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# Riesz and Valiron means and fractional moments

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## 1. *Introduction and results*

We shall be concerned here with two classical families of summability methods, and with links between them, together with applications in probability theory and elsewhere.

First, recall the Riesz (typical) means $R(\lambda_n, 1)$ of order 1 based on real $\lambda_n \uparrow \infty$: if $s_n = \sum_0^n a_k$, we write

$$s_n \to s \quad R(\lambda_n, 1)$$

if

$$\frac{1}{x}\int_0^x \{\sum_{\lambda_n \leqslant y} a_n\}\, dy \to s \quad (x \to \infty).$$

For these methods, and their applications to Dirichlet series and analytic number theory, see [**18**], [**12**] ([**17**], §§4·16, 5·16). We shall be concerned here with the case $\lambda_n = \exp(n^{1-\beta})$, $0 < \beta < 1$.

Secondly, recall the Valiron means $V_\beta$ of order $\beta$: for $0 < \beta < 1$, we write

$$s_n \to s \quad V_\beta$$

if

$$\frac{1}{x^\beta \sqrt{(2\pi)}} \sum_0^\infty s_k \exp\{-\tfrac{1}{2}(x-k)^2/x^{2\beta}\} \to s \quad (x \to \infty)$$

([**35**], ([**17**], §9·16). This notation is the standard one, but unfortunately conflicts with that of [**6**], where some relatives of $V_{\frac{1}{2}}$ are studied.) These methods are useful in the context of Fourier series; see [**20**], [**32**].

The results connecting these methods with each other and with convergence are as follows. We write '$f = O_L(g)$' for '$f/g$ is bounded below'.

THEOREM 1. *If* $0 < \beta < 1$, *and*

$$s_n \to s \quad R(\exp(n^{1-\beta}), 1),$$

*then*

$$s_n \to s \quad V_\beta.$$

*The converse holds under the Tauberian condition*

$$\text{(TC)} \qquad \lim_{h \downarrow 0} \liminf_{x \to \infty} \inf_{u \in [0,h]} \frac{1}{hx^\beta} \sum_{x \leqslant k < x + ux^\beta} s_k > -\infty$$

(*in particular, if* $s_n = O_L(1)$), *and* (TC) *is best-possible here.*

The method of proof here, which is based on that of [**34**], derives originally from that of Hardy and Littlewood in 1916 [**19**] (cf. Meyer-König [**30**]). While Wiener methods can be used in this area ([**36**]; cf. [**4**], [**9**]) it is not known whether Theorem 1 can be so obtained.

THEOREM 2. *If* $0 < \beta < 1$,

$$s_n \to s \quad R(\exp(n^{1-\beta}), 1), \quad \textit{or} \quad s_n \to s \quad V_\beta,$$

*and the Tauberian condition*

$$\lim_{\epsilon \downarrow 0} \ \liminf_{n\to\infty} \ \min_{n \leqslant k < n+\epsilon n^\beta} (s_k - s_n) \geqslant 0$$

*holds* (*in particular, if* $a_n = O_L(n^{-\beta})$), *then*

$$s_n \to s.$$

The Riesz means $R(\exp(n^{1-\beta}), 1)$ are related to Cesàro means. The result below (which, unlike Theorems 1 and 2, has no Tauberian character) relates them, on the one hand to Cesàro convergence at a given rate, and on the other to certain 'delayed averages'.

THEOREM 3. *For* $0 < \beta < 1$, *the following are equivalent*:

$$s_n \to s \quad R(\exp(n^{1-\beta}), 1),$$

$$\frac{1}{n+1}\sum_0^n (s_k + \epsilon_k) = s + o(1/n^{1-\beta}) \quad \textit{as } n \to \infty, \textit{for some } \epsilon_n \to 0,$$

$$\frac{1}{un^\beta}\sum_{n \leqslant k < n+un^\beta} s_k \to s \quad (n \to \infty) \quad \textit{for all } u > 0.$$

Theorem 3 may be extended to the limiting case $\beta = 1$. Here the second statement is ordinary Cesàro convergence, and the third is equivalent to this by a result of Agnew [**1**]. Since Cesàro convergence is just $R(n, 1)$ ([**17**], theorem 58), one may adopt the convention here that $R(\exp(n^{1-\beta}), 1)$ is to be read in the limiting case $\beta = 1$ as $R(n, 1)$. Better, one may use $R(\exp(\int_1^n x^{-\beta} dx), 1)$, which is $R(n, 1)$ if $\beta = 1$, and equivalent to $R(\exp(n^{1-\beta}), 1)$ if $0 < \beta < 1$. This follows from the 'second consistency theorem' for Riesz means, which says that convergence under $R(\lambda_n, 1)$ implies convergence to the same limit under $R(\mu_n, 1)$ if $\log \mu_n = O(\log \lambda_n)$, where $\lambda_n = \lambda(n)$, $\mu_n = \mu(n)$ and $\lambda, \mu$ are 'logarithmico-exponential' functions (Hardy [**16**]), or certain conditions on their derivatives hold (Hirst [**22**], Kuttner [**25**]).

Next, we note the result of varying $\beta$.

THEOREM 4. *If* $0 < \beta < \alpha \leqslant 1$,

$$s_n \to s \quad R(\exp(n^{1-\beta}), 1)$$

*implies*

$$s_n \to s \quad R\left(\exp\left(\int_1^n x^{-\alpha} dx\right), 1\right).$$

This follows from the second statement in Theorem 3, or alternatively from the second consistency theorem. Thus, our means $R(\exp(n^{1-\beta}), 1)$ are ordered by implication (or inclusion) in the direction of increasing $\beta$.

Next, we turn to certain probabilistic results which motivated this study. In what follows, $X, X_0, X_1, \ldots$ will denote independent and identically distributed random variables; we shall be interested in the interplay between moment conditions and rates of convergence. Recall [**17**] the following classical summability methods: $C_\alpha$, the Cesàro method of order $\alpha > 0$, $A$, the Abel method, $E_p$, the Euler method of parameter $p \in (0, 1)$, and $B$, the Borel method. One has

THEOREM L (Lai [28]). *The following are equivalent*:

$$E|X| < \infty, \quad EX = \mu,$$

$$X_n \to \mu \quad \text{a.s.} \quad C_\alpha \quad \textit{for some (all) } \alpha \geq 1,$$

$$X_n \to \mu \quad \text{a.s.} \quad A.$$

THEOREM C (Chow [13]). *The following are equivalent*:

$$\operatorname{var} X < \infty, \quad EX = \mu,$$

$$X_n \to \mu \quad \text{a.s.} \quad E_p \textit{ for some (all) } p \in (0, 1),$$

$$X_n \to \mu \quad \text{a.s.} \quad B,$$

$$X_n \to \mu \quad \text{a.s.} \quad R(e^{\sqrt{n}}, 1),$$

$$X_n \to \mu \quad \text{a.s.} \quad V_{\frac{1}{2}}.$$

For the first four statements here see Chow [13], and for the fifth see [5], theorem 3.

The case $1 < p \leq 2$ of Theorem 5 below provides a range of results intermediate between Theorems L and C:

THEOREM 5. *For $p > 1$, the following are equivalent*:

$$E(|X|^p) < \infty, \quad EX = \mu, \tag{1}$$

$$X_n \to \mu \quad \text{a.s.} \quad R(\exp(n^{1-(1/p)}), 1), \tag{2}$$

$$X_n \to \mu \quad \text{a.s.} \quad V_{1/p}. \tag{3}$$

As before, we can include the limiting case $p = 1$ of the equivalence of (1) and (2). Since existence of one moment implies existence of all lower ones, this gives a probabilistic interpretation to the ordering of Theorem 4.

Another rate-of-convergence complement to the laws of large numbers above is provided by the following result, in which we write $S_n = \sum_1^n X_k$.

THEOREM BK. *For $p \geq 1$, the following are equivalent*:

$$E(|X|^p) < \infty, \quad EX = \mu,$$

$$\textstyle\sum_1^\infty n^{p-2} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \quad \textit{for all } \epsilon > 0,$$

$$\textstyle\sum_1^\infty n^{p-2} P\left(\frac{1}{n} \max_{1 \leq j \leq n} |S_j - j\mu| > \epsilon\right) \quad \textit{for all } \epsilon > 0.$$

See Baum and Katz [2], theorem 3, Chow [13], theorem 2. The cases $p = 1, 2$ are due to Spitzer [33], theorem 4·2, Erdös [15].

## 2. *Proofs*

*Proof of Theorem* 1. That Cesàro-convergence at rate $o(1/n^{1-\beta})$ implies convergence under $V_\beta$ is due to Hyslop ([23], theorem 2). The Abelian part follows from this by Theorem 3 (below).

For the Tauberian part, note first that by the results of [9], $\{s_n\}$ satisfies (TC) if and only if it may be written $s_n = t_n + u_n$, where $t_n \geqslant 0$ and $u_n$ satisfies the third statement of Theorem 3. Using this decomposition, it suffices to restrict attention to the case $s_n \geqslant 0$. Write

$$U(x) := \sum_{0 \leqslant n \leqslant x} s_n$$

(with the convention that empty sums are zero); thus $U$ is non-decreasing and vanishes on $(-\infty, 0)$. The key step remaining is contained in the following result:

Lemma. *Let $\beta \in (0,1)$, $U$ be non-decreasing and vanishing on $(-\infty, 0)$. If*

$$\frac{1}{x^\beta \sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}h^2\}\, dU(x+hx^\beta) \to s \quad (x \to \infty), \tag{4}$$

*then*
$$\frac{1}{x^\beta}\sqrt{\left(\frac{a}{2\pi}\right)} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}ah^2\}\, dU(x+hx^\beta) \to s \quad (x \to \infty) \tag{5}$$

*for any $a \in (0,1)$.*

*Proof.* Write $b = (1-a)^{\frac{1}{2}}$. We begin with the identity

$$\exp\{-\tfrac{1}{2}ah^2\} = \exp\{-\tfrac{1}{2}h^2\} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}t^2 + bht\}\, dt.$$

Inserting this in (5), we obtain

$$\begin{aligned}
&x^{-\beta}\sqrt{\left(\frac{a}{2\pi}\right)} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}h^2\} \left(\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}t^2 + bht\}\, dt\right) dU(x+hx^\beta) \\
&\quad = x^{-\beta}\sqrt{\left(\frac{a}{2\pi}\right)} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}t^2\} \left(\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}h^2 + bht\}\, dU(x+hx^\beta)\right) dt \\
&\quad = \sqrt{\left(\frac{a}{2\pi}\right)} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}t^2\} \left(\frac{x^{-\beta}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}w^2 + \tfrac{1}{2}b^2t^2\}\, dU(x+(w+bt)x^\beta)\right) dt \\
&\quad = \sqrt{\left(\frac{a}{2\pi}\right)} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}at^2\} F(x,t)\, dt, \quad \text{say,}
\end{aligned}$$

where
$$F(x,t) = \frac{x^{-\beta}}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}w^2\}\, dU(x+(w+bt)x^\beta))$$

(as $dU$ is a positive measure, the inversion is justified by Fubini's theorem).

We will prove that, for each $t$,

$$F(x,t) \to s \quad (x \to \infty), \tag{6}$$

and that (using interchangeably the Vinogradov $\ll$ and Landau $O$ symbols)

$$F(x,t) \ll 1 + |t|^\beta \quad \text{uniformly in } x \geqslant 1. \tag{7}$$

From this and dominated convergence, we obtain the existence of the limit in (5), and the required value

$$\sqrt{\left(\frac{a}{2\pi}\right)} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}at^2\}\, s\, dt = s.$$

From (4), we have

$$U(x+x^\beta) - U(x) \ll \int_0^1 \exp\{-\tfrac{1}{2}h^2\}\, dU(x+hx^\beta) \ll x^\beta,$$

which in turn implies

$$U(x+y)-U(x) \ll y+x^{\beta} \quad (x,y \geqslant 1) \tag{8}$$

(cf. [**9**], theorem 9, or [**34**], (12)). Next, fix $t$, and write $z = x+btx^{\beta}$ for large $x$. Then

$$\begin{aligned} F(x,t) &\sim \frac{z^{-\beta}}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}w^2\}\, dU(z+wx^{\beta}) \\ &= \frac{z^{-\beta}}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\left(\frac{y-z}{z^{\beta}}\right)^2\left(\frac{z}{x}\right)^{2\beta}\right\} dU(y) \\ &= \frac{z^{-\beta}}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\left(\frac{y-z}{z^{\beta}}\right)^2(1+O(1/x^{1-\beta}))\right\} dU(y). \end{aligned}$$

We decompose the right into an 'inner' term, $F_1$, where $|y-z| \leqslant z^{(1+3\beta)/4}$, and the corresponding 'outer' one, $F_2$. In $F_1$, the integrand is

$$(1+O(x^{-\frac{1}{2}(1-\beta)})\exp\left\{-\tfrac{1}{2}\left(\frac{y-z}{z^{\beta}}\right)^2\right\} \quad \text{(take logarithms), so}$$

$$F_1 = (1+O(x^{-\frac{1}{2}(1-\beta)}))\frac{z^{-\beta}}{\sqrt{(2\pi)}}\int_{|y-z|\leqslant z^{(1+3)\beta/4}} \exp\left\{-\tfrac{1}{2}\left(\frac{y-z}{z^{\beta}}\right)^2\right\} dU(y).$$

Also

$$\left(\frac{y-z}{z^{\beta}}\right)^2 > z^{\frac{1}{2}(1-\beta)}$$

in the outer range, so the integrand is exponentially small, while by (8) $U(y) \ll 1+y$. Hence $F_2$ is exponentially small, and

$$F(x,t) \sim \frac{z^{-\beta}}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}\left(\frac{y-z}{z^{\beta}}\right)^2\right\} dU(y) \to s,$$

by (4), proving (6).

Finally,

$$F(x,t) \ll x^{-\beta}\sum_{k=-\infty}^{\infty} \exp\{-\tfrac{1}{2}(|k|-1)^2\}(U(x+(k+1+bt)x^{\beta})-U(x+(k+bt)x^{\beta})).$$

Using (8) with $x$ and $y$ replaced by $x+(k+bt)x^{\beta}$ and $x^{\beta}$,

$$\begin{aligned} F(x,t) &\ll x^{-\beta}\sum_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(|k|-1)^2\}(|x+(k+bt)x^{\beta}|^{\beta}+x^{\beta}) \\ &\ll \sum_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(|k|-1)^2\}(1+|k|^{\beta}x^{-\beta(1-\beta)}+|t|^{\beta}x^{-\beta(1-\beta)}) \\ &\ll 1+|t|^{\beta} \quad \text{uniformly for large } x, \end{aligned}$$

proving (7) and completing the proof of the lemma.

The remainder of the proof of Theorem 1 follows from the Lemma as in [**34**], 241–243 ((21) and below – where the argument is set out in detail for the case $\beta = \frac{1}{2}$), using Vitali's convergence theorem and a diagonalization argument.

To see that (TC) cannot be weakened, observe (using Theorem 3 and the decomposition $s_n = t_n + u_n$) that (TC) is implied by the desired conclusion

$$s_n \to s \quad (R(\exp(n^{1-\beta}), 1)).$$

*Proof of Theorem* 2. The 'Valiron' part is due to Hyslop([**23**], theorem 4). The 'Riesz' part is also classical; see Karamata [**24**], or [**4**], 228.

*Proof of Theorem* 3. This is proved in detail for the case $\beta = \frac{1}{2}$ in [**9**], and the general case follows by the same method.

*Proof of Theorem* 5. That (1) is equivalent to

$$\frac{1}{un^{1/p}} \sum_{n \leqslant k < n+un^{1/p}} X_k \to \mu \quad \text{a.s.} \quad (n \to \infty) \quad \text{for all } u > 0 \tag{9}$$

follows by theorem 1 of Chow [**13**]. The equivalence of (9) with (2) follows by Theorem 3. Since (9) includes (TC) with $\beta = 1/p$, the equivalence of (1), (2) and (3) follows by Theorem 1.

## 3. *Remarks*

1. That the Borel method $B$ and $V_{\frac{1}{2}}$ are closely linked has long been known [**19**]. The Borel method is a particular case of a power-series method ([**17**], §4·12, [**11**]). For links between power-series and Valiron methods more generally, see [**17**], §9·16, Valiron [**35**]. For other results in this area, see [**6**], [**7**].

2. By the Borel–Cantelli lemmas, $E(|X|^p) < \infty$ is equivalent to

$$X_n = o(n^{1/p}) \quad \text{a.s.} \tag{10}$$

This is the condition needed to pass from $V_{1/p}$ in (3) to the power-series method above ([**17**], §9·16).

3. It is natural to seek results of iterated-logarithm type in the contexts above, interpreted as rate-of-convergence complements to the law of large numbers. This programme has been carried out by Lai [**27**], who also obtained Strassen versions [**26**]; cf. [**28**], [**10**]. In the same spirit, central limit theorems in this context have been obtained by Embrechts and Maejima [**14**].

4. The context ($\{X_n\}$ independent and identically distributed) of the probabilistic Theorem 5 is much more restrictive than that of its analytic counterpart Theorem 1. It is natural to ask whether probabilistic analogues of Theorem 1 hold more generally. Suppose, for instance, that $\{X_n\}$ is stationary ergodic; one could ask how far independence may be weakened in Theorem 5, e.g. to an appropriate mixing condition. We will not pursue this question here, but note for comparison that versions of Theorem BK have been obtained under mixing conditions; see Berbee [**3**], Peligrad [**31**], Lai [**29**], Hipp [**21**].

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