THE AVERAGE ORDERS OF HOOLEY'S Δ_r-FUNCTIONS

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§1. *Introduction.* In this paper we are concerned with upper bounds for the sums

$$S_r(x) = \sum_{n \leqslant x} \Delta_r(n),$$

where

$$\Delta_r(n) = \max_{u_1, \ldots, u_{r-1}} \operatorname{card}\{d_1, \ldots, d_{r-1} : d_1 \ldots d_{r-1} \mid n, \quad u_i < d_i \leqslant eu_i \quad \forall i\}$$

and $\Delta_2(n)$ is written simply as $\Delta(n)$. These functions were introduced by Hooley [**4**] and applied in a novel way to problems related to Waring's, and in Diophantine approximation. Thus Hooley deduced from his result about $S_2(x)$ that for any irrational θ, real γ, and fixed $\varepsilon > 0$, the inequality

$$\|n^2\theta - \gamma\| < n^{-1/2}(\log n)^{(2/\pi)-(1/2)+\varepsilon}$$

holds for infinitely many n. His result for $S_3(x)$ led to a proof that

$$r_8(n) \ll_\varepsilon n^{5/8}(\log n)^{(\sqrt{3})-1+\varepsilon},$$

where $r_8(n)$ denotes the number of representations of n as the sum of eight positive cubes.

In our previous work [**2**] we concentrated on the case $r = 2$ and improved Hooley's estimate by an elementary method based on a certain iterative procedure. Although we retain the main feature of that memoir here, that is we consider the generalized sums

$$S_r(x, y) = \sum_{n \leqslant x} \Delta_r(n) y^{\omega(n)}, \qquad 0 < y < \infty,$$

(where $\omega(n)$ denotes the number of distinct prime factors of n, and we set $\Delta_1(n) \equiv 1$), we further introduce a reappraisal of Hooley's method which may then be combined with the iteration method to give the best known results.

We define $\alpha(r, y)$ to be the infimum of the numbers ξ for which $S_r(x, y) \ll_\xi x(\log x)^\xi$ and set $A_r = \alpha(r, 1)$: notice that $\alpha(1, y) \equiv y - 1$. Hooley proved that $A_2 \leqslant (4/\pi) - 1$, $A_3 \leqslant \sqrt{3} - 1$, $A_4 \leqslant 1$. In general his method shows that $A_r \leqslant h_r - 1$ where $h_r = h_r(1)$ and $h_r(y)$ is the infimum of the numbers ξ for which

$$\limsup_{X \to \infty} X^{-\xi} \int_{-1}^{1} \ldots \int_{-1}^{1} \exp\left(y \int_1^X |1 + e^{i\theta_1 t} + \ldots + e^{i\theta_{r-1} t}| \frac{dt}{t} \right) d\theta_1 \ldots d\theta_{r-1} < \infty,$$

[MATHEMATIKA, **31** (1984), 98–109]

indeed that $\alpha(r, y) \leqslant h_r(y)-1$. From Jensen's inequality, we have $h_r \geqslant h_r'$ where

$$h_r' = \frac{1}{(2\pi)^{r-1}} \int_0^{2\pi} \dots \int_0^{2\pi} |1+e^{ix_1}+e^{ix_2}+ \dots +e^{ix_{r-1}}| dx_1 \dots dx_{r-1};$$

moreover $h_2 = h_2' = 4/\pi$ (when $r = 2$ the integrand is a periodic function of t). Starting from an identity of Kluyver [**7**, p. 420], Hooley [**5**] expressed h_r' as a single integral involving Bessel functions, *viz*,

$$h_r' = r \int_0^\infty \{J_0(u)\}^{r-1} J_1(u) du/u$$

in order to obtain good numerical estimates for h_r' when r is of modest size. We note that $h_r' \sim \frac{1}{2}\sqrt{(\pi r)}$. To see this, set

$$Z_r = \frac{\sqrt{2}}{\sqrt{r}} (e^{ix_0}+e^{ix_1}+ \dots +e^{ix_{r-1}})$$

where $x_0, \dots, x_{r-1}$ are uniformly distributed random variables on $[0, 2\pi)$, then the expectation of $|Z_r|$ is $(2/r)^{1/2} h_r'$. By the Central Limit Theorem, the random variable $\operatorname{Re} Z_r$ is asymptotically normal, moreover $\arg Z_r$ is manifestly uniformly distributed by symmetry, and independent of $|Z_r|$. Thus (writing Exp for expectation)

$$\operatorname{Exp}|Z_r| \,.\, \operatorname{Exp}|\cos\theta| = \operatorname{Exp}|\operatorname{Re} Z_r| \to \sqrt{(2/\pi)}$$

and since $\operatorname{Exp}|\cos\theta| = 2/\pi$ we have the result stated. More details may be found in [**5**, §3 and **7**, p. 419]. See also Feller [**1**] and Pearson [**6**].

In its present form, the argument on p. 124 of [**4**] is insufficient to prove that $h_r \leqslant \sqrt{r}$ when $r \geqslant 5$ and hence to obtain Theorem 1B. Professor Hooley has been kind enough to inform us that he is in accord with this opinion. We do not have a proof of this theorem but we show below that $h_r \leqslant \frac{1}{4}r+1$ $(r \geqslant 5)$. In [**2**] we demonstrated that $A_2 < 0{\cdot}23454 < h_2'-1 = 0{\cdot}27323\dots$ and we improve that result further here, and obtain new estimates for A_3 and A_4. In particular we find that $A_3 < h_3'-1$, and it may well be that in fact $A_r < h_r'-1$ for every r. However the inequality $A_r \leqslant h_r'-1$ remains unproved for $r \geqslant 4$. With the exception of Corollary 1 and Theorem 2 our results are not final and our object has been to describe the various available methods and how they may be combined.

§2. *Theorems.*

THEOREM 1. *For* $1 \leqslant s \leqslant r$, $y, z > 0$ *we have the iteration inequality*

$$2\alpha(r, y) \leqslant \alpha(r, z)+(s-1)\chi(sz-1)+\alpha(r-s+1, y^2/z)$$

where $\chi(u) = \max(0, u)$, $\alpha(1, y) = y-1$.

This theorem is the basis of the iteration method developed in [2]. It has the following corollaries of which the first is sharp: we remark that it might be possible to employ $\alpha(r, y)$ *directly* in some applications.

COROLLARY 1. *We have* $\alpha(r, y) = y-1$ *for* $y \leqslant \frac{1}{2}$ *and every* r.

COROLLARY 2. *We have* $A_3 \leqslant (\sqrt{7})-2$.

Corollary 1 will be proved below. To obtain the bound for A_3 we then put $r = s = 3$, $y = 1$ and $z = 1/\sqrt{7}$ in the iteration inequality.

THEOREM 2. *We have* $\alpha(r, y) = ry-r$ *for* $y \geqslant 2$ *and every* r.

After first refining Hooley's method, and then combining it with the iteration method, we obtain

THEOREM 3. *We have* $A_2 < 0{\cdot}21969$.

A similar hybrid approach yields

THEOREM 4. *We have* $A_3 < 0{\cdot}55153$, $A_4 < 0{\cdot}92752$.

§3. *Two lemmas.*

LEMMA 1. *For* $r \geqslant 2$, $\sigma, z > 0$ *let*

$$J_r(\sigma, z) = \int_0^1 \dots \int_0^1 \prod_{j<r} |\sigma + i\theta_j|^{-2z} \prod_{k<l<r} |\sigma + i(\theta_l - \theta_k)|^{-2z} d\theta_1 \dots d\theta_{r-1}.$$

Then

$$J_r(\sigma, z) \ll_r \left\{\left(\frac{1}{\sigma}\right)^{\chi(rz-1)} \log\frac{1}{\sigma}\right\}^{r-1}$$

where $\chi(u) = \max(0, u)$ *throughout this paper.*

Proof. By symmetry we may restrict the range of integration so that $\theta_1 \leqslant \theta_2 \leqslant \dots \leqslant \theta_{r-1}$ and we put $x_1 = \theta_1$, $x_i = \theta_i - \theta_{i-1}$ for $2 \leqslant i < r$. We have

$$|\sigma + i\theta_j| \geqslant |\sigma + ix_m|, \qquad m = m(j)$$

$$|\sigma + i(\theta_l - \theta_k)| \geqslant |\sigma + ix_m|, \qquad m = m(k, l)$$

provided $m(j) \leqslant j$, $k < m(k, l) \leqslant l$, and our idea is to majorize $J_r(\sigma, z)$ by an integral involving only factors of the form $|\sigma + ix_m|^{-2z}$ in which each x_m appears equally often. Since there are $\binom{r}{2}$ factors altogether and $r-1$ values of m, each x_m should appear $\frac{1}{2}r$ times. If r is even, we set $m(j) = 1$ $(1 \leqslant j \leqslant \frac{1}{2}r)$, $m(j) = j$ $(\frac{1}{2}r < j < r)$ and $m(k, l) = k+1$ $(1 \leqslant l-k \leqslant \frac{1}{2}r)$, $m(k, l) = l$ $(\frac{1}{2}r < l-k < r)$. Thus 1 appears $\frac{1}{2}r$ times as $m(j)$, $1 \leqslant j \leqslant \frac{1}{2}r$. If $2 \leqslant m \leqslant \frac{1}{2}r$, then m appears $\frac{1}{2}r$ times as $m = m(m-1, l)$,

$m \leqslant l < m+(\frac{1}{2}r)$. If $m > \frac{1}{2}r$ it appears $r-m$ times as $m = m(m-1, l)$, $m \leqslant l < r$, $m-\frac{1}{2}r-1$ times as $m = m(k, m)$, $0 < k < m-\frac{1}{2}r$, and once as $m(j), j = m$. Now let r be odd. We set $m(j) = 1$ $(1 \leqslant j \leqslant [\frac{1}{2}r])$, and $m(j) = j$ $([\frac{1}{2}r]+1 < j < r)$; and $m(k, l) = k+1$ $(1 \leqslant l-k \leqslant [\frac{1}{2}r])$, and $m(k, l) = l$, $([\frac{1}{2}r]+1 < l-k < r)$. So far each x_m appears $[\frac{1}{2}r]$ times. In the remaining cases, when j or $l-k = [\frac{1}{2}r]+1 = t$ say, we write

$$|\sigma+i\theta_t| \geqslant |\sigma+ix_1|^{1/2}|\sigma+ix_2|^{1/2}$$

$$|\sigma+i(\theta_l-\theta_k)| \geqslant |\sigma+ix_{2l-r}|^{1/2}|\sigma+ix_{2l-r+1}|^{1/2}, \quad l > t.$$

We now have

$$J_r(\sigma, z) \leqslant \int_0^1 \dots \int_0^1 \prod_{m<r} |\sigma+ix_m|^{-rz} dx_1 \dots dx_{r-1}$$

$$\ll \left\{\left(\frac{1}{\sigma}\right)^{\chi(rz-1)} \log\frac{1}{\sigma}\right\}^{r-1}$$

as required. We note that the exponent of $1/\sigma$ is sharp: we need only integrate over the range $\max\theta_j \leqslant \sigma$.

LEMMA 2. *For* $0 < \sigma < 1$, $y > 0$, *let*

$$M_r(\sigma, y) = \sum_{n=1}^{\infty}{}' \frac{\Delta_r(n)y^{\omega(n)}}{n^{1+\sigma}}$$

where ′ denotes, throughout the paper, that the sum is restricted to squarefree n, *and let* $\gamma(r, y)$ *denote the infimum of the numbers* ξ *such that* $M_r(\sigma, y) \ll_{\xi} (1/\sigma)^{\xi}$. *Then* $\gamma(r, y) = \alpha(r, y)+1$.

Proof. The inequality $\gamma \leqslant \alpha+1$ follows by partial summation. Next, we observe that for any prime p and integer m (divisible by p or not)

$$\Delta_r(p^s m) \leqslant \binom{r+s-1}{r-1}\Delta_r(m), \qquad s \in \mathbb{Z}^+,$$

by a generalization of the argument in [2, Lemma 7]. Thus, for any integers n, m such that $m \mid n$,

$$\Delta_r(n) \leqslant \Delta_r(m)\tau_r(n/m)$$

where $\tau_r(n)$ denotes the number of ways that n may be written as the product of r factors. Hence

$$S_r(x, y) < \sum_{m<x}{}' \Delta_r(m)y^{\omega(m)}\left\{\sum_{n<x} \tau_r(n/m) : \sum_{p\mid n} p = m\right\}.$$

The inner sum does not exceed $Q^r(x, m)$ where

$$Q(x, m) = \text{card}\left\{n < x : \prod_{p\mid n} p = m\right\}.$$

It is known ([3], the proof is due to Erdős) that for every $t > 0$,

$$\sum_{m<x}' Q^t(x,m) \ll_t x .$$

Let $h > 1$ and $k = 2h/(h-1)$. By Hölder's inequality

$$S_r(x,y) \leqslant \sum_{m<x}' \left(\Delta_r(m) y^{\omega(m)}\right)^{1/h} \left(\tau_r(m) y^{\omega(m)}\right)^{2/k} Q^r(x,m)$$

$$\leqslant \left(\sum_{m<x}' \Delta_r(m) y^{\omega(m)}\right)^{1/h} \left(\sum_{m<x}' \tau_r^2(m) y^{2\omega(m)}\right)^{1/k} \left(\sum_{m<x}' Q^{rk}(x,m)\right)^{1/k}$$

$$\ll_h x^{1-(1/h)} (\log x)^{c/k} \left(\sum_{m<x}' \Delta_r(m) y^{\omega(m)}\right)^{1/h}$$

for suitable $c = c(r, y)$. We let $h \to 1$, $k \to \infty$ and conclude that, if the last sum in brackets is $\ll_\xi x(\log x)^\xi$, then $\alpha(r, y) \leqslant \xi$. Plainly

$$\sum_{m<x}' \Delta_r(m) y^{\omega(m)} \ll x^{1/2} + \frac{1}{\log x} \sum_{m<x}' \Delta_r(m) y^{\omega(m)} \log m .$$

The sum on the right does not exceed (on writing $\log m = \Sigma\{\log p, p \mid m\}$, $m = np$)

$$y \sum_{p<x} \log p \sum_{n<x/p}' \Delta_r(np) y^{\omega(n)} \ll \sum_{n<x}' \Delta_r(n) y^{\omega(n)} \sum_{p<x/n} \log p$$

$$\ll x \sum_{n<x}' \Delta_r(n) \frac{y^{\omega(n)}}{n} \ll x M_r\left(\frac{1}{\log x}, y\right).$$

Hence $M_r(\sigma, y) \ll (1/\sigma)^\xi$ implies $\alpha(r, y) \leqslant \xi - 1$, and so $\alpha(r, y) \leqslant \gamma(r, y) - 1$. This gives the result stated.

§4. *Proof of Theorem* 1. This depends on the inequality

$$\Delta_r^2(n\,; u_1, \ldots, u_{r-1})$$

$$\leqslant \sum_{[\delta_1\delta_2\ldots\delta_{r-1},\, \delta_1'\delta_2'\ldots\delta_{r-1}'] \mid n}^{*} \Delta_r\left(\frac{n}{[\delta_1\delta_2\ldots\delta_{r-1},\, \delta_1'\delta_2'\ldots\delta_{r-1}']}, \frac{u_1}{\delta_1}, \ldots, \frac{u_{r-1}}{\delta_{r-1}}\right)$$

where the sum is over all sets of numbers $\delta_1, \delta_2, \ldots, \delta_{r-1}, \delta_1', \delta_2', \ldots, \delta_{r-1}'$ such that $[\delta_1\delta_2\ldots\delta_{r-1}, \delta_1'\delta_2'\ldots\delta_{r-1}'] \mid n$ and the $*$ denotes that, for each i, $0 < i < r$, we have $|\log \delta_i/\delta_i'| \leqslant 1 = (\delta_i, \delta_i')$. For the left-hand side above is the number of sets $d_1, d_2, \ldots, d_{r-1}$, $d_1', d_2', \ldots, d_{r-1}'$ such that for each i, $u_i < d_i$, $d_i' \leqslant eu_i$ and both $d_1 d_2 \ldots d_{r-1} \mid n$ and $d_1' d_2' \ldots d_{r-1}' \mid n$. Set $t_i = (d_i, d_i')$ and $\delta_i = d_i/t_i$, $\delta_i' = d_i'/t_i$. Plainly $|\log \delta_i/\delta_i'| \leqslant 1 = (\delta_i, \delta_i')$. Both $\delta_1\delta_2\ldots\delta_{r-1} \mid n$ and $\delta_1'\delta_2'\ldots\delta_{r-1}' \mid n$ so that $[\delta_1\delta_2\ldots\delta_{r-1}, \delta_1'\delta_2'\ldots\delta_{r-1}'] \mid n$ and $t_1 t_2 \ldots t_{r-1} \mid n/[\delta_1\delta_2\ldots\delta_{r-1}, \delta_1'\delta_2'\ldots\delta_{r-1}']$. Given the δ's, the number of sets $d_1, \ldots, d_{r-1}'$ from which they derive does not exceed the number of choices of the t's, and we note that, for each i, $u_i/\delta_i < t_i \leqslant eu_i/\delta_i$, which gives the result stated. An immediate corollary is that

$$\Delta_r^2(n) \leqslant \sum_{[\delta_1\delta_2 \ldots \delta_{r-1}, \delta_1'\delta_2' \ldots \delta_{r-1}'] \mid n}^{*} \Delta_r(n/[\delta_1\delta_2 \ldots \delta_{r-1}, \delta_1'\delta_2' \ldots \delta_{r-1}']) .$$

Now let $s \leqslant r$ and suppose $u_1, u_2, \ldots, u_{r-1}$ chosen so that $\Delta_r(n; u_1, u_2, \ldots, u_{r-1}) = \Delta_r(n)$. Then

$$\Delta_r(n) = \sum_{\substack{d_s d_{s+1} \ldots d_{r-1} \mid n \\ u_i < d_i \leqslant eu_i\ (i \geqslant s)}} \Delta_s\left(\frac{n}{d_s d_{s+1} \ldots d_{r-1}}, u_1, u_2, \ldots, u_{s-1}\right).$$

This makes sense in the case $s = r$ provided the empty product $d_s d_{s+1} \ldots d_{r-1}$ is interpreted as 1. The Cauchy–Schwarz inequality gives

$$\Delta_r^2(n) \leqslant \Delta_{r-s+1}(n) \sum_{\substack{d_s \ldots d_{r-1} \mid n \\ u_i < d_i \leqslant eu_i\ (i \geqslant s)}} \Delta_s^2\left(\frac{n}{d_s d_{s+1} \ldots d_{r-1}}, u_1, \ldots, u_{s-1}\right).$$

We now apply the inequality stated at the beginning of the section, with s in place of r. We obtain

$$\begin{aligned} \frac{\Delta_r^2(n)}{\Delta_{r-s+1}(n)} &\leqslant \sum_{[\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}'] \mid n}^{*} \sum_{\substack{d_s \ldots d_{r-1} \mid n/[\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}'] \\ u_i < d_i \leqslant eu_i\ (i \geqslant s)}} \Delta_s\left(n/[\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}'] d_s \ldots d_{r-1}, \frac{u_1}{\delta_1}, \ldots, \frac{u_{s-1}}{\delta_{s-1}}\right) \\ &\leqslant \sum_{[\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}'] \mid n}^{*} \Delta_r(n/[\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}']) . \end{aligned}$$

This inequality could be applied to $S_r(x, y)$ but it is easier to work with $M_r(\sigma, y)$, using Lemma 2. By the Cauchy–Schwarz inequality, for $z > 0$ we have

$$M_r^2(\sigma, y) \leqslant \sum_{n=1}^{\infty}{}' \frac{\Delta_r^2(n) z^{\omega(n)}}{\Delta_{r-s+1}(n) n^{1+\sigma}} \sum_{n=1}^{\infty}{}' \frac{\Delta_{r-s+1}(n)}{n^{1+\sigma}} \left(\frac{y^2}{z}\right)^{\omega(n)}$$

and note that the second sum on the right is

$$\ll_\varepsilon \left(\frac{1}{\sigma}\right)^{\alpha(r-s+1,\, y^2/z)+1+\varepsilon} .$$

The first sum on the right is

$$\begin{aligned} &\leqslant \sum{}^{**} \frac{z^{\omega([\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}'])}}{[\delta_1 \ldots \delta_{s-1}, \delta_1' \ldots \delta_{s-1}']^{1+\sigma}} \sum_{m=1}^{\infty}{}' \frac{\Delta_r(m) z^{\omega(m)}}{m^{1+\sigma}} \\ &\ll_\varepsilon \left(\frac{1}{\sigma}\right)^{\beta(s, z)+\alpha(r, z)+1+\varepsilon} \end{aligned}$$

where the extra * denotes that $[\delta_1 \dots \delta_{s-1}, \delta'_1 \dots \delta'_{s-1}]$ is squarefree and $\beta(s, z)$ is the infimum of the numbers for which this sum is $\ll_{\xi} (1/\sigma)^{\xi}$. We deduce that

$$2\alpha(r, y) \leqslant \alpha(r, z) + \beta(s, z) + \alpha(r-s+1, y^2/z)$$

so that it remains to estimate $\beta(s, z)$. The last sum on the left above does not exceed

$$\sum_{d=1}^{\infty} \frac{\mu^2(d) d^{1+\sigma}}{z^{\omega(d)}} \sum_{\substack{d \mid \delta_1 \dots \delta_{s-1} \\ d \mid \delta'_1 \dots \delta'_{s-1}}}^{**} \frac{z^{\omega(\delta_1 \dots \delta'_{s-1})}}{(\delta_1 \dots \delta'_{s-1})^{1+\sigma}}$$

and the extra * now means that both $\delta_1 \dots \delta_{s-1}$ and $\delta'_1 \dots \delta'_{s-1}$ are squarefree. Let $d = m_1 m_2 \dots m_{s-1} = m'_1 m'_2 \dots m'_{s-1}$ be factorizations for which $(m_i, m'_i) = 1$ for each i, of which there are $\{(s-1)(s-2)\}^{\omega(d)}$. We put $\delta_i = m_i t_i$, $\delta'_i = m'_i t'_i$ so that the sum above does not exceed

$$\sum_{d=1}^{\infty} \frac{\mu^2(d) z^{\omega(d)}}{d^{1+\sigma}} \sum_{\substack{m_1 m_2 \dots m_{s-1} = d \\ m'_1 m'_2 \dots m'_{s-1} = d}} \prod_{j<s} \left\{ \sum_{t_j=1}^{\infty}{}' \sum_{t'_j=1}^{\infty}{}' \frac{z^{\omega(t_j t'_j)}}{(t_j t'_j)^{1+\sigma}} w\left(\frac{m_j t_j}{m'_j t'_j}\right) \right\}$$

where $(m_j, m'_j) = 1$ in the second sum and w is any non-negative weight such that $w(u) \geqslant 1$ for $e^{-1} \leqslant u \leqslant e$. A suitable w is given by the formula

$$w(e^x) = \left(\frac{\sin \frac{1}{2}x}{x \sin \frac{1}{2}}\right)^2 = \frac{1}{4 \sin^2 \frac{1}{2}} \int_{-1}^{1} (1-|\theta|) e^{i\theta x} d\theta .$$

Hence the inner sum above is

$$\ll \int_{-1}^{1} \dots \int_{-1}^{1} \prod_{p \mid d} \{|p^{i\theta_1} + \dots + p^{i\theta_{s-1}}|^2 - (s-1)\} \prod_{j<s} \left\{ \sum_{t_j}{}' \sum_{t'_j}{}' \frac{z^{\omega(t_j t'_j)}}{(t_j t'_j)^{1+\sigma}} \left(\frac{t_j}{t'_j}\right)^{i\theta_j} (1-|\theta_j|) d\theta_j \right\}.$$

Summing over d, we obtain

$$\ll \int_{-1}^{1} \dots \int_{-1}^{1} \prod_{p} \left(1 + z \frac{|p^{i\theta_1} + \dots + p^{i\theta_{s-1}}|^2 - (s-1)}{p^{1+\sigma}}\right) \prod_{j<s} \left\{ \left| \sum_{t=1}^{\infty} \frac{z^{\omega(t)}}{t^{1+\sigma+i\theta_j}} \right|^2 (1-|\theta_j|) d\theta_j \right\}$$

$$\ll \int_{-1}^{1} \dots \int_{-1}^{1} \prod_{k<l} |\zeta(1+\sigma+i\theta_k - i\theta_l)|^{2z} \prod_{j} |\zeta(1+\sigma+i\theta_j)|^{2z} d\theta_1 \dots d\theta_{s-1}$$

$$\ll \int_{0}^{1} \dots \int_{0}^{1} \prod_{k<l} |\sigma + i\theta_k - i\theta_l|^{-2z} \prod_{j} |\sigma + i\theta_j|^{-2z} d\theta_1 \dots d\theta_{s-1} ,$$

using standard information about the Riemann ζ-function. Plainly the region where the θ's have the same sign dominates. This integral is $J_r(\sigma, z)$ and we apply Lemma 1, whence

$$\beta(s, z) \leqslant (s-1)\chi(sz-1).$$

This completes the proof.

Corollary 1 is almost immediate. We set $s = 2$ and take $z = y$. This gives $\alpha(r, y) \leqslant \alpha(r-1, y)$ for $y \leqslant 1/2$. Since $\alpha(1, y) = y-1$ (for all y) we may proceed by induction on r.

Proof of Theorem 2. Since $\Delta_r(n) \geqslant \max(1, \tau_r(n)/(\log n)^{r-1})$ we have $\alpha(r, y) \geqslant y-1+(r-1)\chi(y-1)$. We have to establish equality when $y \geqslant 2$ and we show that for such y, the stronger inequality $h_r(y) \leqslant ry-r+1$ holds. For $a > 0$ we have $u \leqslant (u^2+a^2)/(2a)$ whence

$$\int_1^X |1+e^{i\theta_1 t}+\ldots+e^{i\theta_r t}| \frac{dt}{t} \leqslant \frac{a}{2}\log X + \frac{1}{2a}\int_1^X |\ldots|^2 \frac{dt}{t}$$

$$\leqslant \left(\frac{a}{2}+\frac{r}{2a}\right)\log X + \frac{1}{a}\sum_{j<r}\log\left(\frac{X}{1+|\theta_j|X}\right) + \frac{1}{a}\sum_{k<l<r}\log\left(\frac{X}{1+|\theta_l-\theta_k|X}\right) + O_r(1).$$

We multiply by y and exponentiate, and we need ξ so large that

$$X^{-\xi+((a/2)+(r/2a))y} J_r\left(\frac{1}{X}, \frac{y}{2a}\right) = O(1), \qquad X \to \infty.$$

By Lemma 1,

$$\xi > \left(\frac{a}{2}+\frac{r}{2a}\right)y+(r-1)\chi\left(\frac{ry}{2a}-1\right)$$

will suffice, and we set $a = r$ to obtain our result.

Notice that we now have $\alpha(r, y) = y-1+(r-1)\chi(y-1)$ for $y \in (0, \frac{1}{2}] \cup [2, \infty)$: it would be desirable to shorten the excluded interval $(\frac{1}{2}, 2)$. When $y < 2$, the optimal value of a is $\max(\sqrt{r}, \frac{1}{2}ry)$ and this yields the upper bound

$$h_r(y) \leqslant \begin{cases} y\sqrt{r}, & \text{if } y \leqslant 2/\sqrt{r}, \\ \frac{1}{4}ry^2+1, & \text{otherwise}. \end{cases}$$

In particular, we deduce that $A_r \leqslant \frac{1}{4}r$ for $r \geqslant 5$.

§5. *Proof of Theorem* 3. We begin with the following result which is obtained by a refinement of Hooley's original method.

LEMMA 3. *We have*

$$\alpha(2, y) \leqslant \begin{cases} (16y-3\pi-4)/(3\pi+2), & \frac{2}{3} \leqslant y \leqslant \pi/(2\pi-4), \\ 2y-2, & y > \pi/(2\pi-4). \end{cases}$$

This improves on the upper bound obtained in [2] when $y > 0{\cdot}95048\ldots = y_0$.

To prove this we introduce a non-negative weight w such that $w(v) \geqslant 1$ for $1/\sqrt{e} < v \leqslant \sqrt{e}$. Then

$$\Delta(n) < \max_u \sum_{d \mid n} w(d/u).$$

Using the same w as in the proof of Theorem 1 we have, for squarefree n,

$$\Delta(n) \ll \max_u \int_{-1}^{1} u^{-i\theta}(1-|\theta|) \prod_{p \mid n} (1+p^{i\theta}) d\theta$$

$$\ll \max_u \int_{-1}^{1} u^{-i\theta}(1-|\theta|) \prod_{p \mid n} (\lambda p^{-i\theta} + 1 + p^{i\theta} + \lambda p^{2i\theta}) d\theta,$$

for $\lambda \geqslant 0$ because the contribution of the extra terms is non-negative. Hence

$$\Delta(n) \ll \int_{-1}^{1} \prod_{p \mid n} |\lambda p^{-i\theta} + 1 + p^{i\theta} + \lambda p^{2i\theta}| d\theta,$$

and

$$M_2(\sigma, y) \ll \int_{-1}^{1} \exp\left\{ y \sum_p \frac{1}{p^{1+\sigma}} |\lambda p^{-i\theta} + 1 + p^{i\theta} + \lambda p^{2i\theta}| \right\} d\theta.$$

We split the integral into two parts I_1 and I_2 according as $|\theta| > \sigma$ or not. Plainly

$$I_1 \leqslant \int_{-\sigma}^{\sigma} \exp\left\{ y \sum_p \frac{2+2\lambda}{p^{1+\sigma}} \right\} d\theta \ll \left(\frac{1}{\sigma}\right)^{(2+2\lambda)y-1}.$$

For $0 \leqslant \lambda \leqslant \frac{1}{3}$, and for real t,

$$|2\cos\tfrac{1}{2}t + 2\lambda \cos\tfrac{3}{2}t| = \frac{4}{\pi}(1-\tfrac{1}{3}\lambda) + \sum_{m=1}^{\infty} a_m(\lambda) \cos mt$$

where $a_m(\lambda) \ll m^{-2}$. Hence, for $\sigma < |\theta| \leqslant 1$,

$$\sum_p \frac{1}{p^{1+\sigma}} |2\cos(\tfrac{1}{2}\theta \log p) + 2\lambda \cos(\tfrac{3}{2}\theta \log p)|$$

$$= \frac{4}{\pi}(1-\tfrac{1}{3}\lambda) \log \zeta(1+\sigma) + \sum_{m=1}^{\infty} a_m(\lambda) \log|\zeta(1+\sigma+im\theta)| + O(1)$$

$$= \frac{4}{\pi}(1-\tfrac{1}{3}\lambda) \log\frac{1}{\sigma} + \sum_{m \leqslant 1/|\theta|} a_m(\lambda) \log\frac{1}{m|\theta|} + O(1).$$

We note that

$$\sum_{m=1}^{\infty} a_m(\lambda) = 2+2\lambda-\frac{4}{\pi}(1-\tfrac{1}{3}\lambda) = a(\lambda)\,,\text{ say}.$$

Therefore

$$I_2 \ll \left(\frac{1}{\sigma}\right)^{(4/\pi)(1-(\lambda/3))y} \int_{\sigma}^{1} \theta^{-a(\lambda)y}\,d\theta\,.$$

If $\frac{2}{3} \leqslant y \leqslant \pi/(2\pi-4)$, we choose λ (in the range $0 \leqslant \lambda \leqslant \frac{1}{3}$) so that $a(\lambda)y = 1$. This makes the exponents of $1/\sigma$ in our bounds for I_1 and I_2 equal, indeed we have

$$M_2(\sigma, y) \ll (1/\sigma)^{(16y-2)/(3\pi+2)}\log(1/\sigma)$$

which gives the result stated. If $y > \pi/(2\pi-4)$ we choose $\lambda = 0$. Thus $a(\lambda)y > 1$ and

$$I_1, I_2 \ll \left(\frac{1}{\sigma}\right)^{2y-1}.$$

This gives the result in this case and completes the proof.

We have now shown that $A_2 \leqslant (12-3\pi)/(3\pi+2) = 0{\cdot}22540\ldots$ and we improve on this by a convexity argument, using the upper bound for $\alpha(2, y)$ obtained in our previous paper when y is somewhat less than 1, and the present Lemma 3 when it is somewhat more. We then estimate $\alpha(2, 1)$ itself by Hölder's inequality. The details are as follows.

Hölder's inequality implies that for each r, $\alpha(r, e^t)$ is a convex function of t. In particular

$$A_2 = \alpha(2,1) \leqslant \left(1-\frac{1}{q}\right)\alpha(2,y)+\frac{1}{q}\alpha(2,y^{1-q})$$

for $y > 0$, $q > 1$. We shall choose y and q so that $\frac{1}{2}\sqrt{3} \leqslant y \leqslant y_0$, $y^{1-q} \leqslant \pi/(2\pi-4)$. We recall from [**2**] that in this range,

$$\alpha(2,y) \leqslant g(y) = \tfrac{3}{2}z^2+2z-\tfrac{11}{8}\,, \qquad 2z^2(z+1) = y^2\,,$$

and, from Lemma 3

$$\alpha(2,y^{1-q}) \leqslant ay^{1-q}-b\,, \qquad a = 16/(3\pi+2)\,, \qquad b = 1+2/(3\pi+2)\,.$$

Hence

$$A_2 \leqslant f(y,q) = \left(1-\frac{1}{q}\right)(\tfrac{3}{2}z^2+2z-\tfrac{11}{8})+\frac{1}{q}(ay^{1-q}-b)\,,$$

and we note that

$$\frac{\partial f}{\partial y} = \left(1-\frac{1}{q}\right)\left(\frac{y}{z}-ay^{-q}\right),$$

and that the stationary value is a minimum. Given q we fix y so that $y^{1+q} = az$.

Next,

$$\frac{df}{dq} = \frac{\partial f}{\partial q} = \frac{1}{q^2}\left\{-\tfrac{1}{2}z^2 - \tfrac{11}{8} + b - \frac{qy^2}{z}\log y\right\},$$

which is zero when z satisfies

$$-\tfrac{1}{2}z^2 - \tfrac{3}{8} + \frac{2}{(3\pi+2)} - z(z+1)\log\left\{\frac{128}{(3\pi+2)^2(z+1)}\right\} = 0\,.$$

This equation has a root at $z_1 = 0{\cdot}507214\ldots$, the corresponding values of y and q being $y_1 = 0{\cdot}880630\ldots\,\varepsilon(\tfrac{1}{2}\sqrt{3}, y_0)$ and $q_1 = 1{\cdot}690561\ldots$, and we check that $y_1^{1-q_1} < \pi/(2\pi-4)$. We have $A_2 \leqslant f(y_1, q_1) < 0{\cdot}21969$ which is the result stated.

§6. *Proof of Theorem* 4.

LEMMA 4. *For* $y \geqslant \frac{1}{2}$ *and every* r,

$$\alpha(r, y) \leqslant 2^{2^{r-1}-r} y^{2^{r-1}} - \tfrac{1}{2} - (\tfrac{1}{2})^r\,.$$

Proof. This is by induction on r, using Theorem 1 with $s = 2$ and $z = \frac{1}{2}$, which gives

$$2\alpha(r, y) \leqslant -\tfrac{1}{2} + \alpha(r-1, 2y^2)\,,$$

and the fact that $\alpha(1, y) = y - 1$.

Of course this is only useful when y is near $\frac{1}{2}$. In particular, for $r = 3$ we have $\alpha(3, z) \leqslant 2z^4 - \frac{5}{8}$ $(z \geqslant \frac{1}{2})$. From Lemma 3, we have $\alpha(2, 1/z) \leqslant 2/z - 2$ $(z \leqslant 2 - 4/\pi)$ and so, in this range for z, Theorem 1 with $r = 3$, $s = 2$, $y = 1$ gives

$$2A_3 \leqslant (2z^4 - \tfrac{5}{8}) + (2z-1) + \big((2/z) - 2\big)\,.$$

The optimum value of z is $0{\cdot}672047\ldots$ which gives $A_3 < 0{\cdot}55153$ as required.

It will be seen that in the cases $r = 2, 3$ we obtained our best results by a hybrid method. This works for $r = 4$ as well, but with only partial success, since we cannot show that $A_4 \leqslant h_4' - 1$.

First we show that $\alpha(3, y) \leqslant \max\big(y(\sqrt{3}) - 1, 3y - 3\big)$ by a straightforward extension of Hooley's argument given in §4 of his paper. In particular $\alpha(3, \frac{1}{3}(3+\sqrt{3})) \leqslant \sqrt{3}$. We apply Theorem 1 with $r = 3$, $s = 2$, $y = 1$, $z = \frac{1}{2}(3-\sqrt{3})$ so that we have

$$\begin{aligned} 2A_4 &\leqslant \alpha\big(4, \tfrac{1}{2}(3-\sqrt{3})\big) + 2 - \sqrt{3} + \alpha\big(3, \tfrac{1}{3}(3+\sqrt{3})\big) \\ &\leqslant \alpha\big(4, \tfrac{1}{2}(3-\sqrt{3})\big) + 2\,. \end{aligned}$$

We apply Lemma 4 on the right, which yields

$$A_4 \leqslant 3929 + \tfrac{7}{32} - 2268\sqrt{3} < 0{\cdot}92752\,.$$

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