

# On some Tauberian theorems related to the prime number theorem

A. Hildebrand and G. Tenenbaum

## § 1. Introduction.

The main purpose of this paper is to describe the asymptotic behavior of sequences  $\{a_n\}_{n=0}^{\infty}$  satisfying a relation of the form

$$(1.1) \quad a_n = \frac{1}{n} \sum_{1 \leq k \leq n} c_k a_{n-k} + r_n \quad (n \geq 1),$$

with initial condition

$$(1.2) \quad a_0 = r_0.$$

Here  $\{c_n\}_{n=1}^{\infty}$  and  $\{r_n\}_{n=0}^{\infty}$  are given sequences of complex numbers on which we shall impose some mild growth conditions.

The equation (1.1) expresses each term of the sequence  $\{a_n\}$  as an average of the previous terms, weighted by the coefficients  $c_k$ , plus a remainder term  $r_n$ . One might expect that this repeated averaging process induces some degree of regularity on the behavior of  $a_n$ . We shall show that this is indeed the case, even if we impose no regularity conditions on the behavior of  $c_n$ .

Our motivation for this work came from two directions. The first is the theory of multiplicative arithmetic functions, where Wirsing [Wi], Halász [Ha1, Ha2] and others have developed deep and powerful techniques to study the asymptotic behavior of the averages  $m(x) := e^{-x} \sum_{n \leq e^x} f(n)$  of a multiplicative function  $f$ . As was observed by Wirsing, these averages satisfy integral equations of the type

$$(1.3) \quad m(x) = x^{-1} \int_0^x m(y)k(x-y) dy + r(x) \quad (x \geq 0),$$

which are continuous analogs of the discrete equation (1.1). While Wirsing has given results for rather general equations of the form (1.3), subsequent work, and in particular the powerful analytic method of Halász, has been focussed on the narrow context of multiplicative functions, and the full scope of this method remains yet to be exploited. As a first step in this direction, it seemed natural to investigate the case of discrete equations, where the results take a particularly neat and simple form.

Another reason for studying equations of the form (1.1) is that the results have potential applications to certain non-linear relations such as

$$(1.4) \quad na_n + \sum_{1 \leq k < n} a_k a_{n-k} = 2n + O(1) \quad (n \geq 1),$$

which arise in proofs of the prime number theorem and its analogs. Specifically, (1.4) plays a crucial role in Bombieri's proof [Bo1, Bo2] of the prime number theorem for

algebraic function fields. Bombieri gave a Tauberian argument showing that any sequence of nonnegative real numbers  $a_n$  satisfying (1.4) as well as

$$(1.4)' \quad na_{2n} + \sum_{1 \leq k < n} a_{2k} a_{2n-2k} = 2n + O(1) \quad (n \geq 1)$$

necessarily converges to 1. Zhang [Zh] observed that (1.4) alone together with the nonnegativity of  $a_n$  is not sufficient to imply the convergence of  $a_n$ , a counter-example being provided by  $a_n = 1 + (-1)^{n+1}$ .

Zhang [Zh2] sharpened this result by proving that (1.4) together with the condition

$$(1.4)'' \quad \sum_{1 \leq k \leq n} a_{2k} = n + O(1),$$

which is weaker than (1.4)', implies the estimate  $a_n = 1 + O(1/n)$ . Granville [Gr] showed that for any sequence  $\{a_n\}$  of nonnegative numbers satisfying (1.4) with the weaker error term  $o(n)$  either  $a_n$  or  $a_n + (-1)^n$  converges to 1, thereby answering a question of Bombieri ([Bo2], p. 178).

By considering (1.4) as a special case of an equation of the type (1.1), we shall obtain a theorem that describes the asymptotic behavior of the general (nonnegative) solutions of this equation and which contains both Zhang's and Granville's results.

*Notation.* We shall use the symbols  $\ll$ ,  $\gg$ ,  $O(\dots)$ , and  $o(\dots)$  in their usual meaning and write  $\asymp$  if both  $\ll$  and  $\gg$  hold; any dependence of the constants implied by these symbols on other parameters will be explicitly stated or indicated by a subscript.

Ranges like  $k \leq x$  in summations are generally understood to include all *positive* integers satisfying the indicated inequality. If 0 is to be included, we will indicate this explicitly.

We shall use the standard notations

$$e(t) = e^{2\pi it}, \quad \|t\| = \min\{|t - n| : n \in \mathbf{Z}\},$$

and we define

$$(1.5) \quad \log^* t := \max(1, \log t).$$

## § 2. Statement of results.

We assume that  $\{c_n\}_{n=1}^\infty$  and  $\{r_n\}_{n=0}^\infty$  are given sequences of complex numbers and that the sequence  $\{a_n\}_{n=0}^\infty$  is defined (uniquely) by (1.1) and (1.2). Our first and main result is the analog of Halász' celebrated mean value theorem for multiplicative functions [Ha1]. It completely describes the asymptotic behavior of  $a_n$  for arbitrary complex coefficients  $c_k$  of modulus at most 1 under a mild growth condition on the remainder terms  $r_k$ .

**Theorem 1.** *Assume*

$$(2.1) \quad |c_k| \leq 1 \quad (k \geq 1)$$

and

$$(2.2) \quad \lim_{n \rightarrow \infty} r_n = 0, \quad \sum_{n \geq 1} \frac{|r_n|}{n} < \infty.$$

Then one of the following alternatives holds :

(i) The series

$$(2.3) \quad \sum_{n \geq 1} \frac{\operatorname{Re}(1 - c_n e(n\vartheta))}{n}$$

converges for some real number  $\vartheta$ ; in this case we have

$$(2.4) \quad a_n = ae(\varphi(n)) + o(1) \quad (n \rightarrow \infty),$$

where

$$(2.5) \quad \varphi(x) := -\vartheta x + \operatorname{Im} \sum_{1 \leq k \leq x} \frac{c_k e(k\vartheta)}{2\pi k}$$

and

$$(2.6) \quad a := \sum_{k \geq 0} r_k \lambda_k$$

with

$$(2.7) \quad \lambda_0 := e^{-\Lambda}, \quad \lambda_k := k e^{-\Lambda} \int_0^1 e^{\Lambda t} (1-t)t^{k-1} dt \quad (k \geq 1),$$

$$(2.8) \quad \Lambda(t) := \sum_{n \geq 1} \frac{(1 - c_n e(n\vartheta))t^n}{n}, \quad \Lambda := \operatorname{Re} \Lambda(1-) = \sum_{n \geq 1} \frac{\operatorname{Re}(1 - c_n e(n\vartheta))}{n}.$$

(ii) The series (2.3) diverges for all real numbers  $\vartheta$ ; in this case,  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Remarks.* (i) By (2.1), the series (2.3) has only nonnegative terms and hence, for any given  $\vartheta$ , either converges to a finite limit or diverges to infinity. It is easy to see that there exists at most one value of  $\vartheta$  for which (2.3) converges. Moreover, if (2.3) converges for some  $\vartheta$ , then we have  $|\lambda_k| \leq 1/(k+1)$  — see (6.11) —, and in view of (2.2) it follows that the series in (2.6) is absolutely convergent. Thus, the quantities  $a$  and  $\varphi(n)$  are well-defined under the hypothesis of case (i) of the theorem.

(ii) The oscillating factor  $e(\varphi(n))$  in the asymptotic formula (2.4) may be regarded as a perturbation of the function  $e(-n\vartheta)$ . In fact, in the course of the proof of the theorem we shall show that the function  $L(x) := e(\varphi(x) + x\vartheta)$  is “slowly oscillating” in the sense that

$$(2.9) \quad \max_{\varepsilon x \leq y \leq x/\varepsilon} |L(x) - L(y)| = o(1) \quad (x \rightarrow \infty)$$

holds for any fixed  $\varepsilon > 0$ .

(iii) The conditions (2.2) on  $r_n$  are essentially necessary for the conclusion (2.4) to be valid. This is clear in the case of the first condition. To see that the second condition in (2.2) cannot be dropped, it suffices to take  $c_k = 1$  for all  $k$ , in which case it is easily verified that  $a_n$  is given by

$$(2.10) \quad a_n = r_n + \sum_{0 \leq k \leq n-1} \frac{r_k}{k+1}.$$

If the terms  $r_n$  are nonnegative and the series in (2.2) diverges, then the last expression is unbounded as  $n \rightarrow \infty$ , and hence cannot satisfy a relation of the form (2.4).

(iv) The assumption (2.1) on  $c_k$  is the simplest and most natural condition under which the asymptotic behavior of  $a_n$  can be determined, and we shall prove the result only in this case. We remark, however, that this condition can be relaxed and generalized in various ways. For example, one can show that the result remains valid if the coefficients  $c_k$  are bounded by 1 in a suitable average sense, such as

$$(2.11) \quad \frac{1}{x} \sum_{k \leq x} |c_k|^2 \leq 1 + K/(\log x)^2 \quad (x \geq x_0)$$

with some constants  $K$  and  $x_0$ . Moreover, an analogous result describing the asymptotic behavior of  $a_n n^{1-C}$  can be proved in the case when (2.1) (or (2.11)) holds with the bound 1 replaced by an arbitrary constant  $C$ .

We state two corollaries of Theorem 1. The first of these results, an analog of a mean value theorem of Delange [De], gives a necessary and sufficient condition for the convergence of  $a_n$  to a non-zero limit, assuming that the remainder terms  $r_k$  are zero for  $k \geq 1$ . The second result, an analog of Wirsing's mean value theorem [Wi], deals with the case when the coefficients  $c_k$  are real.

**Corollary 1.** *Suppose that (2.1) holds and that  $r_0 \neq 0$  and  $r_k = 0$  for  $k \geq 1$ . Then the limit  $\lim_{n \rightarrow \infty} a_n$  exists and is non-zero, if and only if the series  $\sum_{n \geq 1} (1 - c_n)/n$  converges.*

**Corollary 2.** *Assume that (2.1) and (2.2) hold, and suppose in addition that the numbers  $c_k$  are all real. Then at least one of the two limits  $\lim_{n \rightarrow \infty} a_n$  or  $\lim_{n \rightarrow \infty} (-1)^n a_n$  exists.*

The possibility of two alternatives in the last result represents an important divergence from the case of continuous equations such as (1.3), where, under similar hypotheses, Wirsing's theorem shows that the solution  $m(x)$  is always convergent.

We next turn to quantitative estimates for  $a_n$ . We shall prove several estimates, assuming various kinds of bounds on the coefficients  $c_k$ , but without imposing restrictions on the remainders  $r_k$ . The results are again largely inspired by known estimates from the theory of multiplicative functions.

We shall use henceforth the following notation. Given a sequence  $\{c_k\}_{k=1}^{\infty}$ , we set, for  $x \geq y \geq 0$ ,

$$(2.12) \quad S(x, y; \vartheta) := \operatorname{Re} \sum_{y < k \leq x} \frac{c_k e(\vartheta k)}{k}, \quad S(x; \vartheta) := S(x, 0; \vartheta),$$

$$(2.13) \quad S^*(x, y) := \max_{\vartheta \in \mathbf{R}} S(x, y; \vartheta), \quad S^*(x) := S^*(x, 0).$$

**Theorem 2.** *Suppose that, with some positive constants  $C$  and  $K$ , we have*

$$(2.14) \quad |c_k| \leq K \quad (k \geq 1),$$

and

$$(2.15) \quad S^*(x, y) \leq C \log(x/y) + K \quad (x \geq y \geq 1).$$

Then

$$(2.16) \quad |a_n| \ll_{C,K} |r_n| + n^{C-1} \sum_{0 \leq k \leq n-1} \frac{|r_k|}{(k+1)^C} \quad (n \geq 1).$$

Furthermore, if  $c_k \geq 0$  for all  $k$ , then we have

$$(2.16)' \quad |a_n| \ll_{C,K} |r_n| + \frac{1}{n} \sum_{0 \leq k \leq n-1} |r_k| \min \left( \left( \frac{n}{k+1} \right)^C, e^{S^*(n)} \right) \quad (n \geq 1).$$

The estimates (2.16) and (2.16)' are sharp in several respects. In the first place, an application of Theorem 1 shows that if  $r_k = 0$  for  $k \geq 1$ ,  $0 \leq c_k \leq 1$  for all  $k$ , and the sum  $\sum_{k \geq 1} (1 - c_k)/k$  converges, then

$$\lim_{n \rightarrow \infty} a_n = r_0 \exp \left\{ - \sum_{k \geq 1} \frac{1 - c_k}{k} \right\} = r_0 e^{-\gamma} \lim_{n \rightarrow \infty} \frac{e^{S^*(n)}}{n},$$

where  $\gamma$  is Euler's constant. Thus the right-hand side of (2.16)' is in this case within absolute constants from the actual order of magnitude of  $|a_n|$  for large  $n$ .

Moreover, the dependence of (2.16) on the remainders  $r_k$  is also essentially best-possible. This can be seen by taking  $c_k = C$  for all  $k$ . As we have remarked above — see (2.10) —, the right-hand side of (2.16) is exactly equal to  $a_n$  if  $C = 1$  and the numbers  $r_k$  are nonnegative. More generally, it can be shown that, for arbitrary  $C > 0$ , the solution  $a_n$  to (1.1) with  $c_k = C$  is

$$(2.17) \quad a_n = r_n + C \binom{C + n - 1}{n} \sum_{0 \leq k < n} \frac{r_k}{k + 1} \binom{C + k}{k + 1}^{-1}.$$

For nonnegative  $r_k$ , this has the same order of magnitude than the upper bound of (2.16).

It is interesting to note that, in the special case of nonnegative coefficients  $c_k$ , the estimate (2.16)' also holds, under the same hypotheses, for the solutions to equations of the form

$$a_n = \frac{1}{n} \sum_{0 \leq k \leq n-1} c_k a_k + r_n \quad (n \geq 1).$$

This is a consequence of a discrete version of Gronwall's inequality; see Mitrinovic *et al.* [MPF], p. 436. It is tempting to believe that there is a connection between the two estimates. However, we have not succeeded in deducing the bound of the theorem from a Gronwall type inequality, and it seems that the former lies deeper.

A simple way of applying Theorem 2 is to use the inequality

$$(2.18) \quad S^*(x, y) \leq \sum_{y < k \leq x} \frac{|c_k|}{k}$$

and an estimate of the type  $C \log(x/y) + K$  for the right-hand side. This gives an upper bound for  $|a_n|$  in terms of an average of the numbers  $|c_k|$ ,  $k \leq n$ , which is analogous to estimates of Hall [Ha] and Halberstam and Richert [HR] for sums of multiplicative functions. As we remarked, this procedure often leads to sharp estimates, and hence is quite precise, when the coefficients  $c_k$  are nonnegative. In the general case, however, the hypothesis (2.15), which requires only one-sided bounds for sums over  $c_k$ , is much more efficient, and the resulting estimate is of far better quality. The following corollary is particularly useful for applications and often provides improvements on estimates obtained via (2.18). For instance, if the coefficients  $c_k$  are all negative and bounded in modulus by  $|C_-|$ , then the corollary shows that (2.15) and hence (2.16) hold with  $C = \frac{1}{2}|C_-|$ , whereas (2.18) would give (2.16) only with  $C = |C_-|$ .

**Corollary 3.** *If, for some constants  $C_-$ ,  $C_+$ , and  $k_0$ , we have*

$$(2.19) \quad C_- \leq c_k \leq C_+ \quad (k \geq k_0),$$

*then (2.15) and hence (2.16) hold with  $C = \max(C_+, \frac{1}{2}(C_+ - C_-))$ .*

Our next result represents a quantitative version of part (ii) of Theorem 1 and may be viewed as an analog of a quantitative mean value theorem of Halász [Ha2] (see also [HT]). We set

$$(2.20) \quad T(x; \vartheta) := \sum_{k \leq x} \frac{\operatorname{Re}(1 - c_k e(\vartheta k))}{k},$$

and define

$$(2.21) \quad T^*(x) := \min_{\vartheta \in \mathbf{R}} T(x; \vartheta), \quad \Delta_x := e^{-T^*(x)}.$$

**Theorem 3.** *Suppose that (2.14), (2.15) and*

$$(2.22) \quad \max_{\vartheta \in \mathbf{R}} |S(x; \vartheta)| \leq \log x + K \quad (x \geq 1)$$

hold with some constant  $K$  and with  $C = 1$ . Then, we have

$$(2.23) \quad |a_n| \ll_K |r_n| + \sum_{0 \leq k \leq n-1} |r_k| \lambda_k(\Delta_n) \quad (n \geq 1),$$

where  $\Delta_n$  is defined by (2.20) and (2.21) and

$$(2.24) \quad \lambda_0(\Delta) := \Delta \log^*(1/\Delta), \quad \lambda_k(\Delta) := \min \left( \frac{1}{k}, k\Delta \left( \log^* \frac{1}{k\sqrt{\Delta}} \right)^2 \right) \quad (k \geq 1),$$

with  $\log^* t$  being defined by (1.5).

It is not hard to see that, under the condition (2.1), the series (2.3) diverges for every  $\vartheta \in \mathbf{R}$ , if and only if the function  $\Delta_x$  tends to 0 as  $x \rightarrow \infty$ . Theorem 1 shows that in this case  $a_n$  tends to 0, and it is therefore natural to seek a quantitative bound for  $|a_n|$  in terms of the quantity  $\Delta_n$ . The theorem gives a bound of this type.

The hypotheses on  $c_k$  in Theorem 3 are weaker than those of Theorem 1 and in a sense represent the minimal hypotheses under which the estimate of the theorem holds. For simplicity we have confined ourselves to the case when (2.15) and (2.22) holds with the constant  $C = 1$  as a factor of the logarithms, but similar estimates, with  $\lambda_k(\Delta)$  replaced by different weight functions, can be proved in the general case.

The coefficients  $\lambda_k(\Delta)$  defined in (2.24) are close to being best-possible as functions of  $k$  and  $\Delta$ . This can be seen by taking, for given integers  $n_0 > 1$  and  $k_0 \geq 0$ ,

$$(2.25) \quad c_k = \begin{cases} -1 & (k \leq n_0) \\ 1 & (k > n_0) \end{cases}, \quad r_k = \begin{cases} 1 & (k = k_0) \\ 0 & (k \neq k_0) \end{cases}$$

and comparing the estimate (2.23) with the asymptotic formula of Theorem 1. On the one hand, Theorem 1 gives  $\lim_{n \rightarrow \infty} a_n = \lambda_{k_0}$  with

$$\lambda_0 = \exp \left\{ -2 \sum_{k \leq n_0} \frac{1}{k} \right\} \asymp n_0^{-2}$$

and, for  $k_0 \geq 1$ ,

$$\begin{aligned} \lambda_{k_0} &= k_0 \int_0^1 \exp \left\{ -2 \sum_{k \leq n_0} \frac{1-t^k}{k} \right\} (1-t)t^{k_0-1} dt \\ &\asymp k_0 \int_0^1 \min \{ 1, (n_0(1-t))^{-2} \} (1-t)t^{k_0-1} dt \\ &\asymp \min \left( \frac{1}{k_0}, \frac{k_0}{n_0^2} \log^* \frac{n_0}{k_0} \right). \end{aligned}$$

On the other hand, it is not hard to show that, for sufficiently large  $n$ ,  $\Delta_n \asymp n_0^{-2}$ . The right-hand side of (2.23) therefore is of order  $n_0^{-2} \log n_0$  if  $k_0 = 0$ , and

$$\lambda_{k_0}(\Delta_n) \asymp \min \left( \frac{1}{k_0}, \frac{k_0}{n_0^2} \left( \log^* \frac{n_0}{k_0} \right)^2 \right),$$

if  $k_0 \geq 1$ . Thus the bound (2.23) is in this case by at most a logarithmic factor larger than the asymptotic order of magnitude for  $a_n$  given by Theorem 1. In fact, it seems likely that by directly evaluating  $a_n$  for sequences such as (2.25) one can show that, for suitable finite ranges of  $n$ , the bound (2.23) lies within absolute constants of the true order magnitude of  $a_n$ .

Of particular interest is the situation when the coefficients  $c_k$  are real. In Section 7 we shall show that, assuming (2.1) — which implies the hypotheses of Theorem 3 —, we have in this case

$$(2.26) \quad T^*(x) \geq \frac{1}{5} \min \left( T(x, 0), T(x, \frac{1}{2}) \right) + O(1)$$

and thus obtain the following corollary.

**Corollary 4.** *Suppose that the coefficients  $c_k$  are real and satisfy (2.1). Then (2.23) holds with  $\Delta_n = e^{-T(n,0)/5} + e^{-T(n,1/2)/5}$ .*

As we have mentioned above, the proofs of Theorems 2 and 3 are based on analytic methods. The principal tool is an estimate for  $a_n$  in terms of the generating function

$$(2.27) \quad a(z) := \sum_{n=0}^{\infty} a_n z^n.$$

We now state this result, together with a related estimate of independent interest. We set

$$(2.28) \quad M(\varrho) = \sup_{|z|=\varrho} |a(z)| \quad (\varrho > 0),$$

with the convention that  $M(\varrho) = \infty$  if the series  $a(z)$  does not converge for some  $z$  with  $|z| = \varrho$ .

**Theorem 4.** (i) *If  $c_k$  satisfies (2.14) for some constant  $K$ , then*

$$(2.29) \quad |a_n| \ll_K \frac{1}{n} \int_0^{1-1/n} \frac{M(\varrho)}{1-\varrho} d\varrho + \frac{1}{n} \sum_{0 \leq k \leq n-1} |r_k| + |r_n| \quad (n \geq 1).$$

(ii) *If, for some constant  $K$ , we have*

$$(2.30) \quad \sum_{k \leq x} |c_k|^2 \leq Kx \quad (x \geq 1),$$

then

$$(2.31) \quad |a_n| \ll_K \left( \frac{1}{n} \int_0^{1-1/n} M(\varrho)^2 d\varrho \right)^{1/2} + \left( \frac{1}{n} \sum_{0 \leq k \leq n-1} |r_k|^2 \right)^{1/2} + |r_n| \quad (n \geq 1).$$

An application of Cauchy's inequality shows that the right-hand side of (2.29) is bounded from above by the right-hand side of (2.31). Thus, the estimate of part (i) is stronger than that of part (ii). On the other hand, the second estimate holds under a weaker hypothesis on  $c_k$  than the first.

The first part of Theorem 4 will be used in the proofs of Theorems 2 and 3. The second part is analogous to quantitative estimates of Halász and Montgomery for multiplicative functions ([Ha2, Mo]; see also [MV], [Te], [HT]), and its proof is modelled after the Halász-Montgomery argument, as given in [Mo] or [Te] (§ III.4.3).

In our final theorem we apply the above results to quadratic equations of the type (1.4).

**Theorem 5.** *Suppose  $\{a_n\}_{n=1}^\infty$  is a sequence of nonnegative real numbers satisfying*

$$(2.32) \quad na_n + \sum_{k=1}^{n-1} a_k a_{n-k} = 2n + O(R(n)) \quad (n \geq 1),$$

where  $R(n)$  is a positive-valued function with the properties

$$(2.33) \quad R(n) \text{ is nondecreasing,} \quad R(n)/n \text{ is nonincreasing,}$$

$$(2.34) \quad \lim_{n \rightarrow \infty} R(n)/n = 0.$$

Then we have either

$$(2.35) \quad a_n = 1 + O\left(\frac{R(n)}{n}\right) \quad (n \geq 1)$$

or

$$(2.35)' \quad a_n = 1 + (-1)^{n+1} + O\left(\frac{R(n)}{n}\right) \quad (n \geq 1).$$

The error terms in (2.35) and (2.35)' are clearly best-possible, and the condition (2.34) is obviously necessary in order to conclude the convergence of  $a_n$  or  $a_n + (-1)^n$ . In the case of the equation (1.4), the hypotheses of the theorem are satisfied with  $R(n) \equiv 1$ , and we therefore obtain that (2.35) or (2.35)' hold with error terms  $O(1/n)$ .

The result of Theorem 5 is related to the Abstract Prime Number Theorem for additive arithmetical semi-groups (see [Zh1] for a precise definition and a list of references). In this context, it can be shown that the analog of the arithmetical quantity  $\psi(n)/n$  (where  $\psi$  denotes the Chebyshev function) satisfies (2.32) with  $R(n) \equiv 1$ . The fact that then (2.35) or (2.35)' holds has been recently proved in this case [IMW], but it was not previously known that this conclusion merely follows from the Selberg type formula (2.32), without any assumption aside from the nonnegativity of the coefficients.

§ 3. Preliminaries

In this section, we obtain an identity and an upper bound for the generating function  $a(z)$  defined by (2.27), and we prove that the function  $S^*(x)$  introduced in (2.13) is nondecreasing up to an bounded additive error term. We set

$$c(z) := \sum_{n=1}^{\infty} c_n z^n, \quad r(z) := \sum_{n=0}^{\infty} r_n z^n.$$

A priori these series may be divergent for any non-zero value of  $z$ . However, we shall see that, for the proofs of Theorems 1–4, we may assume that each of the series  $a(z)$ ,  $c(z)$ , and  $r(z)$  converges in the disk  $|z| < 1$ .

In the case of Theorem 1, the convergence of  $c(z)$  and  $r(z)$  for  $|z| < 1$  follows from the hypotheses (2.1) and (2.2). The same assumptions together with (1.1) give

$$|a_n| \leq \frac{1}{n} \sum_{0 \leq k \leq n-1} |a_k| + R \quad (n \geq 1)$$

with  $R := \sup_{n \geq 0} |r_n|$ . By a simple induction argument, this implies  $|a_n| \ll_{\varepsilon} (1 + \varepsilon)^n$  for any fixed  $\varepsilon > 0$ , and hence the convergence of  $a(z)$  in  $|z| < 1$ .

Next, observe that  $a_n$  is uniquely determined by the values of  $r_k$  and  $c_k$  for  $k \leq n$ . Thus, in proving the estimates of Theorems 2–4 for a given value of  $n$ , we may assume that  $c_k = r_k = 0$  for  $k > n$ . Under this assumption, the functions  $c(z)$  and  $r(z)$  are polynomials and thus convergent for all  $z$ . Moreover, we have

$$|a_k| \leq \frac{K}{k} \sum_{0 \leq \ell \leq k} |a_{\ell}| \quad (k > n)$$

with  $K := \max_{k \leq n} |c_k|$ , which as before implies the convergence of  $a(z)$  for  $|z| < 1$ .

With these assumptions, we have the following result.

**Lemma 1.** For  $|z| < 1$  we have

$$(3.1) \quad a(z) = r_0 e^{C(z)} + \int_0^z e^{C(z)-C(w)} r'(w) dw,$$

where

$$(3.2) \quad C(z) := \sum_{k=1}^{\infty} k^{-1} c_k z^k.$$

*Proof.* Multiplying (1.1) by  $nz^n$  and summing over  $n \geq 1$ , we obtain

$$(3.3) \quad za'(z) = a(z)c(z) + zr'(z) \quad (|z| < 1).$$

The general solution to this differential equation is of the form

$$a(z) = \left\{ A + \int_0^z e^{-C(w)} r'(w) dw \right\} e^{C(z)}$$

for some constant  $A$ . The required result follows, since, by (1.2),  $a(0) = a_0 = r_0$  and therefore  $A = r_0$ .

The next lemma gives an upper bound for  $M(\varrho) = \max_{|z|=\varrho} |a(z)|$  which will be used in the proofs of Theorems 2 and 3.

**Lemma 2.** *Assume that (2.14) holds for some constant  $K$ . Then we have, for any  $x \geq 1$ ,*

$$(3.4) \quad M(1 - 1/x) \ll_K |r_0| e^{S^*(x)} + \sum_{k \geq 1} |r_k| \int_{k/x}^k e^{S^*(x, k/u)} e^{-u} du.$$

*Proof.* Expanding  $r'(z)$  and interchanging the order of summation and integration in (3.1), we obtain, for  $|z| < 1$ ,

$$(3.5) \quad |a(z)| \leq |r_0| e^{\operatorname{Re} C(z)} + \sum_{k \geq 1} |r_k I_k(z)|,$$

with

$$I_k(z) := k \int_0^z e^{C(z) - C(w)} w^{k-1} dw.$$

Setting  $w = te(\vartheta)$  with  $0 \leq t < 1$  and  $\vartheta \in \mathbf{R}$ , we have by (2.14)

$$\begin{aligned} C(w) &= \sum_{k \leq 1/(1-t)} \frac{c_k e(k\vartheta)}{k} + O_K \left( \sum_{k \leq 1/(1-t)} \frac{1-t^k}{k} + \sum_{k > 1/(1-t)} \frac{t^k}{k} \right) \\ &= S\left(\frac{1}{1-t}; \vartheta\right) + O_K(1). \end{aligned}$$

It follows that for  $z = (1 - 1/x)e(\vartheta)$  and  $w = te(\vartheta)$ ,  $0 \leq t \leq 1 - 1/x$ ,

$$\operatorname{Re} (C(z) - C(w)) = S\left(x, \frac{1}{1-t}; \vartheta\right) + O_K(1) \leq S^*\left(x, \frac{1}{1-t}\right) + O_K(1),$$

and, in particular,  $\operatorname{Re} C(z) \leq S^*(x) + O_K(1)$ . This shows that the coefficient of  $|r_0|$  in (3.5) is of order  $e^{S^*(x)}$  and gives for the integral  $I_k(z)$  the bound

$$\begin{aligned} |I_k(z)| &\ll_K k \int_0^{1-1/x} e^{S^*(x, 1/(1-t))} t^{k-1} dt \\ &\ll k \int_1^x e^{S^*(x, y)} e^{-k/y} \frac{dy}{y^2} \\ &= \int_{k/x}^k e^{S^*(x, k/u)} e^{-u} du. \end{aligned}$$

The desired estimate now follows.

**Lemma 3.** *Under the hypothesis (2.14), we have*

$$(3.6) \quad S^*(y) \leq S^*(x) + O_K(1) \quad (1 \leq y \leq x).$$

*Proof.* Select  $\vartheta_y \in \mathbf{R}$  such that  $S^*(y) = S(y; \vartheta_y)$ . Since the numbers  $c_k$  are bounded, we have  $|\partial S(y; \vartheta)/\partial \vartheta| \ll_K y$  and hence

$$S(y; \vartheta) = S^*(y) + O_K(1) \quad (|\vartheta - \vartheta_y| \leq 1/y).$$

Thus, we may write

$$\begin{aligned} S^*(y) &= \frac{1}{2}y \int_{\vartheta_y-1/y}^{\vartheta_y+1/y} S(y; \vartheta) d\vartheta + O_K(1) \\ &= \frac{1}{2}y \int_{\vartheta_y-1/y}^{\vartheta_y+1/y} S(x; \vartheta) d\vartheta - \frac{1}{2}y \int_{\vartheta_y-1/y}^{\vartheta_y+1/y} S(x, y; \vartheta) d\vartheta + O_K(1). \end{aligned}$$

In this last expression, the first term does not exceed  $S^*(x)$  since, by definition,  $\max_{\vartheta} S(x; \vartheta) = S^*(x)$ ; the second term is equal to

$$-\frac{1}{2}y \operatorname{Re} \int_{\vartheta_y-1/y}^{\vartheta_y+1/y} \sum_{y < k \leq x} \frac{c_k}{k} e(k\vartheta) d\vartheta = O\left(y \sum_{y < k \leq x} \frac{|c_k|}{k^2}\right) = O_K(1).$$

This completes the proof.

#### § 4. Proof of Theorem 4.

We may assume that the series  $a(z)$  converges in the disk  $|z| < 1$ , for otherwise we have, by definition,  $M(\varrho) = \infty$  in some non-trivial interval  $\varrho_0 < \varrho < 1$  and the right-hand sides of the two estimates (2.29) and (2.31) are infinite.

We fix a positive integer throughout the proof. We may suppose that  $n > 1$ , since for  $n = 1$  the estimates to be proved are equivalent to the bound  $|a_1| \ll_K |r_0| + |r_1|$ , which follows immediately from (1.1) and (2.30). We set

$$c_m^* := \begin{cases} c_m & (m \leq n) \\ 0 & (m > n) \end{cases}$$

and

$$d_m := \sum_{k \leq m} c_k^* a_{m-k},$$

so that by (1.1)

$$(4.1) \quad a_m = \frac{1}{m} d_m + r_m \quad (1 \leq m \leq n).$$

We first show that, under the hypothesis (2.30) (and thus, a fortiori, under (2.14)), we have

$$(4.2) \quad \sum_{m \leq 2x} |d_m|^2 \ll_K x M(1 - 1/x)^2 \quad (1 \leq x \leq n).$$

Let the generating functions for the sequences  $\{c_m^*\}$  and  $\{d_m\}$  be denoted by  $c^*(z)$  and  $d(z)$  respectively. We have  $d(z) = c^*(z)a(z)$ , and by Parseval's formula it follows that, for  $0 < \varrho < 1$ ,

$$\begin{aligned} \sum_{m \geq 0} |d_m|^2 \varrho^{2m} &= \int_0^1 |d(\varrho e(\vartheta))|^2 d\vartheta \\ (4.3) \quad &= \int_0^1 |c^*(\varrho e(\vartheta))a(\varrho e(\vartheta))|^2 d\vartheta \\ &\leq M(\varrho)^2 \int_0^1 |c^*(\varrho e(\vartheta))|^2 d\vartheta = M(\varrho)^2 \sum_{m \leq n} |c_m|^2 \varrho^{2m}. \end{aligned}$$

By partial summation and (2.30) we have

$$\begin{aligned} \sum_{m \leq n} |c_m|^2 \varrho^{2m} &= \varrho^{2n} \sum_{m \leq n} |c_m|^2 + 2|\log \varrho| \int_0^n \sum_{m \leq u} |c_m|^2 \varrho^{2u} du \\ &\leq K \left( \varrho^{2n} n + 2|\log \varrho| \int_0^n u \varrho^{2u} du \right) \ll_K \min \left( n, \frac{1}{1-\varrho} \right). \end{aligned}$$

Taking  $\varrho = 1 - 1/x$  and noting that with this choice

$$\sum_{m \leq 2x} |d_m|^2 \ll \sum_{m \geq 0} |d_m|^2 \varrho^{2m},$$

we obtain (4.2).

From (4.1) and (4.2) we deduce first that

$$\begin{aligned} \sum_{1 \leq m \leq n} \frac{|d_m|^2}{m^2} &\leq \frac{1}{n^2} \sum_{m \leq n} |d_m|^2 + 2 \int_1^n \sum_{m \leq 2u} |d_m|^2 \frac{du}{u^3} \\ &\ll_K \frac{1}{n} M(1 - 2/n)^2 + \int_1^n M(1 - 1/u)^2 \frac{du}{u^2} \\ &\ll \int_1^n M(1 - 1/u)^2 \frac{du}{u^2} = \int_0^{1-1/n} M(\varrho)^2 d\varrho, \end{aligned}$$

since, by the monotonicity of the function  $M(\varrho)$  and our assumption  $n \geq 2$ ,

$$M(1 - 2/n)^2 \leq \frac{2}{n} \int_{n/2}^n M(1 - 1/u)^2 du \ll n \int_{n/2}^n M(1 - 1/u)^2 \frac{du}{u^2}.$$

By a further application of (4.1) and (2.30) it follows that

$$\begin{aligned} \sum_{0 \leq k \leq n} |a_k|^2 &\ll \sum_{m \leq n} \frac{|d_m|^2}{m^2} + \sum_{0 \leq m \leq n} |r_m|^2 \\ &\ll_K \int_0^{1-1/n} M(\varrho)^2 d\varrho + \sum_{0 \leq m \leq n} |r_m|^2. \end{aligned}$$

From this estimate, (1.1) and (2.30) we obtain by Cauchy's inequality

$$\begin{aligned} |a_n| &\leq \left( \frac{1}{n} \sum_{0 \leq k \leq n} |a_k|^2 \right)^{1/2} \left( \frac{1}{n} \sum_{k \leq n} |c_k|^2 \right)^{1/2} + |r_n| \\ &\ll_K \left( \frac{1}{n} \int_0^{1-1/n} M(\varrho)^2 d\varrho \right)^{1/2} + \left( \frac{1}{n} \sum_{0 \leq k \leq n-1} |r_k|^2 \right)^{1/2} + |r_n|, \end{aligned}$$

which proves (2.31).

To show (2.29) under the stronger hypothesis (2.14), we first apply (1.1) together with (2.14) and (4.1) to obtain

$$\begin{aligned} |a_n| &\ll_K \frac{1}{n} \sum_{0 \leq m \leq n} |a_m| + |r_n| \\ &\ll \frac{1}{n} \sum_{m \leq n} \frac{|d_m|}{m} + \frac{1}{n} \sum_{0 \leq m \leq n-1} |r_m| + |r_n|. \end{aligned}$$

Next, we use partial summation, Cauchy's inequality, and (4.2) to get

$$\begin{aligned} \sum_{m \leq n} \frac{|d_m|}{m} &= \frac{1}{n} \sum_{m \leq n} |d_m| + \int_1^n \sum_{m \leq u} |d_m| \frac{du}{u^2} \\ &\ll_K M(1 - 2/n) + \int_1^n M(1 - 1/u) \frac{du}{u} \ll \int_0^{1-1/n} \frac{M(\varrho)}{1 - \varrho} d\varrho. \end{aligned}$$

Combining these estimates yields (2.29) and completes the proof of Theorem 4.

§ 5. Proofs of Theorems 2 and 3.

As we have remarked above, we may assume that  $r_k = 0$  for  $k > n$ . Combining the first estimate of Theorem 4 with Lemma 2, we then obtain, under the common hypothesis (2.14),

$$\begin{aligned} (5.1) \quad |a_n| &\ll_K \frac{1}{n} \int_1^n M(1 - 1/x) \frac{dx}{x} + \frac{1}{n} \sum_{0 \leq k \leq n-1} |r_k| + |r_n| \\ &\ll_K \sum_{0 \leq k \leq n-1} |r_k| \left( I_k + \frac{1}{n} \right) + |r_n| \end{aligned}$$

with

$$I_0 := \frac{1}{n} \int_1^n e^{S^*(x)} \frac{dx}{x}, \quad I_k := \frac{1}{n} \int_1^n \frac{1}{x} \int_{k/x}^k e^{S^*(x, k/u)} e^{-u} du dx \quad (k \geq 1).$$

If, in addition to (2.14), the hypothesis (2.15) of Theorem 2 holds with some positive constant  $C$ , then we have

$$e^{S^*(x)} \ll_K x^C, \quad e^{S^*(x, y)} \ll_K (x/y)^C \quad (x \geq y \geq 1)$$

and therefore

$$\begin{aligned} I_0 &\ll_K \frac{1}{n} \int_1^n x^{C-1} dx \ll_{C, K} n^{C-1}, \\ I_k &\ll_K \frac{1}{n} \int_1^n \frac{1}{x} \int_0^\infty \left( \frac{xu}{k} \right)^C e^{-u} du dx \\ &\ll_{C, K} \frac{1}{nk^C} \int_1^n x^{C-1} dx \ll_C \frac{n^{C-1}}{k^C} \quad (k \geq 1). \end{aligned}$$

Hence the coefficients of  $|r_k|$  in (5.1) are bounded by  $\ll n^{C-1}(k+1)^{-C}$  for  $0 \leq k \leq n$ , and we obtain the first estimate, (2.16), of Theorem 2.

To prove (2.16)', we first observe that, if we put  $R_k := |r_k|$ , and let  $\{A_n\}_{n=0}^\infty$  be the sequence defined by  $A_0 = |a_0|$  and

$$(5.2) \quad A_n = \frac{1}{n} \sum_{k=1}^n c_k A_{n-k} + R_n \quad (n \geq 1),$$

then  $|a_n| \leq A_n$  for all  $n$ , since by hypothesis the numbers  $c_k$  are nonnegative. Thus it suffices to prove the required estimate in the case when the numbers  $r_k$  and  $a_k$  are nonnegative.

With these assumptions we have, by (1.1), (1.2) and Lemma 2,

$$(5.3) \quad \begin{aligned} a_n &\leq \frac{1}{n} \sum_{1 \leq k \leq n} K a_{n-k} + r_n \ll_K \frac{1}{n} a \left(1 - \frac{1}{n}\right) + r_n \\ &\ll r_0 \frac{e^{S^*(n)}}{n} + \frac{1}{n} \sum_{k \leq n} r_k \int_{k/n}^k e^{S^*(n, k/u)} e^{-u} du + r_n. \end{aligned}$$

Now, by (2.15) and the nonnegativity of the coefficients  $c_k$ , we have for all  $k \geq 1$

$$S^*(n, k/u) = \sum_{k/u < \ell \leq n} \frac{c_\ell}{\ell} \leq \min(S^*(n), C \log(nu/k)) + O_K(1).$$

Hence the integrals in (5.3) are bounded by

$$\ll_K \int_{k/n}^k \min\left(e^{S^*(n)}, \left(\frac{nu}{k}\right)^C\right) e^{-u} du \ll \min\left(e^{S^*(n)}, \left(\frac{n}{k}\right)^C\right),$$

and we obtain the desired estimate (2.16)'.

Suppose now that the hypotheses of Theorem 3 — i.e. (2.14), (2.15) with  $C = 1$  and (2.22) — are satisfied. We then have

$$(5.4) \quad S^*(x) = \sum_{k \leq x} \frac{1}{k} - T^*(x) = \log x - T^*(x) + O_K(1) \quad (x \geq 1).$$

By Lemma 3, we deduce that, for  $1 \leq y \leq x$ , we have

$$S^*(y) \leq \log x - T^*(x) + O_K(1).$$

Together with (2.15), this yields

$$(5.5) \quad e^{S^*(y)} \asymp y \Delta_y \ll_K \min(y, x \Delta_x) \quad (1 \leq y \leq x),$$

and in turn

$$I_0 \ll_K \frac{1}{n} \int_1^n \min\left(1, \frac{n \Delta_n}{x}\right) dx \ll \Delta_n \log^*(1/\Delta_n) = \lambda_0(\Delta_n).$$

Next, we note that, for  $1 \leq y \leq x$ , we have for some  $\vartheta = \vartheta(x, y)$

$$S^*(x, y) = S(x; \vartheta) - S(y; \vartheta) \leq S^*(x) + \max_{\vartheta \in \mathbf{R}} |S(y, \vartheta)|.$$

Thus, we may deduce from (2.15) and (2.22) that

$$S^*(x, y) \leq \min(\log(x/y), S^*(x) + \log y) + O_K(1).$$

Taking (5.4) into account, we therefore obtain

$$(5.7) \quad e^{S^*(x, y)} \ll_K \min(x/y, xy \Delta_x) \quad (1 \leq y \leq x).$$

This implies for  $k \geq 1$  and  $1 \leq x \leq n$ ,

$$\begin{aligned} \frac{1}{x} \int_{k/x}^k e^{S^*(x,k/u)} e^{-u} du &\ll_K \int_0^\infty \min\left(\frac{u}{k}, \frac{k\Delta_x}{u}\right) e^{-u} du \\ &\ll_K \min\left(\frac{1}{k}, k\Delta_x \log^* \frac{1}{k\sqrt{\Delta_x}}\right) \\ &\ll_K \min\left(\frac{1}{k}, \frac{nk\Delta_n}{x} \log^* \frac{1}{k\sqrt{n\Delta_n/x}}\right), \end{aligned}$$

where in the last step we used the estimate  $\Delta_x \ll_K n\Delta_n/x$ , which follows from (5.5). We hence infer for  $k \geq 1$

$$I_k \ll_K \frac{1}{n} \int_1^n \min\left(\frac{1}{k}, \frac{nk\Delta_n}{x} \log^* \frac{1}{k\sqrt{n\Delta_n/x}}\right) dx \ll \lambda_k(\Delta_n),$$

where  $\lambda_k(\Delta)$  is defined by (2.24). Noting further that, by (5.5) we have  $\Delta_n \gg \Delta_2/n \gg 1/n$ , we see that  $I_k + 1/n \ll \lambda_k(\Delta_n)$  and thus obtain the estimate (2.23) of Theorem 3.

### § 6. Proof of Theorem 1.

If the second alternative of the theorem holds, then the desired conclusion follows almost immediately from Theorem 3. Indeed, in this case the function  $T(x, \vartheta)$  defined in Theorem 3 tends to infinity as  $x \rightarrow \infty$ , for every fixed  $\vartheta$ , and since  $T(x, \vartheta)$  is a continuous function of  $\vartheta$  and a nondecreasing function of  $x$ , it follows by a classical result of calculus (see, for example, [Te], Lemma III.4.4.1) that the function  $T^*(x) = \inf_{\vartheta \in \mathbf{R}} T(x, \vartheta)$  tends to infinity as well, so that  $\lim_{n \rightarrow \infty} \Delta_n = 0$ . In view of the hypothesis (2.2) this implies that the right-hand side of (2.20), and therefore  $a_n$ , tends to 0 as  $n \rightarrow \infty$ .

Assume now that the first alternative is satisfied, i.e., the series (2.3) converges for some real  $\vartheta$ . Replacing  $a_n$ ,  $c_n$ , and  $r_n$  by  $a_n^* = a_n e(-n\vartheta)$ ,  $c_n^* = c_n e(-n\vartheta)$ , and  $r_n^* = r_n e(-n\vartheta)$ , respectively, we may assume that  $\vartheta = 0$ .

We first note that, by Theorem 2 and the hypotheses (2.1) and (2.2), the numbers  $a_n$  are uniformly bounded.

Let

$$L(x) := e(\varphi(x)) = \exp\left\{i \operatorname{Im} \sum_{k \leq x} c_k/k\right\},$$

and define

$$\alpha_n := L(n)^{-1} a_n, \quad \alpha(z) := \sum_{n=0}^{\infty} \alpha_n z^n \quad (|z| < 1).$$

To prove (2.4), we have to show the asymptotic relation

$$(6.1) \quad \alpha_n = a + o(1) \quad (n \rightarrow \infty)$$

with  $a$  defined by (2.6) and (2.7). We shall apply Tauber's classical theorem, which asserts that, if one has

$$(6.2) \quad \lim_{\varrho \rightarrow 1^-} (1 - \varrho)\alpha(\varrho) = a,$$

then (6.1) holds if and only if

$$(6.3) \quad \alpha_n = \frac{1}{n} \sum_{0 \leq k \leq n-1} \alpha_k + o(1) \quad (n \rightarrow \infty)$$

is satisfied.

We write  $c_k = u_k + iv_k$  and put

$$W(x) := \sum_{k \leq x} k^{-1} v_k, \quad V(\varrho) := \sum_{k=1}^{\infty} k^{-1} v_k \varrho^k \quad (0 \leq \varrho < 1),$$

so we have in particular  $L(x) = \exp\{iW(x)\}$ . We first show that  $L(x)$  is a slowly varying function of  $n$ , i.e., satisfies the oscillation condition (2.9).

The convergence of (2.3) with  $\vartheta = 0$  implies

$$(6.4) \quad \sum_{k=1}^{\infty} k^{-1} (1 - u_k) < \infty,$$

and since, by the hypothesis (2.1),  $v_k^2 \leq 1 - u_k^2 \leq 2(1 - u_k)$ , it follows that

$$(6.5) \quad \sum_{k=1}^{\infty} k^{-1} v_k^2 < \infty.$$

Together with the estimate

$$\begin{aligned} \sup_{\varepsilon x \leq y \leq x/\varepsilon} |W(x) - W(y)|^2 &\leq \left( \sum_{\varepsilon x \leq k \leq x/\varepsilon} k^{-1} |v_k| \right)^2 \\ &\leq \sum_{\varepsilon x \leq k \leq x/\varepsilon} k^{-1} \sum_{\varepsilon x \leq k \leq x/\varepsilon} k^{-1} v_k^2 \\ &\ll_{\varepsilon} \sum_{k \geq \varepsilon x} k^{-1} v_k^2, \end{aligned}$$

this yields (2.9).

Next, we show that (6.3) is satisfied. Plainly

$$(6.6) \quad \begin{aligned} \alpha_n - \frac{1}{n} \sum_{0 \leq k \leq n-1} \alpha_k &= L(n)^{-1} \left\{ a_n - \frac{1}{n} \sum_{0 \leq k \leq n-1} a_k \right\} \\ &\quad + \frac{1}{n} \sum_{0 \leq k \leq n-1} a_k (L(n)^{-1} - L(k)^{-1}). \end{aligned}$$

By (2.9) and the fact that  $a_n$  is bounded, we see that the second term on the right tends to zero as  $n \rightarrow \infty$ . Moreover, we have for every  $n$

$$a_n - \frac{1}{n} \sum_{0 \leq k \leq n-1} a_k = r_n - \frac{1}{n} \sum_{k \leq n} a_{n-k} (1 - c_k)$$

and, for any positive real number  $\varepsilon$ ,

$$\begin{aligned} \left( \frac{1}{n} \sum_{k \leq n} |1 - c_k| \right)^2 &\leq \frac{1}{n} \sum_{k \leq n} |1 - c_k|^2 \\ &\leq \varepsilon \sum_{k \leq \varepsilon n} \frac{2(1 - u_k) + v_k^2}{k} + \sum_{k > \varepsilon n} \frac{2(1 - u_k) + v_k^2}{k} \\ &\ll \varepsilon + o(1), \end{aligned}$$

in view of (6.4) and (6.5). Thus the first term on the right of (6.6) tends to zero as well, and (6.3) follows.

It remains to prove (6.2). We begin by showing that

$$(6.7) \quad (1 - \varrho)\alpha(\varrho) = e^{-iV(\varrho)}(1 - \varrho)a(\varrho) + o(1) \quad (\varrho \rightarrow 1-).$$

We have

$$\begin{aligned} (1 - \varrho)|\alpha(\varrho) - e^{-iV(\varrho)}a(\varrho)| &= (1 - \varrho) \left| \sum_{n=0}^{\infty} a_n \varrho^n (e^{i(V(\varrho) - W(n))} - 1) \right| \\ &\leq (1 - \varrho) \left| \sum_{n=0}^{\infty} |a_n| \varrho^n |W(n) - V(\varrho)| \right| \\ &\ll (1 - \varrho) \sum_{n=0}^{\infty} \varrho^n \left( \sum_{k \leq n} k^{-1} |v_k| (1 - \varrho^k) + \sum_{k > n} k^{-1} \varrho^k |v_k| \right) \\ &\ll \sum_{k=1}^{\infty} k^{-1} |v_k| \varrho^k (1 - \varrho^k). \end{aligned}$$

For any fixed  $\varepsilon > 0$ , the last expression is bounded by

$$\begin{aligned} &\ll \sum_{k \leq \varepsilon/(1-\varrho)} k^{-1} (1 - \varrho^k) + \left( \sum_{k > \varepsilon/(1-\varrho)} k^{-1} v_k^2 \sum_{k > \varepsilon/(1-\varrho)} k^{-1} \varrho^{2k} \right)^{1/2} \\ &\ll \varepsilon + \left( \varepsilon^{-1} \sum_{k > \varepsilon/(1-\varrho)} k^{-1} v_k^2 \right)^{1/2} = \varepsilon + o(1) \end{aligned}$$

as  $\varrho \rightarrow 1-$ , where we have used the estimate

$$\sum_{k > \varepsilon/(1-\varrho)} k^{-1} \varrho^{2k} \leq (1 - \varrho) \varepsilon^{-1} \sum_{k=1}^{\infty} \varrho^{2k} \leq \varepsilon^{-1}.$$

Since  $\varepsilon$  can be taken arbitrarily small, this establishes (6.7).

We are now in a position to complete the proof of (6.2). By Lemma 1 we have

$$(6.8) \quad (1 - \varrho)a(\varrho) = r_0(1 - \varrho)e^{C(\varrho)} + (1 - \varrho) \sum_{k \geq 1} r_k I_k(\varrho),$$

where

$$I_k(\varrho) = \int_0^{\varrho} e^{C(\varrho) - C(t)} k t^{k-1} dt.$$

With  $\Lambda$  and  $\Lambda(t)$  defined by (2.8) (with  $\vartheta = 0$ ), we have for  $0 \leq t < 1$

$$C(t) = \sum_{k \geq 1} \frac{c_k t^k}{k} = -\log(1 - t) - \Lambda(t).$$

Taking into account that  $V(\varrho) = -\text{Im } \Lambda(\varrho)$  and  $\lim_{\varrho \rightarrow 1-} \text{Re } \Lambda(\varrho) = \Lambda$ , which follows from (2.3), we obtain

$$(6.9) \quad (1 - \varrho)e^{C(\varrho)} = e^{-\Lambda(\varrho)} = e^{-\Lambda + iV(\varrho)} + o(1) = e^{iV(\varrho)} \lambda_0 + o(1) \quad (\varrho \rightarrow 1-),$$

and, for every fixed  $k \geq 1$ ,

$$\begin{aligned}
 (1 - \varrho)I_k(\varrho) &= e^{-\Lambda(\varrho)} \int_0^\varrho e^{\Lambda(t)} k(1-t)t^{k-1} dt \\
 (6.10) \qquad &= e^{-\Lambda+iV(\varrho)} \int_0^1 e^{\Lambda(t)} k(1-t)t^{k-1} dt + o(1) \\
 &= e^{iV(\varrho)} \lambda_k + o(1) \quad (\varrho \rightarrow 1-),
 \end{aligned}$$

where  $\lambda_k$  is defined as in the theorem. Since

$$\begin{aligned}
 (6.11) \qquad (1 - \varrho)|I_k(\varrho)| &\leq \int_0^\varrho e^{\operatorname{Re}(\Lambda(t)-\Lambda(\varrho))} k(1-t)t^{k-1} dt \\
 &\leq \int_0^1 k(1-t)t^{k-1} dt = \frac{1}{k+1},
 \end{aligned}$$

and, by (2.2),  $\sum_{k \geq 0} |r_k|/(k+1) < \infty$ , the series in (6.8) converges uniformly for  $0 < \varrho < 1$ . By (6.9) and (6.10) it follows that

$$(1 - \varrho)a(\varrho) = e^{iV(\varrho)} \sum_{k \geq 0} r_k \lambda_k + o(1) = e^{iV(\varrho)} a + o(1) \quad (\varrho \rightarrow 1-).$$

Combining this relation with (6.7), we obtain (6.2) as required. This completes the proof of (6.1).

## § 7. Proof of the corollaries.

*Proof of Corollary 1.* If  $a_n$  converges to a non-zero limit, then Theorem 1 implies that the series (2.3) converges for some real number  $\vartheta$ , and without loss of generality we may assume that  $0 \leq \vartheta < 1$ . Furthermore, the relation (2.4) gives

$$1 = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} e(\varphi(n+1) - \varphi(n)) = e(-\vartheta),$$

by (2.9). This implies  $\vartheta = 0$  and the convergence of the series  $\sum_{k \geq 1} \operatorname{Re}(1 - c_k)/k$ . A further application of (2.4) then yields the existence of  $\lim_{n \rightarrow \infty} \varphi(n)$  and thus the convergence of the series  $\sum_{k \geq 1} \operatorname{Im}(1 - c_k)/k$ . Hence the series  $\sum_{k \geq 1} (1 - c_k)/k$  converges, as claimed.

Conversely, if the latter series converges, then so does (2.3) with  $\vartheta = 0$  and the limit  $\varphi = \lim_{n \rightarrow \infty} \varphi(n)$  exists. Relation (2.4) then implies that  $a_n$  converges to  $ae(\varphi)$  with  $a = \sum_{k \geq 0} r_k \lambda_k$ . Since, by hypothesis,  $r_0 \neq 0$  and  $r_k = 0$  for  $k \geq 1$ , the limit is non-zero.

*Proof of Corollary 2.* In case (ii) of Theorem 1 the result is trivially valid. Suppose therefore that the series (2.3) converges for some real number  $\vartheta \in [0, 1)$ . We shall show below that, in the case the coefficients  $c_n$  are real and satisfy (2.1), this can only happen for  $\vartheta = 0$  or  $\vartheta = \frac{1}{2}$ . The function  $\varphi(n)$  is therefore either identically 0 or equal to  $-n/2$ . In the first case, (2.4) yields  $a_n = a + o(1)$ , and in second case  $a_n = a(-1)^n + o(1)$  as  $n \rightarrow \infty$ . The conclusion of the corollary then follows.

To prove the above claim, we note that by the hypothesis  $|c_k| \leq 1$  we have for any  $x \geq 2$

$$\begin{aligned}
 \sum_{k \leq x} \frac{\operatorname{Re}(1 - c_k e(k\vartheta))}{k} &\geq \sum_{k \leq x} \frac{1 - |\cos(2\pi k\vartheta)|}{k} \\
 &\geq \sum_{k \leq x} \frac{1}{k} - \sqrt{\sum_{k \leq x} \frac{\cos^2(2\pi k\vartheta)}{k} \sum_{k \leq x} \frac{1}{k}} \\
 (7.1) \qquad &= \left(1 - \sqrt{\frac{1}{\log x} \sum_{k \leq x} \frac{\cos^2(2\pi k\vartheta)}{k}}\right) \log x + O(1) \\
 &= \left(1 - \sqrt{\frac{1}{2} + \frac{\log \min(x, 1/\|2\vartheta\|)}{2 \log x}}\right) \log x + O(1),
 \end{aligned}$$

where in the last step we have used the identity  $\cos^2(2\pi k\vartheta) = \frac{1}{2}(1 + \cos(4\pi k\vartheta))$  and the relation

$$(7.2) \qquad \sum_{k \leq x} \frac{\cos(2\pi k\alpha)}{k} = \log \min(x, 1/\|\alpha\|) + O(1),$$

which holds uniformly for all real numbers  $\alpha$  and  $x \geq 1$ . This relation is easily proved by showing that, with  $x_1 := \min(x, 1/\|\alpha\|)$  the sum over the range  $k \leq x_1$  is, up to an error term  $O(1)$ , equal to  $\sum_{k \leq x_1} 1/k$ , whereas the sum over the remaining range is uniformly bounded.

*Proof of Corollary 3.* We have to show that the condition (2.19) implies (2.15) with  $C = \max(C_+, \frac{1}{2}(C_+ - C_-))$  and a constant  $K$  depending on  $C$  and  $k_0$ . In the case  $C_+ \geq -C_-$  we have  $C = C_+$  and (2.19) implies  $|c_k| \leq C$  for  $k \geq k_0$ , so that (2.15) holds trivially. Suppose therefore that  $C_+ < -C_-$  and set  $C_0 := \frac{1}{2}(C_+ + C_-)$ . We then have  $C_0 < 0$  and  $C = C_+ - C_0$ , so the inequality in (2.19) is equivalent to  $|c_k - C_0| \leq C$ . Hence the left-hand side of (2.15) does not exceed

$$C \sum_{y < k \leq x} \frac{1}{k} + C_0 \sum_{y < k \leq x} \frac{\cos(2\pi k\vartheta)}{k} + O_{C, k_0}(1).$$

By (7.2) the last sum is bounded from below by an absolute constant, and since  $C_0$  is negative, we obtain the required estimate (2.15).

*Proof of Corollary 4.* This result is a consequence of Theorem 4 and (2.26), and it only remains to prove the latter estimate. Thus we have to show that for any real coefficients  $c_k$  of modulus  $\leq 1$  the function  $T(x, \vartheta) := \operatorname{Re} \sum_{k \leq x} (1 - c_k e(k\vartheta))/k$  satisfies

$$(7.3) \qquad T(x, \vartheta) \geq \frac{1}{5} \min(T(x, 0), T(x, \frac{1}{2})) + O(1) \quad (x \geq 1, \vartheta \in \mathbf{R})$$

We fix  $x$  and  $\vartheta$  and set

$$L := \log x, \quad \mu := \frac{\log \min(1/\|2\vartheta\|, x)}{\log x}.$$

By (7.1) we have

$$(7.3) \qquad T(x, \vartheta) \geq \left(1 - \sqrt{(1 + \mu)/2}\right) L + O(1).$$

On the other hand, using (7.2) we obtain

$$\begin{aligned}
(7.4) \quad T(x, \vartheta) &= \sum_{k \leq x} k^{-1}(1 - c_k) + \sum_{k \leq x} c_k k^{-1}(1 - \cos(2\pi k\vartheta)) \\
&\geq T(x, 0) - \sum_{k \leq x} k^{-1}(1 - \cos(2\pi k\vartheta)) \\
&\geq T(x, 0) - \log x + \log \min(x, 1/\|\vartheta\|) + O(1)
\end{aligned}$$

and

$$\begin{aligned}
(7.5) \quad T(x, \vartheta) &= \sum_{k \leq x} k^{-1}(1 - c_k(-1)^k) + \sum_{k \leq x} k^{-1}c_k((-1)^k - \cos(2\pi k\vartheta)) \\
&\geq T(x, \frac{1}{2}) - \sum_{k \leq x} k^{-1}(1 - \cos(2\pi k(\vartheta + \frac{1}{2}))) \\
&\geq T(x, \frac{1}{2}) - \log x + \log \min(x, 1/\|\vartheta + \frac{1}{2}\|) + O(1).
\end{aligned}$$

Now set  $T_0(x) := \min(T(x, 0), T(x, \frac{1}{2}))$ , so that (2.26) may be written as

$$(7.6) \quad T(x, \vartheta) \geq \frac{1}{5}T_0(x).$$

Since  $\|2\vartheta\| \gg \min(\|\vartheta\|, \|\vartheta + \frac{1}{2}\|)$ , we infer from (7.4) and (7.5) that

$$(7.7) \quad T(x, \vartheta) \geq T_0(x) - (1 - \mu) \log x + O(1).$$

This implies (7.6) if  $(1 - \mu)L \leq \frac{4}{5}T_0(x)$ . In the remaining case we have  $\mu < 1 - \frac{4}{5}T_0(x)/L$ , and applying (7.3) we obtain

$$T(x, \vartheta) \geq \left(1 - \sqrt{1 - \frac{2}{5}T_0(x)/L}\right) L + O(1) \geq \frac{1}{5}T_0(x) + O(1),$$

which again gives (7.4).

### § 8. Proof of Theorem 5.

We fix a function  $R(n)$  satisfying the hypotheses of the theorem and a sequence  $\{a_n\}_{n=1}^{\infty}$  of nonnegative numbers satisfying (2.32). The constants implied in the estimates of this section may depend on these data.

We normalize  $a_n$  by setting  $a_n = 1 + b_n$ . By the nonnegativity of  $a_n$  and (2.32) we have

$$(8.1) \quad |b_n| \leq 1 + O\left(\frac{R(n)}{n}\right),$$

and, in particular,

$$(8.1)' \quad |b_n| \leq 1 + o(1) \quad (n \rightarrow \infty).$$

Noting that, by the hypotheses on  $R(n)$ ,

$$(8.2) \quad R(n) \geq R(1) > 0 \quad (n \geq 1),$$

we can rewrite equation (2.32) as

$$(8.3) \quad b_n = \frac{1}{n} \sum_{k \leq n-1} (-2 - b_k)b_{n-k} + O\left(\frac{R(n)}{n}\right).$$

We need to show that either

$$(8.4) \quad b_n = O\left(\frac{R(n)}{n}\right)$$

or

$$(8.5) \quad b_n = (-1)^{n+1} + O\left(\frac{R(n)}{n}\right)$$

holds. As a first step we shall show, using Theorem 3 and Corollary 3, that  $b_n$  satisfies either

$$(8.4)' \quad b_n = o(1) \quad (n \rightarrow \infty)$$

or

$$(8.5)' \quad b_n = (-1)^{n+1} + o(1) \quad (n \rightarrow \infty).$$

A relatively simple elementary argument will then allow us to sharpen (8.4)' and (8.5)' to (8.4) and (8.5), respectively.

We begin with a lemma which gives an estimate for the summatory function  $B_n := \sum_{k \leq n} b_k$ . This is in fact a consequence of a result of Erdős [Er]. However, since the latter result is proved in [Er] only in a special case corresponding to  $R(n) \equiv 1$ , we shall give here an independent proof based on Theorem 3.

**Lemma 4.** *We have  $B_n = o(n)$  as  $n \rightarrow \infty$ . Moreover, if*

$$(8.6) \quad \limsup_{n \rightarrow \infty} |b_n| < 1,$$

then we have  $B_n = O(R(n))$ .

*Proof.* From (8.3) we obtain

$$nB_n - \sum_{m \leq n-1} B_m = \sum_{m \leq n} mb_m = \sum_{k \leq n-1} (-2 - b_k)B_{n-k} + O\left(\sum_{m \leq n} R(m)\right).$$

Since  $R(m)$  is nondecreasing, it follows that

$$(8.7) \quad B_n = \frac{1}{n} \sum_{k \leq n-1} (-1 - b_k)B_{n-k} + O(R(n))$$

holds for all  $n \geq 1$ , with the convention that  $B_0 = 0$ . This equation is of the form (1.1) with  $a_n = B_n$ ,  $c_k = -1 - b_k$  and  $|r_k| \ll R(k)$ . By (8.1)', the hypothesis (2.19) of Corollary 3 is satisfied, for any fixed  $\varepsilon > 0$ , with  $C_- = -2 - \varepsilon$ ,  $C_+ = \varepsilon$ ,  $C = (C_+ - C_-)/2 = 1 + \varepsilon$ , with a suitable constant  $k_0 = k_0(\varepsilon)$ . Since  $|r_n| \ll R(n) = o(n)$  as  $n \rightarrow \infty$ , we obtain, on taking  $\varepsilon < 1$ ,

$$\begin{aligned} |B_n| &\ll R(n) + n^\varepsilon \sum_{k \leq n-1} \frac{R(k)}{k^{1+\varepsilon}} \\ &= o(n) + o\left(n^\varepsilon \sum_{k \leq n-1} k^{-\varepsilon}\right) = o(n) \quad (n \rightarrow +\infty), \end{aligned}$$

which proves the first assertion of the lemma. If (8.6) holds, then (2.19) holds with some constant  $C = (C_+ - C_-)/2 < 1$ , and we obtain

$$\begin{aligned} |B_n| &\ll R(n) + n^{C-1} \sum_{k \leq n-1} \frac{R(k)}{k^C} \\ &\leq R(n) + R(n)n^{C-1} \sum_{k \leq n-1} k^{-C} \ll R(n), \end{aligned}$$

using the monotonicity of  $R(n)$ . This proves the second assertion of the lemma.

**Lemma 5.** Let  $\{a_n\}_{n=0}^\infty$ ,  $\{c_n\}_{n=1}^\infty$  and  $\{r_n\}_{n=0}^\infty$  denote real sequences satisfying (1.1), (1.2) and  $|c_n| \leq K + o(1)$  with some constant  $K < \sqrt{2}$ . Suppose further that the following conditions hold :

- (i)  $r_n \rightarrow 0 \quad (n \rightarrow \infty)$ ,
- (ii)  $\limsup_{n \rightarrow \infty} n^{-1} \sum_{k < n} c_k < 1$ ,
- (iii)  $\limsup_{n \rightarrow \infty} n^{-1} \sum_{k < n} (-1)^k c_k < 1$ .

Then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* We show that the sequence  $\{c_k\}$  must satisfy condition (2.15), i.e.

$$(8.8) \quad \sup_{\vartheta \in \mathbf{R}} \sum_{y < k \leq x} \frac{c_k \cos(2\pi k\vartheta)}{k} \leq C \log(x/y) + O(1) \quad (1 \leq y \leq x),$$

with some constant  $C < 1$ . By Theorem 2, this implies

$$|a_n| \ll |r_n| + n^{C-1} \sum_{k \leq n-1} \frac{|r_k|}{k^C} = o(1) \quad (n \rightarrow \infty),$$

i.e. the conclusion of the lemma.

To prove (8.8), we fix a real number  $\vartheta$  and set  $x_0 := 1/\|2\vartheta\|$ , with the convention that  $x_0 = \infty$  if  $\vartheta \in \frac{1}{2}\mathbf{Z}$ . It clearly suffices to prove the desired inequality for the two cases  $1 \leq y \leq x \leq x_0$  and  $x \geq y \geq x_0$ .

Suppose first that  $1 \leq y \leq x \leq x_0$ . By the definition of  $x_0$  we have  $\vartheta = \frac{1}{2}(n + 1/x_0)$  for some integer  $n$ , so that  $e(\vartheta) = \varepsilon + O(1/x_0)$  with  $\varepsilon = \varepsilon(\vartheta) = \pm 1$ . We then have

$$\sum_{y < k \leq x} \frac{c_k \cos(2\pi k\vartheta)}{k} = \sum_{y < k \leq x} \frac{c_k \varepsilon^k + O(k/x_0)}{k} = \sum_{y < k \leq x} \frac{c_k \varepsilon^k}{k} + O(1).$$

By (ii) or (iii) — according to the value of  $\varepsilon$  — and partial summation, this does not exceed  $C \log(x/y) + O(1)$  for some positive constant  $C < 1$  and hence satisfies (8.8).

If  $x \geq y \geq x_0$ , we use the bound  $c_k \cos(2\pi k\vartheta) \leq K |\cos(2\pi k\vartheta)| + o(1)$  and apply Cauchy's inequality and (7.2) as above — cf. (7.1) — to obtain for any positive  $\eta$

$$\begin{aligned} \sum_{y < k \leq x} \frac{c_k \cos(2\pi k\vartheta)}{k} &\leq (K + \eta) \sum_{y < k \leq x} \frac{|\cos(2\pi k\vartheta)|}{k} + O_\eta(1) \\ &\leq \frac{(K + \eta)}{\sqrt{2}} \log(x/y) + O_\eta(1), \end{aligned}$$

which again implies (8.8) for suitable  $\eta$ . This completes the proof of Lemma 5.

We now embark on the proof of (8.4)' or (8.5)'. By Lemma 4, relation (8.3) reduces to

$$(8.9) \quad b_n = -\frac{1}{n} \sum_{k \leq n-1} b_k b_{n-k} + O\left(\frac{R(n)}{n}\right),$$

if (8.6) holds, and in any case implies

$$(8.9)' \quad b_n = -\frac{1}{n} \sum_{k \leq n-1} b_k b_{n-k} + o(1) \quad (n \rightarrow \infty).$$

We will show that the latter relation, together with (8.1)', implies either (8.4)' or (8.5)'.

We first prove that the conclusion certainly holds in the form (8.4)' if we have

$$(8.10) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{k \leq x} |b_k|^2 < 1.$$

Setting  $f_n := n^{-1} \sum_{k \leq n} |b_k|^2$ , we obtain from (8.9)' by Cauchy's inequality

$$(8.11) \quad |b_n| \leq f_{n-1} + o(1) \quad (n \rightarrow \infty).$$

It follows that, for any fixed  $\varepsilon > 0$  and a suitable number  $n_0(\varepsilon)$ ,

$$f_n = \left(1 - \frac{1}{n}\right) f_{n-1} + \frac{1}{n} |b_n|^2 \leq \left(1 - \frac{1 - f_{n-1}}{n}\right) f_{n-1} + \frac{\varepsilon}{n} \quad (n \geq n_0(\varepsilon)).$$

Choosing  $\varepsilon \in (0, \frac{1}{4})$  and  $n_0 \geq n_0(\varepsilon)$  such that  $f_{n_0-1} \leq 1 - 2\varepsilon$  (which is possible in view of (8.10), provided  $\varepsilon$  is sufficiently small), we deduce

$$f_n \leq \left(1 - \frac{2\varepsilon}{n}\right) f_{n-1} + \frac{\varepsilon}{n} \leq 1 - 2\varepsilon$$

for  $n = n_0$ . By induction, we obtain the same inequality for all  $n \geq n_0$ , whence  $\limsup_{n \rightarrow \infty} f_n < 1$ . A further application of (8.11) then gives

$$\limsup_{n \rightarrow \infty} f_n \leq \limsup_{n \rightarrow \infty} |b_n|^2 \leq \limsup_{n \rightarrow \infty} f_n^2 < 1,$$

which implies (8.4)'.

Next, we note that (8.4)' also holds if we have

$$(8.12) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} (-1)^{k+1} b_k < 1.$$

Indeed, in view of (8.9)', we can in this case apply Lemma 5 with  $a_n = b_n$  and  $c_n = -b_n$ : the condition on  $K$  follows from (8.1)', assumption (i) from (8.9)', assumption (ii) from Lemma 4, and assumption (iii) is exactly (8.12).

It remains to deal with the case when neither (8.10) nor (8.12) are fulfilled. Setting  $b_n = (-1)^{n+1}(1 - \beta_n)$ , we have by (8.1)

$$(8.13) \quad O\left(\frac{R(n)}{n}\right) \leq \beta_n \leq 2 + O\left(\frac{R(n)}{n}\right),$$

and in particular

$$(8.13)' \quad o(1) \leq \beta_n \leq 2 + o(1) \quad (n \rightarrow \infty).$$

The hypothesis that (8.10) and (8.12) are not satisfied then amounts to assuming

$$(8.14) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{k \leq x} \min(|\beta_k|, 2 - |\beta_k|) = 0$$

and

$$(8.15) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{k \leq x} |\beta_k| = 0.$$

We shall show that under these conditions we have

$$(8.16) \quad \beta_n = o(1) \quad (n \rightarrow \infty),$$

which is equivalent to (8.5)'.

Rewriting (8.9)' in terms of  $\beta_k$ , we obtain, by a straightforward computation,

$$\beta_n = \frac{1}{n} \sum_{k \leq n-1} (2 - \beta_k) \beta_{n-k} + o(1) \quad (n \rightarrow +\infty).$$

By (8.13)' this implies

$$(8.17) \quad |\beta_n| \leq \frac{2}{n} \sum_{k \leq n} |\beta_k| + o(1) \quad (n \rightarrow \infty).$$

Let  $g_x := x^{-1} \sum_{k \leq x} |\beta_k|$  and set  $\lambda := 11/10$ . By (8.17) and (8.13)' we have, for sufficiently large  $x$  and  $x \leq m \leq \lambda x$ ,

$$|\beta_m| \leq 2g_m + 1/11 \leq 2g_x + (2/x) \sum_{x < k \leq \lambda x} |\beta_k| + 1/11 \leq 2g_x + 3/5.$$

If now

$$(8.18) \quad g_x \leq 1/10,$$

then it follows that  $|\beta_m| \leq 1$  holds for each  $m$  in the interval  $[x, \lambda x]$ . Therefore

$$\sum_{x \leq m \leq \lambda x} |\beta_m| = \sum_{x \leq m \leq \lambda x} \min(|\beta_m|, 2 - |\beta_m|),$$

which by (8.14) is  $\leq x/100$ , provided  $x$  is sufficiently large. This in turn implies

$$g_{\lambda x} = \frac{1}{\lambda} g_x + \frac{1}{\lambda x} \sum_{x < m \leq \lambda x} |\beta_m| \leq \frac{1}{10\lambda} + \frac{1}{100\lambda} = \frac{1}{10},$$

and hence (8.18) with  $x$  replaced by  $\lambda x$ . Since by (8.15),  $\liminf_{x \rightarrow \infty} g_x = 0$ , we obtain by induction a sequence  $\{x_n\}_{n=0}^{\infty}$  of the form  $x_n = \lambda^n x_0$ , such that (8.18) holds for each  $x_n$ . Since

$$\begin{aligned} \max_{x_n \leq x \leq x_{n+1}} g_x &\leq g_{x_n} + \frac{1}{x_n} \sum_{x_n < k \leq x_{n+1}} |\beta_k| \leq g_{x_n} + \frac{2(x_{n+1} - x_n)}{x_n} + o(1) \\ &\leq \frac{1}{10} + 2(\lambda - 1) + o(1) = \frac{3}{10} + o(1) \quad (n \rightarrow \infty) \end{aligned}$$

it follows that  $\limsup_{x \rightarrow \infty} g_x \leq 3/10$ . By (8.17) this implies  $\limsup_{k \rightarrow \infty} |\beta_k| \leq 3/5$ . Using (8.14) as before, we deduce from this that  $\lim_{x \rightarrow \infty} g_x = 0$ , and hence obtain (8.16), by another application of (8.17).

We have now shown that either (8.4)' or (8.5)' holds, and it remains to sharpen these asymptotic relations to the quantitative estimates (8.4) or (8.5) respectively.

Assume first that (8.4)' is satisfied. We can then apply the second part of Lemma 4 and obtain (8.9). Define  $\lambda_n$  by  $b_n = \lambda_n R(n)/n$ , and set  $\lambda_n^* := \max_{k \leq n} |\lambda_k|$ . To prove (8.4), we need to show that  $\lambda_n^*$  is bounded. From (8.9) and the monotonicity of  $R(n)$  we get

$$\begin{aligned} |\lambda_n| &\leq \frac{2}{R(n)} \sum_{k \leq n/2} |b_k b_{n-k}| + O(1) \\ &\leq \frac{2}{R(n)} \sum_{k \leq n/2} |b_k| \lambda_n^* \frac{R(n-k)}{n-k} + O(1) \\ &\leq \lambda_n^* \frac{4}{n} \sum_{k \leq n/2} |b_k| + O(1). \end{aligned}$$

By (8.4)' the last sum is of order  $o(n)$ , and we obtain

$$|\lambda_n| = o(\lambda_n^*) + O(1) \quad (n \rightarrow \infty).$$

This implies  $\lambda_n^* = O(1)$  and hence (8.4).

Suppose now that (8.5)' holds. Defining  $\beta_n$  as above by  $b_n = (-1)^{n+1}(1 - \beta_n)$ , we have  $\beta_n = o(1)$ , and we obtain from (8.3)

$$(8.19) \quad \beta_n = \frac{1}{n} \sum_{k \leq n-1} \beta_k \left( 2 + (-1)^{n-k} - \beta_{n-k} \right) + O\left(\frac{R(n)}{n}\right).$$

To eliminate the term  $(-1)^{n-k}$  we replace  $n$  by  $n+1$  and add the resulting equation to (8.19). Setting  $\gamma_n = \beta_n + \beta_{n+1}$ , we then obtain, after some simplification,

$$(8.20) \quad \begin{aligned} \gamma_n &= \frac{1}{n} \sum_{k \leq n-1} \beta_k (4 - \gamma_{n-k}) + O\left(\frac{R(n) + R(n+1)}{n}\right) \\ &= \frac{2}{n} \sum_{k \leq n} \gamma_k + r_n, \end{aligned}$$

where

$$(8.21) \quad |r_n| \ll \frac{R(n) + R(n+1)}{n} + \frac{1}{n} \sum_{k \leq n} |\beta_k| (|\beta_{n-k}| + |\beta_{n+1-k}|).$$

In this form, (8.20) together with the initial condition  $\gamma_0 = r_0$  has the closed form solution

$$(8.22) \quad \gamma_n = r_n + 2(n+1) \sum_{k=0}^{n-1} \frac{r_k}{(k+1)(k+2)},$$

given by (2.17) with  $C = 2$ . (This may be easily verified by showing that the difference  $(n+1)\gamma_{n+1} - n\gamma_n$  computed via (8.20) is equal to the corresponding expression computed by means of (8.22).) Since  $\gamma_n$  and  $r_n$  are bounded as  $n \rightarrow \infty$ , the series on the right of (8.22), when extended to infinity, must have sum zero. It follows that

$$(8.23) \quad \left| \sum_{k=0}^{n-1} \frac{r_k}{(k+1)(k+2)} \right| = \left| \sum_{k \geq n} \frac{r_k}{(k+1)(k+2)} \right| \leq r_n^* \sum_{k \geq n} \frac{1}{(k+1)(k+2)} = \frac{r_n^*}{n+1},$$

where  $r_n^* := \sup_{k \geq n} |r_k|$ . Now set  $\beta_n^* := \sup_{k \geq n} |\beta_k|$ . By (8.21), (8.16) and the monotonicity of  $R(n)/n$  we have

$$|r_n| \ll \frac{R(n)}{n} + \frac{1}{n} \beta_{n/2}^* \sum_{k \leq (n+1)/2} |\beta_k| \ll \frac{R(n)}{n} + o(\beta_{n/2}^*) \quad (n \rightarrow \infty),$$

and hence

$$|r_n^*| \ll \frac{R(n)}{n} + o(\beta_{n/2}^*) \quad (n \rightarrow \infty).$$

Noting further that by (8.13)

$$\beta_n = \gamma_n - \beta_{n+1} \leq \gamma_n + O\left(\frac{R(n)}{n}\right)$$

we deduce from (8.22) and (8.23) that, for suitable positive constants  $C$  and  $n_0$ ,

$$|\beta_n| \leq C \frac{R(n)}{n} + \frac{1}{4} \beta_{n/2}^* \quad (n \geq n_0).$$

This implies

$$\begin{aligned} \beta_t^* = \max_{n \geq t} |\beta_n| &\leq C \max_{n \geq t} \frac{R(n)}{n} + \frac{1}{4} \max_{n \geq t} \beta_{n/2}^* \\ &\leq C \frac{R(t)}{t} + \frac{1}{4} \beta_{t/2}^* \quad (t \geq n_0), \end{aligned}$$

using again the monotonicity of  $R(n)/n$ . Iterating this inequality yields

$$(8.24) \quad \beta_t^* \leq C \sum_{k=0}^{k_0} \frac{R(t2^{-k})}{t2^{-k}4^k} + \frac{1}{4^{k_0+1}} \beta_{t2^{-k_0-1}}^*,$$

where  $k_0$  is defined by  $t2^{-k_0-1} < n_0 \leq t2^{-k_0}$ . The right-hand side of (8.24) is

$$\ll \frac{R(t)}{t} + 4^{-k_0} \ll \frac{R(t)}{t} + \frac{1}{t^2} \ll \frac{R(t)}{t},$$

by the monotonicity of  $R(n)$  and the bound  $R(t) \geq R(1) > 0$  for  $t \geq 1$ . This yields (8.5) and thus completes the proof of Theorem 5.

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A. Hildebrand  
 Department of Mathematics  
 University of Illinois  
 Urbana, IL 61801  
 USA

G. Tenenbaum  
 Université Nancy 1  
 Département de Mathématiques  
 BP 239  
 54506 Vandœuvre Cedex  
 France