, actually equal to its lower asymptotic density. This means that it is in general rather delicate to decide whether $\mathcal{M}(\mathcal{A})$ has or not a density for a given sequence $\mathcal{A}$.

One of the aims of this paper is to investigate further the structure of those sequences $\mathcal{A}$ such that $\mathcal{M}(\mathcal{A})$ has an asymptotic density. Paradoxically, we call these sequences *Besicovitch sequences* because we believe, although Besicovitch's contribution was to show that not all sequences have this property, that it is better to base our definition on a positive property rather than on a negative one.

Important progress was made by Erdős in 1948 [Er] with the following criterion. Given an integer sequence $\mathcal{A}$ and a number b, which does not necessarily belong to $\mathcal{A}$, we denote by $M(x, b; \mathcal{A})$ the number of integers $n \leqslant x$ such that $b|n$ and n has no divisor in $\mathcal{A}$ strictly less that b — when b does belong to $\mathcal{A}$, this means that b is the smallest divisor of n which belongs to $\mathcal{A}$.

Theorem 0 (Erdős). *Let $\mathcal{A}$ be a sequence of integers exceeding* 1. *Then $\mathcal{A}$ is a Besicovitch sequence if, and only if,*

$$\lim_{\varepsilon\to 0}\limsup_{x\to+\infty} x^{-1} \sum_{\substack{x^{1-\varepsilon}<a\leqslant x\\ a\in\mathcal{A}}} M(x, a; \mathcal{A}) = 0. \tag{1.1}$$

We shall give a simple proof of this theorem in the next section. It was used in [Er] to show that $\mathcal{E} := \{ab : 1 \leqslant a < b \leqslant 2a\}$ is Besicovitch. The proof that $\mathcal{E}$ is actually a *Behrend sequence*, i.e. a sequence which set of multiples has asymptotic density 1 (see [Ha,HT2]), had to wait until 1983 [MT].

Our first two results concern stability of the Besicovitch property under union and intersection. In the former case the answer is positive without any restriction.

Theorem 1. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be Besicovitch sequences. Then $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a Besicovitch sequence.*

This is an obvious consequence of Theorem 0 since, for all $x \geqslant 1$ and $a \in \mathcal{A}_1 \cup \mathcal{A}_2$,

$$M(x, a; \mathcal{A}_1 \cup \mathcal{A}_2) \leqslant \min\bigl\{M(x, a; \mathcal{A}_1), M(x, a; \mathcal{A}_2)\bigr\}.$$

Recall that a sequence of integers is called *primitive* when none of its elements divides any other. Given an arbitrary sequence $\mathcal{A}$, it is always possible to find a (unique) primitive subsequence $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{M}(\mathcal{A}') = \mathcal{M}(\mathcal{A})$ — see [HT1], p. 48 or [HR], chapter V. It is easy to find counterexamples showing that the Besicovitch property is not stable under intersection for non primitive sequences. Indeed, suppose $\mathcal{A}$ is a non Besicovitch sequence. Put $\mathcal{A}_0 := 2\mathcal{A} = \{2a : a \in \mathcal{A}\}$. Then, since $\mathcal{M}(\mathcal{A}_0) = 2\mathcal{M}(\mathcal{A})$, it is plain that $\mathcal{A}_0$ is also non Besicovitch. If we now set $\mathcal{A}_1 := \{n : n > 2\}$, $\mathcal{A}_2 := \{2\} \cup \mathcal{A}_0$, we see that $\mathcal{M}(\mathcal{A}_1) = \mathcal{A}_1$ has density 1, and that $\mathcal{M}(\mathcal{A}_2) = 2\mathbb{Z}^+$ has density $\frac{1}{2}$, so $\mathcal{A}_1$ and $\mathcal{A}_2$ are both Besicovitch ; however, $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{A}_0$ is not. Our next result, the proof of which necessitates much more elaborate tools, shows that counterexamples may be found even among primitive sequences.

Theorem 2. *There exist pairs $\{\mathcal{A}_1, \mathcal{A}_2\}$ of primitive Besicovitch sequences such that $\mathcal{A}_1 \cap \mathcal{A}_2$ is not a Besicovitch sequence.*

For any integer $k \geqslant 3$, it is possible to construct by the same method k primitive Besicovitch sequences with a non Besicovitch intersection, but such that all the intersections of $k-1$ sequences is Besicovitch.

Our third result gives a simple sufficient condition that a sequence be Besicovitch. It contains as typical special cases sequences composed of pairwise coprime elements and sequences composed of integers with a bounded number of prime factors. In the following statement and in the sequel of the paper, we denote by $\Omega(n)$ (resp. $\omega(n)$) the total number of prime factors of an integer n, counted with (resp. without) multiplicity.

Theorem 3. *Let $k \in \mathbb{Z}^+$ be fixed, and let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be an arbitrary sequence of integers such that*

$$\max_{1\leqslant i<j} \omega\Big((a_i, a_j)\Big) \leqslant k \qquad (j = 1, 2, \ldots). \tag{1.2}$$

Then $\mathcal{A}$ is a Besicovitch sequence.

Moreover, the above statement is optimal in the sense that, given an arbitrary sequence $k_j \to +\infty$, there is a non Besicovitch sequence $\mathcal{A}$ satisfying

$$\Omega(a_j) \leqslant k_j \qquad (j = 1, 2, \ldots). \tag{1.3}$$

Since the left-hand side of (1.3) is trivially at least as large as that of (1.2) for every j, we see that the second part of the statement implies that the upper bound in (1.2) cannot be replaced by a quantity tending to infinity with j.

Related to this result, we have an annoying problem. When $\mathcal{A}$ consists solely of integers having a fixed number of prime factors (so that, by Theorem 3, $\mathcal{M}(\mathcal{A})$ has a density), we cannot find in general a simple criterion for $\mathcal{A}$ to be a Behrend sequence, that is for $\mathcal{M}(\mathcal{A})$ to be of density 1. The corresponding question for a sequence of primes is easy, the required criterion being

$$\sum_{a\in\mathcal{A}} a^{-1} = +\infty.$$

Ruzsa (private communication) has very recently settled the case of two prime factors. He has shown that $\mathcal{A} \subset \{n : \Omega(n) = 2\}$ is Behrend if, and only if, for any partition $\mathbb{P} = \mathcal{P} \cup \mathcal{P}'$ of the primes such that $\sum_{p\in\mathcal{P}} 1/p < \infty$, we have

$$\sum_{\substack{a=pq\in\mathcal{A}\\ p\in\mathcal{P}',\, q\in\mathcal{P}'}} \frac{1}{a} = \infty. \tag{1.4}$$

We now make an easy, but useful, observation concerning sequences of the type $\mathcal{A} = \cup_j (T_j, 2T_j] \cap \mathbb{Z}^+$. These so-called "block" sequences play an important role in the whole theory. Besicovitch's original counterexample is of this form; also the structure involved is sufficiently smooth to leave some hope of finding a relatively simple criterion for such a sequence to be Behrend — see in particular the necessary condition given in Theorem 1 of [HT2]. Our remark is that such a sequence $\mathcal{A}$ is Behrend as soon as it is Besicovitch. Indeed, if $M(x)$ denotes the counting function of $\mathcal{M}(\mathcal{A})$, the Besicovitch property implies that $M(x) \sim \alpha x$ as $x \to +\infty$ for some $\alpha \in [0,1]$. But then

$$2\alpha T_j \sim M(2T_j) = M(T_j) + T_j + O(1) \sim (1+\alpha)T_j \qquad (j \to +\infty).$$

Hence we must have $\alpha = 1$.

Let $\tau(n,\mathcal{A})$ denote the number of divisors of n which belong to $\mathcal{A}$. Then $\mathcal{M}(\mathcal{A})$ is exactly the sequence of integers n such that $\tau(n,\mathcal{A}) > 0$. More generally, let $\mathcal{A}^{(k)}$ denote the kth derived sequence of $\mathcal{A} = \{a_1, a_2, \ldots\}$, namely

$$\mathcal{A}^{(k)} := \{[a_{i_0}, a_{i_1}, \ldots, a_{i_k}] : 1 \leqslant i_0 < i_1 < \ldots < i_k\}.$$

Then $\mathcal{M}\big(\mathcal{A}^{(k)}\big) = \{n : \tau(n,\mathcal{A}) > k\}$. A surprising property of Behrend sequences is that any of their derivatives is still a Behrend sequence. In other words, we have

$$\tau(n,\mathcal{A}) \to +\infty \qquad \text{pp} \tag{1.5}$$

whenever $\mathcal{A}$ is Behrend. (Here as in previous works, we use the notation pp — *presque partout* — to indicate that a relation holds on a set of asymptotic density one .) Relation (1.5) was proved in [HT2], as a simple consequence of the Davenport–Erdős theorem [DE1] and Behrend's inequality [Beh] — see (5.1) below.

We now investigate various quantitative forms of (1.5). Here and throughout, we denote by $\mathrm{d}\mathcal{A}$ (resp. $\overline{\mathrm{d}}\mathcal{A}$, $\underline{\mathrm{d}}\mathcal{A}$) the asymptotic (resp. upper, lower asymptotic) density of an integer sequence $\mathcal{A}$. Similarly, we let $\delta\mathcal{A}$ denote the logarithmic density of $\mathcal{A}$, with parallel notations for the upper and lower variants. We then write

$$t_k(\mathcal{A}) := 1 - \delta\mathcal{M}(\mathcal{A}^{(k)}) = 1 - \underline{\mathrm{d}}\mathcal{M}(\mathcal{A}^{(k)}), \tag{1.6}$$

where the second equality follows from the Davenport–Erdős theorem. Thus, we have

$$t_k(\mathcal{A}) = \delta\{n : \tau(n,\mathcal{A}) \leqslant k\} = \overline{\mathrm{d}}\{n : \tau(n,\mathcal{A}) \leqslant k\}, \tag{1.7}$$

and (1.5) means that $t_0(\mathcal{A}) = 0$ implies $t_k(\mathcal{A}) = 0$ for every k. We improve on this in the following result.

Theorem 4. *Let* $\varphi_k(\sigma) := \sup\{t_k(\mathcal{A}) : t_0(\mathcal{A}) \leqslant \sigma\}$ $(0 \leqslant \sigma \leqslant 1)$. *Then, for each* $k \in \mathbb{Z}^+$, φ_k *is a continuous function of* σ *and we have*

$$\varphi_k(\sigma) \leqslant (k+2)\sigma^{1/(k+1)} \qquad (0 \leqslant \sigma \leqslant 1). \tag{1.7}$$

Of course, this inequality is useful only for rather small values of σ, the right-hand side being larger than 1 for $\sigma > (k+2)^{-k-1}$. This raises the question of evaluating the greatest lower bound ϱ_k of the set of $\sigma \in [0,1]$ such that $\varphi_k(\sigma) = 1$. Since $t_k(\mathcal{A}) = 1$ if, and only if, $|\mathcal{A}| \leqslant k$, it is obvious that $\varrho_k \leqslant \pi_k$, where

$$\pi_k := \inf\{t_0(\mathcal{A}) : |\mathcal{A}| \leqslant k\}, \tag{1.8}$$

but it is not at all clear that equality holds. We can however achieve this in the case when the elements are relatively prime — see Corollary to Theorem 6 below. Since $\pi_k < 1$ and $\varphi_k([\pi_k, 1]) = \{1\}$, any polynomial (or even analytic) upper bound for $\varphi_k(\sigma)$ will take values exceeding 1. It is clear that $\varphi_k(\sigma)$ is a non-decreasing function of σ, but it is worthwhile to notice that $t_0(\mathcal{A}) < t_0(\mathcal{B})$ does not imply $t_k(\mathcal{A}) < t_k(\mathcal{B})$. A counterexample is $\mathcal{A} := \{3, 5, 7\}$, $\mathcal{B} := \{2, 4\}$, when

$$t_0(\mathcal{A}) = \tfrac{16}{35}, \quad t_0(\mathcal{B}) = \tfrac{1}{2}, \quad t_1(\mathcal{A}) = \tfrac{92}{105}, \quad t_1(\mathcal{B}) = \tfrac{3}{4}.$$

We now evaluate π_k in the following fairly intuitive result. We write

$$\{p_1 = 2, p_2 = 3, \ldots\}$$

for the increasing sequence of primes, and let the letter p denote generically a prime number.

Theorem 5. *Let π_k be defined by* (1.8). *Then we have*

$$\pi_k = \prod_{p \leqslant p_k} (1 - 1/p). \tag{1.9}$$

This means that, if one wants to sieve out a set of density as large as possible using only k integers, the set of the first k primes is the more efficient choice.

Our last two theorems concern the case when the elements of $\mathcal{A}$ are pairwise coprime. We define

$$t_{k,n} := t_k(\{2, 3, 5, \ldots, p_n\}), \tag{1.10}$$

the density of the integers divisible by at most k among the first n primes. We also let $\varphi_k^*(\sigma)$ be the function defined as in the statement of Theorem 4 with the extra condition that the elements of $\mathcal{A}$ should be relatively prime. We stated in Theorem 4, and prove in Section 5, that φ_k is a continous function of σ. The same argument yields that this is equally valid for φ_k^* ; we skip the details which are identical, *mutatis mutandis*, to those appearing in Section 5. We first give an inequality for $t_k(\mathcal{A})$ that yields a sharp upper bound for $\varphi_k^*(\sigma)$.

Theorem 6. *Let the elements of $\mathcal{A}$ be pairwise coprime and $t_0(\mathcal{A}) = \sigma$. Then, for $k \geqslant 1$ and each $n \geqslant 1$, we have*

$$t_k(\mathcal{A}) \leqslant \frac{(\sigma - \pi_{n+1})t_{k,n} + (\pi_n - \sigma)t_{k,n+1}}{\pi_n - \pi_{n+1}} \qquad (\pi_{n+1} \leqslant \sigma \leqslant \pi_n). \tag{1.11}$$

Corollary. *We have for $k \geqslant 1$, $n \geqslant 1$, $\pi_{n+1} \leqslant \sigma \leqslant \pi_n$,*

$$\varphi_k^*(\sigma) \leqslant \frac{(\sigma - \pi_{n+1})t_{k,n} + (\pi_n - \sigma)t_{k,n+1}}{\pi_n - \pi_{n+1}} \tag{1.12}$$

with equality at the end-points. In particular, $\varphi_k^*(\sigma) < 1$ for all $\sigma < \pi_k$.

Since $\varphi_k^*(\sigma)$ is a monotonic function, this gives lower estimates as well. In some applications, however, explicit analytic bounds for $\varphi_k^*(\sigma)$ will be more useful.

Theorem 7. *We have for all* $\sigma \in]0,1]$

$$\sigma \sum_{j=0}^{k} \frac{1}{j!}\Big(\log\frac{1}{\sigma}\Big)^j \leqslant \varphi_k^*(\sigma) \leqslant e^C \sigma \sum_{j=0}^{k} \frac{1}{j!}\Big(\log\frac{1}{\sigma}\Big)^j, \tag{1.12}$$

where $C := \sum_p \{1/(p-1) + \log(1-1/p)\} = 0.45743\ldots$.

Here again, since $\varphi_k^*(\sigma) = 1$ for $\sigma \geqslant \pi_k$, we see that the factor e^C in (1.12) cannot be replaced by an expression tending to 1 too quickly as a function of k. Nevertheless, σ must be extravagantly small before (1.12) yields $\varphi_k^*(\sigma) < 1$: whereas, by Mertens' theorem, $\pi_k \sim e^{-\gamma}/\log k$, classical estimates on the exponential series show that the right-hand side of (1.12) equals 1 when $\sigma = \exp\{-k - C_1\sqrt{k}\}$ with $C_1 = C_1(k)$ positive and bounded.

Our upper and lower bounds for $\varphi_k(\sigma)$ and $\varphi_k^*(\sigma)$ are all concave functions of σ, and this might suggest that these functions are themselves concave. This is not true of φ_k^*. We can show in fact that φ_k^* possesses intervals of constancy. We postpone the proof, which we hope to extend to φ_k. There seems little reason to suppose that these functions behave very differently, if indeed they are distinct.

It is possible to give heuristic arguments which suggest that $\varphi_k(\sigma) = \varphi_k^*(\sigma)$ identically. We can prove nothing like this. A first step towards such a result would be to show that $\varrho_k = \pi_k$, that is

$$\varphi_k(\sigma) < 1 \quad \text{for all} \quad \sigma < \pi_k.$$

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2. A short proof of Erdős' criterion

The proof we present here of Theorem 1 is technically much easier than in Erdős' original setting. However, the basic ideas remain essentially the same.

The following lemma will be very useful. It is a sligtltly improved version of Theorem 07 of [HT1] (which states the same estimate with an unspecified constant c_0 intead of $\frac{1}{2}$ in the exponential) and may be proved by an easy application of Rankin's method.

Lemma 2.1. *We have uniformly for* $x \geqslant z \geqslant y \geqslant 2$

$$\Big|\Big\{n \leqslant x : \prod_{\substack{p^\nu || n \\ p \leqslant y}} p^\nu > z\Big\}\Big| \ll x \exp\Big\{-\frac{\log z}{2\log y}\Big\}.$$

Given an integer sequence $\mathcal{A}$, we write $d_1(n,\mathcal{A})$ for the smallest divisor of n which belongs to $\mathcal{A}$, with the convention that $d_1(n,\mathcal{A}) = \infty$ if $n \notin \mathcal{M}(\mathcal{A})$. We let a denote generically an integer that belongs to $\mathcal{A}$ and, for each a, we let $B(a)$ be the density of the set of those n such that $d_1(n,\mathcal{A}) = a$. By the Davenport-Erdős theorem [DE1,DE2], we have

$$\delta\mathcal{M}(\mathcal{A}) = \underline{\mathrm{d}}\mathcal{M}(\mathcal{A}) = \sum_{a\in\mathcal{A}} B(a). \tag{2.1}$$

We let $M(x)$ denote the counting function of $\mathcal{M}(\mathcal{A})$ and note that

$$M(x) = \sum_{a\leqslant x} M(x,a;\mathcal{A}) \tag{2.2}$$

We shall show that (1.1) is necessary and sufficient for $\mathcal{M}(\mathcal{A})$ to have a density.

First, let us establish the necessity. We assume that $\mathrm{d}\mathcal{M}(\mathcal{A})$ exists, and by (2.1) we have $\mathrm{d}\mathcal{M}(\mathcal{A}) = \delta\mathcal{M}(\mathcal{A})$, so we may write in view of (2.2)

$$x^{-1}\sum_{a\leqslant x} M(x,a;\mathcal{A}) = \delta\mathcal{M}(\mathcal{A}) + o(1) \qquad (x\to+\infty). \tag{2.3}$$

Let $T>0$ be fixed. Then we have as $x\to+\infty$

$$x^{-1}\sum_{a\leqslant T} M(x,a;\mathcal{A}) = \sum_{a\leqslant T} B(a) + o(1). \tag{2.4}$$

Subtracting this from (2.3) and using (2.1), we obtain

$$x^{-1}\sum_{T<a\leqslant x} M(x,a;\mathcal{A}) = \sum_{a>T} B(a) + o(1).$$

Hence (1.1) holds, in the stronger form

$$\lim_{T\to+\infty}\limsup_{x\to+\infty} x^{-1}\sum_{T<a\leqslant x} M(x,a;\mathcal{A}) = 0.$$

To prove the sufficiency, we again consider (2.2). Given arbitrary $\varepsilon>0$, we can find $T_0 = T_0(\varepsilon)$ such that for all $T>T_0$ we have

$$\delta\mathcal{M}(\mathcal{A}) - \varepsilon \leqslant \sum_{a\leqslant T} B(a) \leqslant \delta\mathcal{M}(\mathcal{A}).$$

Thus, in view of (2.4), we deduce that for fixed, positive ε, T with $T>T_0(\varepsilon)$ and suitable $x_0(\varepsilon,T)$

$$\Big|x^{-1}\sum_{a\leqslant T} M(x,a;\mathcal{A}) - \delta\mathcal{M}(\mathcal{A})\Big| \leqslant 2\varepsilon \qquad \big(x>x_0(\varepsilon,T)\big).$$

Moreover the hypothesis (1.1) tells us that the subsum of (2.2) corresponding to $x^{1-\varepsilon}<a\leqslant x$ is at most $\eta(\varepsilon)x$ with $\eta(\varepsilon)\to0$ as $\varepsilon\to0$. Thus it is enough to establish that, given arbitrary $\varepsilon>0$ and suitable $T=T(\varepsilon)>T_0(\varepsilon)$, we have for $x>x_1(\varepsilon)$

$$\sum_{T<a\leqslant x^{1-\varepsilon}} M(x,a;\mathcal{A}) \ll \varepsilon x. \tag{2.5}$$

Indeed, assuming this for the moment, we get for fixed ε and large x

$$x^{-1}M(x) = \delta\mathcal{M}(\mathcal{A}) + O(\varepsilon) + O(\eta(\varepsilon)),$$

so the desired result follows by letting successively $x\to+\infty$ and $\varepsilon\to0$.

The left-hand side of (2.5) is equal to the number of those integers $n\leqslant x$ such that $T<d_1(n,\mathcal{A})\leqslant x^{1-\varepsilon}$. We now decompose canonically every integer n in the form $n=n_1n_2$, where n_1 is the largest divisor of n all of whose prime factors are $\leqslant x^{\varepsilon^2}$. By Lemma 2.1, the number of integers n such that $n_1>x^{\varepsilon/2}$ is $\ll e^{-1/(4\varepsilon)}x$. This is of smaller order of magnitude than the left-hand side of (2.5), and we may hence discard the corresponding set of integers. Thus, if we set

$$M'(x,a) := |\{n\leqslant x : n_1\leqslant x^{\varepsilon/2},\, d_1(n,\mathcal{A})=a\}|,$$

we only need to prove that

$$\sum_{T<a\leqslant x^{1-\varepsilon}} M'(x,a) \ll \varepsilon x. \tag{2.6}$$

In view of the convergence of the series $\sum_a B(a)$, this is an immediate consequence of the following bound, of independent interest,

$$M'(x,a) \ll_\varepsilon B(a)x \qquad (a \leqslant x^{1-\varepsilon}), \tag{2.7}$$

which we shall next establish.

We observe that we have for each $a \leqslant x$

$$B(a) = \prod_{p\leqslant x}(1-\frac{1}{p}) \sum_{\substack{P(m)\leqslant x\\ d_1(am,\mathcal{A})=a}} \frac{1}{am}, \tag{2.8}$$

where, here and in the sequel, we let $P(m)$ denote the largest prime factor of m, with the convention that $P(1)=1$. Formula (2.8) follows by a simple computation on noticing that $d_1(n,\mathcal{A})=a$ holds if, and only if, n may be (uniquely) decomposed in the form $n=amh$ with $d_1(am,\mathcal{A})=a$, $P(m)\leqslant x$, and $p|h \Rightarrow p>x$. Now any integer n counted by $M'(x,a)$ with $a\leqslant x^{1-\varepsilon}$ may be written as $n=aN$, with $N_1 \leqslant a_1N_1 = n_1 \leqslant x^{\varepsilon/2}$, so that $aN_1 \leqslant x^{1-\varepsilon/2}$, and $a=d_1(n,\mathcal{A})=d_1(aN,\mathcal{A})=d_1(aN_1,\mathcal{A})$. By the classical upper bounds of the sieve, we hence obtain for $a\leqslant x^{1-\varepsilon}$

$$\begin{aligned} M'(x,a) \leqslant \sum_{\substack{N\leqslant x/a\\ aN_1\leqslant x^{1-\varepsilon/2}\\ d_1(aN_1,\mathcal{A})=a}} 1 &= \sum_{\substack{am\leqslant x^{1-\varepsilon/2}\\ d_1(am,\mathcal{A})=a\\ P(m)\leqslant x^{\varepsilon^2}}} \sum_{\substack{\ell\leqslant x/(am)\\ p|\ell\Rightarrow p>x^{\varepsilon^2}}} 1 \\ &\ll x \prod_{p\leqslant x^{\varepsilon^2}} (1-\frac{1}{p}) \sum_{\substack{P(m)\leqslant x^{\varepsilon^2}\\ d_1(am,\mathcal{A})=a}} \frac{1}{am} \ll x\,\varepsilon^{-2}B(a). \end{aligned}$$

This completes the proof.

3. Proof of Theorem 2

We shall actually construct primitive Behrend sequences $\mathcal{A}_1$, $\mathcal{A}_2$ such that $\mathcal{A}_1\cap\mathcal{A}_2$ is not a Besicovitch sequence. The starting idea is to take a classical non-Besicovitch primitive sequence, say $\mathcal{B}^*$, and extend it in two disjoint ways in order to produce two Behrend sequences with intersection $\mathcal{B}^*$. The difficulty is of course to keep these sequences primitive.

The construction of $\mathcal{B}^*$ is as follows. We set $\delta := 1-(1+\log_2 2)/\log 2 = 0.08607\ldots$, and put $\varepsilon_j := j^{-2/\delta}$ for $j=1,2\ldots$. We also give ourselves a sequence T_j with $T_1>e^4$ and $T_{j+1}>T_j^4$ for $j\geqslant 1$. We then define, for suitable $J\in\mathbb{Z}^+$,

$$\begin{aligned} \mathcal{B}_j &:= \{n : T_j < n \leqslant T_j^{1+\varepsilon_j},\ p|n \Rightarrow T_j^{\varepsilon_j^2} < p \leqslant T_j^{1/3}\} \qquad (j\geqslant J), \\ \mathcal{B} &:= \bigcup_{j=J}^{\infty} \mathcal{B}_j. \end{aligned}$$

and let $\mathcal{B}^*$ be the unique primitive subsequence of $\mathcal{B}$ such that $\mathcal{M}(\mathcal{B})=\mathcal{M}(\mathcal{B}^*)$.

Lemma 3.1. *For large enough J, the sequence $\mathcal{B}^*$ is not a Besicovitch sequence.*

Proof. Let $M(x)$ be the counting function of $\mathcal{M}(\mathcal{B}^*)$. We have for $j > J$

$$M(T_{j+1}) \leqslant \sum_{k=J}^{j} N_k(T_{j+1}), \tag{3.1}$$

where $N_k(x)$ denotes the number of integers $n \leqslant x$ having at least a divisor in the range $(T_k, T_k^{1+\varepsilon_k}]$. Since $T_{j+1} > T_j^4 > T_j^{2(1+\varepsilon_j)}$, Theorem 21(iii) of [HT1] gives that

$$N_k(T_{j+1}) \ll T_{j+1}\,\varepsilon_k^{\delta} = T_{j+1}\,k^{-2} \qquad (1 \leqslant k \leqslant j).$$

Inserting this estimate in (3.1) yields

$$M(T_{j+1}) \ll T_{j+1}/J.$$

Hence, given any positive real number ε, we can choose an integer J such that

$$\underline{\mathrm{d}}\mathcal{M}(\mathcal{B}^*) = \liminf x^{-1}M(x) < \varepsilon. \tag{3.2}$$

Next, we show that there is an absolute positive constant c_0 such that

$$\overline{\mathrm{d}}\mathcal{M}(\mathcal{B}^*) > c_0. \tag{3.3}$$

The first step is to observe that, if we decompose canonically an integer n in the form $n = n_1 n_2$ where n_1 is the largest divisor of n with $P(n_1) \leqslant T_j^{\varepsilon_j^2}$, then we have

$$n_1 \leqslant T_j^{\varepsilon_j/2} \tag{3.4}$$

for all integers $n \leqslant T_j^{1+\varepsilon_j}$ but at most $\ll T_j^{1+\varepsilon_j}\exp\{-1/(4\varepsilon_j)\}$ exceptions. This follows immediately from Lemma 2.1. Now, let us consider an integer n satisfying (3.4) and

$$T_j^{1+\varepsilon_j/2} < n \leqslant T_j^{1+\varepsilon_j}, \quad P(n) \leqslant T_j^{1/3}. \tag{3.5}$$

Then we may infer that

$$T_j < n_2 \leqslant T_j^{1+\varepsilon_j}, \quad p|n_2 \Rightarrow T_j^{\varepsilon_j^2} < p \leqslant T_j^{1/3},$$

so that $n \in \mathcal{M}(\mathcal{B}) = \mathcal{M}(\mathcal{B}^*)$. Since the number of integers $n \leqslant T_j^{1+\varepsilon_j}$ satisfying (3.5) is $\gg T_j^{1+\varepsilon_j}$ (for large J, the implied constant may in fact be anything smaller that $\varrho(3)$, where ϱ is the Dickman function), we see that, for some absolute constant $c_0 > 0$, we may write

$$M(T_j^{1+\varepsilon_j}) \geqslant c_0 T_j^{1+\varepsilon_j}.$$

This implies (3.3) and completes the proof of Lemma 3.1.

We now proceed to the construction of $\mathcal{A}_1$ and $\mathcal{A}_2$. The letter q being reserved, until the end of this section, to denote prime numbers, we put, for $\ell = 1, 2$,

$$\mathcal{C}_\ell := \{pq : \exp\{(\log p)^3\} < q \leqslant \exp\{(\log p)^4\},\ q \equiv (-1)^\ell (\operatorname{mod} 4)\},$$

and define $\mathcal{A}_\ell := \mathcal{B}^* \cup \mathcal{C}_\ell \quad (\ell = 1, 2)$.

Lemma 3.2. *For large J, the sequences $\mathcal{A}_1$ and $\mathcal{A}_2$ are primitive.*

Proof. It is clear that $\mathcal{C}_\ell$ is primitive, since any two of its elements have the same number of prime factors. Next, we observe that $\mathcal{B}^* \subset \mathcal{B} \subset \{n : \Omega(n) \geqslant 3\}$, so no element of $\mathcal{B}^*$ can divide an element of $\mathcal{C}_\ell$. Finally, we note that, if we set $P_j := T_j^{\varepsilon_j^2}$, then

$$\varepsilon_j^{-2} = j^{4/\delta} < 4^j \, j^{-4/\delta} < \log P_j = j^{-4/\delta} \log T_j$$

for large j, hence any integer $n \in \mathcal{B}^*$ has all its prime factors in an interval of the form $(P_j, \exp\{(\log P_j)^2\}]$, for some j. This implies that no element of $\mathcal{C}_\ell$ can divide an element of $\mathcal{B}^*$, and thereby completes the proof of the lemma.

We clearly have $\mathcal{A}_1 \cap \mathcal{A}_2 = \mathcal{B}^*$, since the congruence condition in the definition of $\mathcal{C}_\ell$ implies that $\mathcal{C}_1 \cap \mathcal{C}_2 = \varnothing$. Thus Theorem 2 follows from Lemmas 3.1, 3.2 and the following result.

Lemma 3.3. *$\mathcal{C}_1$ and $\mathcal{C}_2$ are Behrend sequences.*

Proof. Let $I_x := (\log x, \sqrt{x}\,]$. Then it follows easily by Turán's method that

$$\sum_{p \mid n, \, p \in I_x} 1 = (1 + o(1)) \log_2 x \tag{3.6}$$

for all integers $n \leqslant x$ but at most $o(x)$ exceptions. Now, fix $\ell = 1$ or 2 and define, for each prime p, the arithmetic function

$$\chi_p(n) := \begin{cases} 1 & \text{if } q \equiv (-1)^\ell \pmod 4 \text{ and } \exp\{(\log p)^3\} < q \leqslant \exp\{(\log p)^4\} \Rightarrow q \nmid n \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$n \notin \mathcal{M}(\mathcal{C}_\ell) \iff \min_{p \mid n} \chi_p(n) = 1. \tag{3.7}$$

Let us denote by $E_\ell(x)$ the number of integers $n \leqslant x$ which do not belong to $\mathcal{M}(\mathcal{C}_\ell)$. We deduce from (3.6) and (3.7) that

$$E_\ell(x) \leqslant \sum_{n \leqslant x} \sum_{p \mid n, \, p \in I_x} \chi_p(n) + o(x). \tag{3.8}$$

Indeed, (3.6) implies that the number of n such that the inner sum is empty is $o(x)$, and (3.7) yields that this inner sum is at least 1 for those of the remaining integers that are counted in $E_\ell(x)$. The double sum in (3.8) equals

$$\sum_{p \in I_x} \sum_{m \leqslant x/p} \chi_p(m).$$

By the sieve, this is

$$\begin{aligned} &\ll \frac{x}{p} \exp\Big\{ - \sum_{\substack{q \equiv (-1)^\ell \pmod 4 \\ \exp\{(\log p)^3\} < q \leqslant \exp\{(\log p)^4\}}} \frac{1}{q} \Big\} \\ &\ll x p^{-3/2}, \end{aligned}$$

where we have used a classical weighted form of Dirichlet's theorem on primes in arithmetic progressions. Inserting this estimate in (3.8), we obtain $E_\ell(x) = o(x)$. This finishes the proof.

4. Proof of Theorem 3

The following lemma will be useful. We write $P^-(m)$ for the smallest prime factor of an integer m, with the convention that $P^-(1)=1$.

Lemma 4.1. *For* $0<\eta<\frac12$, $k\in\mathbb{Z}^+$ *and* $z>1$, *let* $V_k(z;\eta)$ *denote the number of integers* $m\leqslant z$ *such that* $\omega(m)\leqslant k$ *and* $P^-(m)>z^\eta$. *Then we have, for suitable* $z_0(\eta)$,

$$V_k(z;\eta)\ll_k \frac{z}{\log z}\Big(\log\frac1\eta\Big)^{k-1}\qquad\big(z>z_0(\eta)\big),\tag{4.1}$$

Proof. The contribution to $V_k(z;\eta)$ of integers m which do not exceed $\sqrt z$ or are non squarefree is at most

$$\sqrt z+z\sum_{p>z^\eta}p^{-2}\ll z^{1-\eta},$$

which is acceptable if $z_0(\eta)$ is sufficiently large. Let $V'_k(z;\eta)$ denote the contribution of the remaining integers. Since $P(m)\geqslant m^{1/k}>z^{1/(2k)}$ for any m counted by $V'_k(z;\eta)$, we may write

$$\begin{aligned}\frac{\log z}{2k}V'_k(z;\eta)&\leqslant\sum_{m\leqslant z}\mu(m)^2\log P(m)\leqslant\sum_{1\leqslant j\leqslant k}\ \sum_{\substack{z^\eta<p_1,\dots,p_j\leqslant z\\ p_1\cdots p_j\leqslant z}}\log p_j\\
&\ll z+\sum_{2\leqslant j\leqslant k}\ \sum_{\substack{z^\eta<p_1,\dots,p_{j-1}\leqslant z\\ p_1\cdots p_{j-1}\leqslant z^{1-\eta}}}\ \sum_{p_j\leqslant z/p_1\cdots p_{j-1}}\log p_j\\
&\ll z+\sum_{2\leqslant j\leqslant k}\ \sum_{z^\eta<p_1,\dots,p_{j-1}\leqslant z}\frac{z}{p_1\cdots p_{j-1}}\ll z\Big(\log\frac1\eta\Big)^{k-1}.\end{aligned}$$

This implies the announced estimate (4.1).

We now embark on the proof of Theorem 3. We first show that any sequence $\mathcal{A}$ satisfying (1.2) is Besicovitch. We use Theorem 0 and set out to majorize the right-hand side of (1.1). Given $\varepsilon\in(0,1/4]$ and $x>x_0(\varepsilon)$, we let S be the number of integers $n\leqslant x$ having at least one divisor $a\in\mathcal{A}$ such that $x^{1-\varepsilon}<a\leqslant x$. Then we plainly have

$$\sum_{\substack{x^{1-\varepsilon}<a\leqslant x\\ a\in\mathcal{A}}}M(x,a;\mathcal{A})\leqslant S.\tag{4.2}$$

Each n counted in S may be written as $n=ab$ with $a\in\mathcal{A}$, $x^{1-\varepsilon}<a\leqslant x$, and we assume throughout that a is as small as possible, so that the decomposition $n=ab$ is unique. We further decompose a in the form $a=u_av_a$ where u_a is the largest divisor of a all of whose prime factors are $\leqslant x^{\varepsilon^2}$. We split S in the form

$$S=S_1+S_2+S_3$$

with

$$\begin{aligned}S_1&:=|\{ab\leqslant x:x^{1-\varepsilon}<a\leqslant x,\,u_a>x^\varepsilon\}|,\\
S_2&:=|\{ab\leqslant x:x^{1-\varepsilon}<a\leqslant x,\,u_a\leqslant x^\varepsilon,\,\omega(v_a)\leqslant k\}|,\\
S_3&:=|\{ab\leqslant x:x^{1-\varepsilon}<a\leqslant x,\,u_a\leqslant x^\varepsilon,\,\omega(v_a)>k\}|.\end{aligned}$$

By Lemma 2.1, we have

$$S_1\ll xe^{-1/(4\varepsilon)}.\tag{4.3}$$

If $n=ab$ is counted by S_2 or S_3, then

$$x^{1-2\varepsilon} < v_a = a/u_a \leqslant x.$$

We have in the first instance

$$\begin{aligned}
S_2 &\ll \sum_{n\leqslant x} \sum_{\substack{v|n,\, x^{1-2\varepsilon}<v\leqslant x\\ \omega(v)\leqslant k,\, P^-(v)>x^{\varepsilon^2}}} 1\\
&\ll x \sum_{\substack{x^{1-2\varepsilon}<v\leqslant x\\ \omega(v)\leqslant k,\, P^-(v)>x^{\varepsilon^2}}} v^{-1} = x\int_{x^{1-2\varepsilon}}^{x} z^{-1}\,\mathrm{d}W_k(z) \qquad (4.4)\\
&\leqslant x\Big\{x^{-1}W_k(x) + \int_{x^{1-2\varepsilon}}^{x} z^{-2}W_k(z)\,\mathrm{d}z\Big\},
\end{aligned}$$

by partial summation, with

$$W_k(z) := \sum_{\substack{v\leqslant z\\ \omega(v)\leqslant k,\, P^-(v)>x^{\varepsilon^2}}} 1 \leqslant V_k(z;\varepsilon^2) \qquad (z\leqslant x).$$

Inserting this inequality in the last upper bound of (4.4) and appealing to (4.1), we deduce that, for suitable $x_0(\varepsilon)$ and $x>x_0(\varepsilon)$, we have

$$S_2 \ll_k x\Big\{\frac{1}{\log x} + \int_{x^{1-2\varepsilon}}^{x} \frac{\mathrm{d}z}{z\log z}\Big\}\Big(\log\frac{1}{\varepsilon}\Big)^{k-1} \ll_k x\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)^{k-1}. \qquad (4.5)$$

The reader will notice that so far we haven't used condition (1.2). We now do this in the estimation of S_3. We have

$$S_3 \leqslant x\sum_{a\in\mathcal{A}_3} a^{-1} = \int_{x^{1-\varepsilon}}^{x} z^{-1}\,\mathrm{d}A_3(z)$$

with $\mathcal{A}_3 := \{a\in\mathcal{A} : x^{1-\varepsilon} < a \leqslant x,\ u_a\leqslant x^{\varepsilon},\ \omega(v_a)>k\}$. Now consider the mapping $m : \mathcal{A}\cap\{n : \omega(n)>k\} \to \mathbb{Z}^+$ that associates to any element a of the source its largest divisor with exactly $k+1$ distinct prime factors. By condition (1.2), m is injective. Moreover, we obviously have $m(a)\leqslant a$ for all a. Finally, since $m(a)|v_a$ for $a\in\mathcal{A}_3$, we have $P^-(m(a)) > x^{\varepsilon^2}$ whenever $a\in\mathcal{A}_3$. This implies that the counting function $A_3(z)$ of $\mathcal{A}_3$ satisfies $A_3(z)\leqslant V_{k+1}(z;\varepsilon^2)$ for $z\leqslant x$. By a computation parallel to (4.4)-(4.5) but with $V_{k+1}(z;\varepsilon^2)$ in place of $V_k(z;\varepsilon^2)$, we hence obtain that the bound (4.5) for S_2 is equally valid for S_3, provided the exponent $k-1$ is replaced by k. Summarizing our estimates for S_j $(j=1,2,3)$, we finally obtain

$$S \ll_k x\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)^{k} \qquad \big(x>x_0(\varepsilon)\big). \qquad (4.6)$$

Inserting this in (4.2) and letting successively x tend to ∞ and ε tend to 0, we obtain the desired conclusion.

To show the second part of Theorem 3, we use the family of integer sequences $\mathcal{B}$ constructed in section 3. We plainly have, for any choice of the T_j satisfying $T_{j+1} > T_j^4$,

$$\Omega(b_s) \leqslant \varepsilon_j^{-2} = j^{4/\delta} \tag{4.7}$$

where b_s denotes a generic element of $\mathcal{B}$ and $j = j(s)$ is uniquely defined by $T_j < b_s \leqslant T_j^{1+\varepsilon_j}$. But, given any sequence $k_s \to +\infty$, we can select a sequence T_j tending to infinity so fast that (4.7) implies $\Omega(b_s) \leqslant k_s$. The corresponding sequence $\mathcal{B}$ then provides the required counterexample.

5. Proof of Theorem 4

We shall use several times Behrend's inequality [Beh], viz

$$t_0(\mathcal{A} \cup \mathcal{B}) \geqslant t_0(\mathcal{A})t_0(\mathcal{B}), \tag{5.1}$$

valid for all integer sequences $\mathcal{A}$, $\mathcal{B}$, with equality if (but not only if) one has $(a, b) = 1$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. This was originally established as a generalization of a result of Heilbronn [Hei] and Rohrbach [Roh], corresponding essentially to the case when $\mathcal{B}$ is reduced to a single element. Although not final, the Heilbronn–Rohrbach inequality is quite often all that is required in applications. We therefore seize the opportunity to present a very short proof which seems to have escaped attention so far.

Put $\mathcal{T}(\mathcal{A}) := \mathbb{Z}^+ \smallsetminus \mathcal{M}(\mathcal{A})$, and $\mathcal{A}' := \{a/(a, b) : a \in \mathcal{A}\}$, where b is a given integer exceeding 1 and not belonging to $\mathcal{A}$. We then have the obvious partition

$$\mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{A} \cup \{b\}) \cup \Big(\mathcal{T}(\mathcal{A}) \cap b\mathbb{Z}^+\Big). \tag{5.2}$$

Furthermore, the definition of $\mathcal{A}'$ yields that $mb \in \mathcal{M}(\mathcal{A})$ if and only if $m \in \mathcal{M}(\mathcal{A}')$, so we may write

$$\mathcal{T}(\mathcal{A}) \cap b\mathbb{Z}^+ = b\mathcal{T}(\mathcal{A}'). \tag{5.3}$$

We infer that

$$t_0(\mathcal{A}) = \delta\mathcal{T}(\mathcal{A}) = t_0(\mathcal{A} \cup \{b\}) + \frac{1}{b}t_0(\mathcal{A}'). \tag{5.4}$$

On noticing that $\mathcal{T}(\mathcal{A}') \subset \mathcal{T}(\mathcal{A})$, and hence that $t_0(\mathcal{A}') \leqslant t_0(\mathcal{A})$, we obtain from (5.4) the validity of (5.1) in the special case when $\mathcal{B} = \{b\}$. The case of equality stated above is immediate since $\mathcal{A}' = \mathcal{A}$ if $(a, b) = 1$ for all $a \in \mathcal{A}$.

Another tool which will be needed in the proof is the formula

$$t_0(\mathcal{A}) = \lim_{n \to +\infty} t_0\big(\{a_j : 1 \leqslant j \leqslant n\}\big), \tag{5.5}$$

valid for any strictly increasing integer sequence $\mathcal{A} = \{a_1, a_2, \ldots\}$. This is part of the Davenport–Erdős theorem [DE1], [DE2].

We now embark on the proof of Theorem 4, and start with the upper bound (1.7). We split $\mathcal{A} = \{a_1, a_2, \ldots\}$ into a disjoint union of $k + 1$ subsequences $\mathcal{A}_j$, $0 \leqslant j \leqslant k$, and observe that $\tau(n, \mathcal{A}) \leqslant k$ implies $\min \tau(n, \mathcal{A}_j) = 0$, whence

$$t_k(\mathcal{A}) \leqslant \sum_{0 \leqslant j \leqslant k} t_0(\mathcal{A}_j). \tag{5.6}$$

Let

$$\tfrac{1}{2} > u_0 > u_1 > \ldots > u_{k-1} > \sigma \geqslant t_0(\mathcal{A}). \tag{5.7}$$

We begin by defining m_j, for each j, $0 \leqslant j < k$, to be the greatest integer such that if $\mathcal{B}_j := \{a_1, a_2, \ldots, a_{m_j}\}$, then

$$t_0(\mathcal{B}_j) > u_j. \tag{5.8}$$

In view of (5.5) and (5.7), $\mathcal{B}_j$ is non-empty and finite. By hypothesis

$$t_0\Big(\mathcal{B}_j \cup \{a_{m_j+1}\}\Big) \leqslant u_j \tag{5.9}$$

and the Heilbronn–Rohrbach inequality yields

$$u_j \geqslant t_0(\mathcal{B}_j)\big(1 - 1/a_{m_j+1}\big) \geqslant \prod_{1 \leqslant i \leqslant m_j+1} (1 - 1/a_i). \tag{5.10}$$

Since $\mathcal{A}$ has no repeated elements, the product on the right is at least $1/a_{m_j+1}$, whence $a_{m_j+1} \geqslant 1/u_j$ and the left-hand inequality in (5.10) implies

$$t_0(\mathcal{B}_j) \leqslant \frac{u_j}{1 - u_j} \qquad (0 \leqslant j < k). \tag{5.11}$$

We set $\mathcal{A}_0 := \mathcal{B}_0$, $\mathcal{A}_1 := \mathcal{B}_1 \smallsetminus \mathcal{B}_0$, $\mathcal{A}_2 := \mathcal{B}_2 \smallsetminus \mathcal{B}_1, \ldots, \mathcal{A}_k := \mathcal{A} \smallsetminus \mathcal{B}_{k-1}$. Behrend's inequality gives

$$t_0(\mathcal{B}_1) \geqslant t_0(\mathcal{A}_1)t_0(\mathcal{B}_0), \quad t_0(\mathcal{B}_2) \geqslant t_0(\mathcal{A}_2)t_0(\mathcal{B}_1), \ldots, \quad t_0(\mathcal{A}) \geqslant t_0(\mathcal{A}_k)t_0(\mathcal{B}_{k-1}), \tag{5.12}$$

and we deduce from (5.8), (5.11) and (5.12) that

$$t_0(\mathcal{A}_0) \leqslant \frac{u_0}{1 - u_0}, \quad t_0(\mathcal{A}_j) \leqslant \frac{u_j/u_{j-1}}{1 - u_j} \quad (1 \leqslant j < k), \quad t_0(\mathcal{A}_k) \leqslant \sigma/u_{k-1}. \tag{5.13}$$

We insert these inequalities into (5.6) to obtain

$$t_k(\mathcal{A}) \leqslant \frac{u_0}{1 - u_0} + \sum_{1 < j \leqslant k-1} \frac{u_j/u_{j-1}}{1 - u_j} + \frac{\sigma}{u_{k-1}}. \tag{5.14}$$

We select $u_j := \sigma^{(j+1)/(k+1)}$ $(0 \leqslant j < k)$. (The optimal choice does not significantly improve the final result.) We obtain

$$t_k(\mathcal{A}) \leqslant \frac{(k+1)\sigma^{1/(k+1)}}{1 - \sigma^{1/(k+1)}} \tag{5.15}$$

and observe that when the right-hand side does not exceed 1, it also does not exceed the upper bound stated in the theorem. The result follows.

It remains to establish the continuity of $\varphi_k(\sigma)$. This is a consequence of (1.7) when $\sigma = 0$, and so, because φ_k is non-decreasing, it will be sufficient to show that

$$\varphi_k(u\sigma) \geqslant \varphi_k(\sigma) + u - 1 \qquad (0 < u, \sigma \leqslant 1). \tag{5.16}$$

We have $\varphi_k(\sigma) \geqslant \varphi_0(\sigma) = \sigma$. (Indeed for any given $\sigma \in [0,1]$ we can find a sequence $\mathcal{A}$ with $t_0(\mathcal{A}) = \sigma$, e.g.by taking $\mathcal{A}$ a subsequence of the primes.) Let $0 < \varepsilon < \varphi_k(\sigma)$. By definition, there exists a sequence $\mathcal{A}(\varepsilon)$ such that $t_0\big(\mathcal{A}(\varepsilon)\big) \leqslant \sigma$, $t_k\big(\mathcal{A}(\varepsilon)\big) > \varphi_k(\sigma) - \varepsilon$. Hence, by (1.5), $\mathcal{A}(\varepsilon)$ is not Behrend , and we have

$$\sum_{p\in\mathcal{A}(\varepsilon)} 1/p < \infty. \tag{5.17}$$

In view of (5.17), there exist a set of primes $\mathcal{P}(\varepsilon)$ which does not intersect $\mathcal{A}(\varepsilon)$ and satisfies

$$\prod_{p\in\mathcal{P}(\varepsilon)} (1-1/p) = u. \tag{5.18}$$

We put $\mathcal{B}(\varepsilon) = \mathcal{P}(\varepsilon) \cup \mathcal{A}(\varepsilon)$, and we have $t_0\big(\mathcal{B}(\varepsilon)\big) = ut_0\big(\mathcal{A}(\varepsilon)\big) \leqslant u\sigma$. Moreover if $\tau(n,\mathcal{B}(\varepsilon)) > k \geqslant \tau(n,\mathcal{A}(\varepsilon))$, then $n \in \mathcal{M}\big(\mathcal{P}(\varepsilon)\big)$, whence the density of such n does not exceed $1-u$ and

$$t_k\big(\mathcal{B}(\varepsilon)\big) \geqslant t_k\big(\mathcal{A}(\varepsilon)\big) + u - 1 \geqslant \varphi_k(\sigma) + u - 1 - \varepsilon. \tag{5.19}$$

This implies (5.16) as required.

6. Proofs of Theorems 5 and 6

Theorem 5 is a simple consequence of Behrend's inequality (5.1). We have to show that if $|\mathcal{A}| \leqslant k$ then $t_0(\mathcal{A})$ is not less than the right-hand side of (1.9). Let $\mathcal{P} := \{P(a) : a \in \mathcal{A}\}$, and for each prime $p \in \mathcal{P}$, let

$$\mathcal{A}_p := \{a \in \mathcal{A} : P(a) = p\}.$$

Then $|\mathcal{P}| \leqslant k$ and $t_0(\mathcal{A}_p) \geqslant 1 - 1/p$. Behrend's inequality now gives

$$t_0(\mathcal{A}) \geqslant \prod_{p\in\mathcal{P}} t_0(\mathcal{A}_p) \geqslant \prod_{p\in\mathcal{P}} (1-1/p)$$

and the result follows.

The proof of Theorem 6 is much more delicate. The main step is the following probabilistic lemma.

Lemma 6.1. *Let* $\{\beta_j\}_{j=1}^{m+1}$ *be a finite, non-increasing sequence of elements of* $[0,1[$ *and let* $\{Y_j\}_{j=1}^{m+1}$ *be the sequence of independent Bernoulli random variables defined by*

$$P(Y_j = 1) = 1 - P(Y_j = 0) = \beta_j \qquad (1 \leqslant j \leqslant m+1). \tag{6.1}$$

Set $Y := \sum_{j=1}^{m+1} Y_j$. *Furthermore, let* $X := \sum_{j=1}^{\infty} X_j$ *be an almost surely convergent series of independent Bernoulli random variables satisfying*

$$\begin{aligned} &\text{(a)} \quad \{P(X_j = 1)\}_{j=1}^{\infty} \quad \textit{is non-increasing},\\ &\text{(b)} \quad P(X_j = 1) \leqslant \beta_j \qquad (1 \leqslant j \leqslant m),\\ &\text{(c)} \quad P(X = 0) \leqslant P(Y = 0). \end{aligned}$$

Then we have

$$P(X \leqslant k) \leqslant P(Y \leqslant k) \qquad (k = 0, 1, 2, \ldots). \tag{6.2}$$

Proof. We first consider the case when the series $\sum X_j$ is finite, say $X = \sum_{j=1}^{n} X_j$. If $n \leqslant m$, conditions (b) and (c) imply $\beta_j = P(X_j = 1)$ for $j \leqslant n$ and $\beta_j = 0$ for $n < j \leqslant m+1$, whence X and Y have the same law and (6.2) holds trivially. Thus we may assume $n \geqslant m+1$, and in fact, in view of (a),

$$P(X_{m+1} = 1) > 0. \tag{6.3}$$

Let $k \in \mathbb{Z}^+$ be given. We set out to prove (6.2). If $k > m$, the right-hand side is equal to 1 and the conclusion holds trivially. We hence assume

$$1 \leqslant k \leqslant m. \tag{6.4}$$

We put

$$\alpha_j := P(X_j = 1) \quad (1 \leqslant j \leqslant n), \qquad \boldsymbol{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n),$$

and consider the continuous function

$$P_k = P_k(\boldsymbol{\alpha}) := P(X \leqslant k).$$

We have to show that the supremum of $P_k(\boldsymbol{\alpha})$ under the constraints (a), (b) and (c) is atttained when

$$\boldsymbol{\alpha} = \boldsymbol{\beta} := (\beta_1, \dots, \beta_m, \beta_{m+1}, 0, \dots, 0). \tag{6.5}$$

The supremum is plainly attained for some $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}}$. We argue by contradiction and assume $\overline{\boldsymbol{\alpha}} \neq \boldsymbol{\beta}$.

Put $X^{(i)} := X - X_i = \sum_{1 \leqslant j \leqslant n,\, j \neq i} X_i$ $(1 \leqslant i \leqslant n)$. We have

$$\begin{aligned} P_k(\boldsymbol{\alpha}) =& P(X \leqslant k) = P(X^{(i)} \leqslant k-1) + P(X^{(i)} = k,\, X_i = 0) \\ =& P(X^{(i)} \leqslant k-1) + (1-\alpha_i) P(X^{(i)} = k), \end{aligned}$$

so

$$\frac{\partial P_k}{\partial \alpha_i}(\boldsymbol{\alpha}) = -P(X^{(i)} = k). \tag{6.6}$$

By (6.3) and (6.4) we have $\overline{\alpha}_{k+1} \geqslant \overline{\alpha}_{m+1} > 0$, i.e. at least $k+1$ of the $\overline{\alpha}_j$ are non-zero. This implies by (6.6)

$$\frac{\partial P_k}{\partial \alpha_i}(\overline{\boldsymbol{\alpha}}) < 0 \qquad (1 \leqslant i \leqslant n). \tag{6.7}$$

We now observe that, if $\alpha_j = \beta_j$ for all $j \leqslant m+1$, then $X \geqslant \widetilde{Y}$ almost surely, where $\widetilde{Y}$ has the same law as Y, and (6.2) follows trivially. We can hence discard this alternative and define the least integer h such that $\overline{\alpha}_h \neq \beta_h$. We then have $\overline{\alpha}_h < \beta_h$, $h \leqslant m+1$. Also, we let ℓ be the greatest integer for which $\overline{\alpha}_\ell > 0$. We must have $\ell > h$, else (c) would not hold for $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}}$.

It may be that $\overline{\alpha}_\ell < \overline{\alpha}_h$. In this case, we obtain a contradiction by considering the n-tuple $\boldsymbol{\alpha}$ such that $\alpha_j = \overline{\alpha}_j$ $(j \neq h, \ell)$, $\alpha_h = \overline{\alpha}_h + \varepsilon$, $\alpha_\ell = \overline{\alpha}_\ell - \varepsilon$, where ε is small and positive. We then have

$$\begin{aligned} P(X=0) =& P_0(\boldsymbol{\alpha}) = P_0(\overline{\boldsymbol{\alpha}}) \Big(1 - \frac{\varepsilon}{1-\overline{\alpha}_h}\Big)\Big(1 + \frac{\varepsilon}{1-\overline{\alpha}_\ell}\Big) \\ <& P_0(\overline{\boldsymbol{\alpha}}) \Big(1 - \frac{\varepsilon^2}{(1-\overline{\alpha}_h)^2}\Big) < P(Y=0), \end{aligned}$$

hence condition (c) is still satisfied for $\boldsymbol{\alpha}$. Moreover, if $h > 1$, we have $\overline{\alpha}_h < \beta_h \leqslant \beta_{h-1} = \overline{\alpha}_{h-1}$, so the coordinates of $\boldsymbol{\alpha}$ still form a non-increasing sequence provided ε is sufficiently small. Finally, the first order expansion of $P_k(\boldsymbol{\alpha})$ is

$$P_k(\boldsymbol{\alpha}) \simeq P_k(\overline{\boldsymbol{\alpha}}) + \varepsilon\Big(\frac{\partial P_k}{\partial \alpha_h}(\overline{\boldsymbol{\alpha}}) - \frac{\partial P_k}{\partial \alpha_\ell}(\overline{\boldsymbol{\alpha}})\Big).$$

By (6.6), this is $> P_k(\overline{\boldsymbol{\alpha}})$ since $\overline{\alpha}_h > \overline{\alpha}_\ell$ and

$$P(X^{(i)} = k) = P(X = 0) \sum_{\substack{i_1 < \ldots < i_k \\ i_j \neq i}} \frac{\alpha_{i_1}}{1 - \alpha_{i_1}} \cdots \frac{\alpha_{i_k}}{1 - \alpha_{i_k}},$$

whence $P(X^{(h)} = k) < P(X^{(\ell)} = k)$. This again contradicts the optimality of $\overline{\boldsymbol{\alpha}}$, and we deduce that $\overline{\alpha}_h = \overline{\alpha}_\ell$.

In this case, we adopt a similar policy, that is we slightly increase $\overline{\alpha}_h$ and slightly decrease $\overline{\alpha}_\ell$, however we have to examine the relative behaviour of P_0 and P_k more precisely. We avoid computations with higher variations by making α_h and α_ℓ functions of a parameter u in such a way that $P_0(\boldsymbol{\alpha})$ is independent of u, viz

$$1 - \alpha_h = (1 - \overline{\alpha}_h)e^{-u}, \quad 1 - \alpha_\ell = (1 - \overline{\alpha}_\ell)e^{u}.$$

Put $X^{(h,\ell)} := X - X_h - X_\ell$. We have

$$\begin{aligned} P_k(\boldsymbol{\alpha}) =& P(X \leqslant k) \\ =& P(X^{(h,\ell)} \leqslant k-2) \\ &+ P(X^{(h,\ell)} = k-1)\{\alpha_h(1-\alpha_\ell) + \alpha_\ell(1-\alpha_h)\} \\ &+ P(X^{(h,\ell)} = k)(1-\alpha_h)(1-\alpha_\ell). \end{aligned}$$

Only the middle term on the right depends on u. We have $P(X^{(h,\ell)} = k-1) > 0$ because $k \geqslant 1$ and $\overline{\alpha}_j > 0$ for $1 \leqslant j \leqslant k+1$. The quantity inside curly brackets equals

$$\begin{aligned} &\Big(1 - (1-\overline{\alpha}_h)e^{-u}\Big)e^{u}(1-\overline{\alpha}_\ell) + \Big(1 - (1-\overline{\alpha}_\ell)e^{u}\Big)e^{-u}(1-\overline{\alpha}_h) \\ &\qquad\qquad = 2(1-\overline{\alpha}_h)\{\overline{\alpha}_h + \cosh u - 1\} > 2\overline{\alpha}_h(1-\overline{\alpha}_h). \end{aligned}$$

Thus, the constraints allow a small positive value of u which increases $P_k(\overline{\boldsymbol{\alpha}})$ and this is the desired contradiction. Therefore, we have proved $\overline{\boldsymbol{\alpha}} = \boldsymbol{\beta}$, as required.

It remains to extend the result to the case of an infinite series X, that is $\alpha_j = P(X_j = 1) > 0$ for all $j \in \mathbb{Z}^+$. The hypothesis on almost sure convergence is equivalent to

$$\sum_{j=1}^{\infty} \alpha_j < \infty. \tag{6.8}$$

As previously, we may assume that $\alpha_h < \beta_h$ for some h, $1 \leqslant h \leqslant m+1$, since otherwise $X \geqslant \widetilde{Y}$ almost surely. Suppose $\beta_j = \beta_h$ for $h \leqslant j \leqslant s$, and either $s = m+1$ or $\beta_s > \beta_{s+1}$. For small positive δ, if we change β_j into $\beta_j - \delta$ in the definition of Y_j for $h \leqslant j \leqslant s$ and leave all the other Y_i unchanged, the random variable Y is changed into a variable $Y(\delta)$ satisfying

$$P(X = 0) \leqslant P(Y(\delta) = 0)/(1 + \delta). \tag{6.9}$$

Moreover, the sequence $\{P(Y(\delta)=1)\}_{j=1}^{m+1}$ is still non-increasing.

Let ε be given, $0<\varepsilon<1$ and $n>m$ be so large that $\sum_{j>n}\alpha_j<\varepsilon$. We write $X=X'+X''$, with $X':=\sum_{j=1}^{n}X_j$. By Markov's inequality, we have

$$P(X''\neq 0)<\varepsilon,$$

whence

$$P(X'=0)=P(X=0)/P(X''=0)<P(X=0)/(1-\varepsilon). \tag{6.10}$$

For sufficiently small δ and $\varepsilon<\delta/2$, we deduce from (6.9) and (6.10) that $P(X'=0)\leqslant P(Y(\delta)=0)$. Moreover, by the choice of h we may also impose, by choosing δ suitably, that the random variables X' and $Y(\delta)$ satisfy hypothesis (b) of the lemma. Since X' is a finite series, we deduce from the first part of the proof that

$$P(X'\leqslant k)\leqslant P(Y(\delta)\leqslant k)\qquad (k=0,1,2,\ldots). \tag{6.11}$$

However $P(X\leqslant k)\leqslant P(X'\leqslant k)$ for all k and the right-hand side of (6.11) is a continuous function of δ. Thus, letting $\delta\to 0$ in (6.11) yields the required result. This finishes the proof of Lemma 6.1.

We are now in a position to complete the proof of Theorem 6. Let $\mathcal{A}=\{a_1,a_2,\ldots\}$ have pairwise coprime elements larger than 1 and $t_0(\mathcal{A})=\sigma>0$. Clearly

$$\sum_{i=1}^{\infty}1/a_i<\infty \tag{6.12}$$

and this easily yields, using the inclusion-exclusion principle, that

$$t_0(\mathcal{A})=\prod_{i=1}^{\infty}(1-1/a_i). \tag{6.13}$$

We now put $\vartheta_j(\mathcal{A}):=\delta\{n:\tau(n,\mathcal{A})=j\}$, so that

$$t_k(\mathcal{A})=\sum_{0\leqslant j\leqslant k}\vartheta_j(\mathcal{A}). \tag{6.14}$$

Any integer n such that $\tau(n,\mathcal{A})=j$ may be uniquely written in the form $n=a_{i_1}^{m_1}\ldots a_{i_j}^{m_j}b$ where $i_1<\ldots<i_j$, the exponents m_i are positive, and b is divisible by none of the a_j. A simple sieve process then yields

$$\vartheta_j(\mathcal{A})=t_0(\mathcal{A})\sum_{i_1<\ldots<i_j}\frac{1}{a_{i_1}-1}\cdots\frac{1}{a_{i_j}-1}=\sum_{i_1<\ldots<i_j}\frac{1}{a_{i_1}}\cdots\frac{1}{a_{i_j}}\prod_{i\neq i_1,\ldots,i_j}\Big(1-\frac{1}{a_i}\Big). \tag{6.15}$$

Thus, we obtain that

$$t_k(\mathcal{A})=P(X\leqslant k) \tag{6.16}$$

where X is the sum of the almost surely convergent series of independent Bernoulli random variables X_j defined by

$$P(X_j=1)=1-P(X_j=0)=1/a_j\qquad (j=1,2,\ldots). \tag{6.17}$$

We obviously have

$$a_j \geqslant p_j \qquad (j = 1, 2, \dots)$$

where p_j denotes the jth prime number. Thus, under the assumption $\pi_{n+1} \leqslant \sigma \leqslant \pi_n$ of our theorem, we obtain by Lemma 6.1 that

$$t_k(\mathcal{A}) \leqslant P(Y \leqslant k) \qquad (k = 0, 1, 2, \dots) \tag{6.18}$$

where $Y := \sum_{j=1}^{n+1} Y_j$ is the sum of independent Bernoulli random variables defined by

$$P(Y_j = 1) = 1/p_j \quad (1 \leqslant j \leqslant n), \quad P(Y_{n+1} = 1) = 1 - \sigma/\pi_n. \tag{6.19}$$

The right-hand side of (6.18) depends linearly on σ and is equal to $t_{k,n}$ when $\sigma = \pi_n$, to $t_{k,n+1}$ when $\sigma = \pi_{n+1}$. Hence it must be equal to the right-hand side of (1.11). This is all we need.

Remark. Let ϱ_k^* denote the infimum of the set of real σ such that $\varphi_k^*(\sigma) = 1$. The corollary to Theorem 6 implies that

$$\varrho_k^* = \pi_k. \tag{6.20}$$

It is however interesting to note that this may be proved by a simple direct analysis which does not require as sharp a bound as (1.12). We can actually prove a result slightly more precise than (6.20), namely that, for any increasing sequence $\mathcal{A} = \{a_1, a_2, \dots\}$ composed of pairwise coprime integers, the inequality

$$t_k(\mathcal{A}) > 1 - \varepsilon \qquad (0 < \varepsilon \leqslant \tfrac{1}{2}) \tag{6.21}$$

implies

$$t_0(\mathcal{A}) \geqslant \Big(1 - c_k\, \varepsilon^{1/(k+1)}\Big) \prod_{1 \leqslant j \leqslant k} \big(1 - \frac{1}{a_j}\big), \tag{6.22}$$

where c_k depends only on k. This implies (6.20) since the product on the right is evidently at least π_k. We supply the details in the end of the next section.

7. Proof of Theorem 7

Let $\mathcal{A} = \{a_1, a_2, \dots\}$ be a sequence of pairwise coprime integers exceeding 1 and $0 < t_0(\mathcal{A}) \leqslant \sigma$. By (6.14) and (6.15), we have

$$t_k(\mathcal{A}) = t_0(\mathcal{A}) \sum_{0 \leqslant j \leqslant k} S_j(\mathcal{A}), \tag{7.1}$$

where $S_j(\mathcal{A})$ is the jth elementary symmetric function of the numbers $1/(a_i - 1)$ $(i = 1, 2, \dots)$. By Lemma 13, p.147, of [HR], we have

$$\frac{1}{j!} S_1(\mathcal{A})^j \Big\{1 - \binom{j}{2} S_1(\mathcal{A})^{-2} \sum_{i=1}^{\infty} (a_i - 1)^{-2}\Big\} \leqslant S_j(\mathcal{A}) \leqslant \frac{1}{j!} S_1(\mathcal{A})^j. \tag{7.2}$$

We first prove the lower bound of (1.12). Let T be a large parameter, and $U = U(\sigma)$ be defined by the inequalities

$$(1 - 1/T)\sigma \leqslant \prod_{T < p \leqslant U} (1 - 1/p) \leqslant \sigma.$$

We select $\mathcal{A} := \{p : T < p \leqslant U\}$. By the prime number theorem, we have, for fixed σ and $T \to +\infty$,

$$S_1(\mathcal{A}) = \sum_{T<p\leqslant U} (p-1)^{-1} = \log(1/\sigma) + o(1).$$

By (7.1) and (7.2), we hence obtain

$$t_k(\mathcal{A}) \geqslant \sigma \sum_{0\leqslant j\leqslant k} \frac{1}{j!}\Big(\log\frac{1}{\sigma}\Big)^j + o(1).$$

This plainly yields the required lower bound.

We now turn our attention to the upper estimate. Since $a_i \geqslant p_i$ for all i, we have

$$S_1(\mathcal{A}) = \sum_{i\geqslant 1} \frac{1}{a_i - 1} \leqslant -\sum_{i\geqslant 1} \log(1 - 1/a_i) + C,$$

where C is defined as in the statement of the theorem. Together with (6.13), (7.1) and (7.2), this yields

$$t_k(\mathcal{A}) \leqslant t_0(\mathcal{A}) \sum_{0\leqslant j\leqslant k} \frac{1}{j!}\Big(\log\frac{1}{t_0(\mathcal{A})} + C\Big)^j. \tag{7.3}$$

Since this is an increasing function of $t_0(\mathcal{A})$, we obtain

$$t_k(\mathcal{A}) \leqslant e^C\sigma \sum_{0\leqslant j\leqslant k} \frac{1}{j!}\big(\log\frac{1}{\sigma}\big)^j$$

on replacing $t_0(\mathcal{A})$ by $e^C\sigma$ in the right-hand side. The result follows.

We now prove that (6.21) implies (6.22). By (6.13), it is sufficient to show that

$$R_k := \sum_{j>k} 1/a_j \ll_k \varepsilon^{1/(k+1)}. \tag{7.4}$$

We first observe that by (6.21) the right-hand side of (7.3) exceeds $\frac{1}{2}$, whence

$$t_0(\mathcal{A}) \gg_k 1. \tag{7.5}$$

Next, we apply (6.15) with $j = k+1$ and write down the lower bound obtained by fixing in the right-hand side $i_1 = 1, \ldots, i_k = k$ and letting i_{k+1} run through $\{k+1, k+2, \ldots\}$. Taking (7.5) into account, we obtain

$$\vartheta_{k+1}(\mathcal{A}) \gg_k R_k/(a_1 \ldots a_k), \tag{7.6}$$

whence, by (6.21),

$$a_k^{-k} R_k \ll_k \varepsilon. \tag{7.7}$$

If $a_k \leqslant (2k+2)/R_k$, this plainly implies (7.4). Otherwise, the partial sums $\sum_{k<j\leqslant n} 1/a_j$ increase by amounts less than $R_k/(2k+2)$, and we can find integers $1 \leqslant j_1 < j_2 < \ldots < j_{k+1}$ such that

$$\frac{R_k}{2k+2} < \sum_{j_t<j\leqslant j_{t+1}} \frac{1}{a_j} \leqslant \frac{R_k}{k+1} \qquad (1 \leqslant t \leqslant k+1),$$

with the convention that $j_{k+2} = \infty$. Then

$$\Big(\frac{R_k}{2k+2}\Big)^{k+1} \leqslant \prod_{1\leqslant t\leqslant k+1} \sum_{j_t<j\leqslant j_{t+1}} \frac{1}{a_j} \leqslant S_{k+1}(\mathcal{A}) \ll_k \varepsilon,$$

where the last inequality follows from (6.21) and (7.5). This yields (7.4) again and hence completes the proof.

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