

A remark on Sarnak's conjecture

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Abstract. We investigate Sarnak's conjecture on the Möbius function in the special case when the test function is the indicator of a level set of an additive function with restricted values at primes.

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1. Introduction and statements of results

According to a general pseudo-randomness principle related to a famous conjecture of Chowla [1] and recently considered by Sarnak [7], the Möbius function μ does not correlate with any function ξ of low complexity. In other words,

$$(1.1) \quad \sum_{n \leq x} \mu(n)\xi(n) = o\left(\sum_{n \leq x} |\xi(n)|\right) \quad (x \rightarrow \infty).$$

There are many ways of constructing functions of low complexity. Sarnak and others use return times of sampling sequences of a dynamical system, which leads to a natural measure of the complexity. Here we propose to follow another path by selecting the test-function as the indicator of a level set of a real additive function f . Since computing the value of such indicator at an integer n rests on the factorization of n , it is certainly not deterministic in the sense introduced by Sarnak. However, as we shall see, this setting provides a continuous measure of correlations as f departs from a constant multiple of the total number of prime factors function.

It is known since Halász [5] that, for any additive function f , we have

$$(1.2) \quad Q(x; f) := \sup_{m \in \mathbb{R}} \sum_{\substack{n \leq x \\ f(n)=m}} 1 \ll \frac{x}{\sqrt{1 + E(x)}}$$

where we have put

$$E(x) := \sum_{\substack{p \leq x \\ f(p) \neq 0}} \frac{1}{p}.$$

Here and in the sequel, the letter p denotes a prime number.

The estimate (1.2) is known to be optimal in this generality since the two sides achieve the same order of magnitude when $f(n)$ is equal to the total number of prime factors of n , counted with or without multiplicity.

As a first investigation of the above described problem, we would like to show that

$$Q(x; f, \mu) := \sup_{m \in \mathbb{R}} \left| \sum_{\substack{n \leq x \\ f(n)=m}} \mu(n) \right|$$

is generically smaller than the right-hand side of (1.2). Of course we have to avoid the case when $f(p)$ is constant, for then $\mu(n)$ does not oscillate on the set of squarefree integers n with $f(n) = m$. Therefore we seek an estimate which matches (1.2) when $f(p)$ is close to a constant and has smaller order of magnitude otherwise.

When $f(p)$ is restricted to assume the values 0 or 1 only, we thus expect a significant improvement over (1.2) when

$$(1.3) \quad F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p}$$

is large. Indeed, in this simple case we obtain the following estimate.

Theorem 1.1. *Let f denote a real additive arithmetic function such that $f(p) \in \{0, 1\}$ for all p . Then, with the above notation and $c = (2\pi - 4)/(3\pi - 2) \approx 0.30751$, we have*

$$(1.4) \quad Q(x; f, \mu) \ll \frac{x\{1 + F(x)\}e^{-cF(x)}}{\sqrt{1 + E(x)}}.$$

For simplicity, let us retain in the sequel the hypothesis $f(p) \in \{0, 1\}$.⁽¹⁾ Under the assumption that $F(x)$, as defined in (1.3) above, grows sufficiently slowly, we may prove an estimate that is valid for each m in a large range around the mean, and so may be stated in the exact frame of Sarnak's conjecture.

Let us denote by $N_m(x; f)$ the number of squarefree integers not exceeding x such that $f(n) = m$. It follows from results of Halász [3], [4], and Sárközy [6] that, given any $\kappa \in]0, 1[$, we have

$$(1.5) \quad N_m(x; f) \asymp x \frac{E(x)^m}{m!} e^{-E(x)} \quad (\kappa E(x) \leq m \leq E(x)/\kappa).$$

Moreover, Halász announced (see [2], p.312) the possibility to obtain, in the same range for m , an asymptotic formula for $N_m(x; f)$, a result which actually follows, as shown in [10], from a general effective mean value estimate for multiplicative functions established in the same work—see below.

This supports the hope to obtain an asymptotic formula for

$$N_m(x; f, \mu) := \sum_{\substack{n \leq x \\ f(n)=m}} \mu(n)$$

which directly compares to (1.5). In view of (1.1), we may assume with no loss of generality that f is strongly additive. We obtain the following result. Here and in the sequel we let \log_k denote the k -fold iterated logarithm.

Theorem 1.2. *Let $\kappa \in]0, 1[$ and let f denote a strongly additive function such that $f(p) \in \{0, 1\}$ for all primes p . Assume furthermore that*

$$(1.6) \quad F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p} \ll \log_3 x \quad (x \rightarrow \infty)$$

$$(1.7) \quad \sum_{\exp\{(\log x)/(\log_2 x)^D\} < p \leq y} \frac{\{1 - f(p)\} \log p}{p} \ll \frac{\log y}{(\log_2 x)^{c_0}} \quad (x^{1/(\log_2 x)^D} < y \leq x)$$

1. All our results could be straightforwardly adapted to the case when $f(p)$ is restricted to a fixed, finite set, or even to a set of moderate size depending on x .

where D and c_0 are positive constants. Provided D is sufficiently large and uniformly in the range $\kappa E(x) \leq m \leq E(x)/\kappa$, we have

$$(1.8) \quad N_m(x; f, \mu) = (-1)^m N_m(x; f) \left\{ \lambda_f e^{-2F(x)} + O\left(\frac{1}{(\log_2 x)^b}\right) \right\},$$

with

$$(1.9) \quad \lambda_f := \prod_{f(p)=0} \frac{1-1/p}{1+1/p} e^{2/p}, \quad b := \frac{1}{2} \min\{1, c_0 \kappa / (4 - \kappa)\}.$$

To fix ideas, note that a strongly additive function f such that $f(p) \in \{0, 1\}$ satisfies hypotheses (1.6) and (1.7) as soon as

$$\sum_{p \leq y} \{1 - f(p)\} \log p \ll \frac{y}{(\log_2 y)^{\max(1, c_0)}}.$$

The proof of Theorem 1.2 rests on the following recent result of the second author [10] (theorem 1.4), for the statement of which we introduce further notation. We let $\mathcal{M}(A, B)$ designate the class of those complex-valued multiplicative functions g such that

$$(1.10) \quad \max_p |g(p)| \leq A, \quad \sum_{p, \nu \geq 2} \frac{|g(p^\nu)| \log p^\nu}{p^\nu} \leq B,$$

and, for $\mathfrak{b} \in \mathbb{R}$, we write

$$(1.11) \quad \beta_0 = \beta_0(\mathfrak{b}, A) := 1 - \frac{\sin(2\pi \mathfrak{b}/A)}{2\pi \mathfrak{b}/A}.$$

Moreover, given a complex-valued function g , we put $w_g := 1$ if g is real, $w_g := \frac{1}{2}$ otherwise, and write

$$M(x; g) := \sum_{n \leq x} g(n), \quad Z(x; g) := \sum_{p \leq x} \frac{g(p)}{p}.$$

Theorem 1.3 ([10]). *Let*

$$\mathfrak{a} \in]0, \frac{1}{4}], \quad \mathfrak{b} \in [\mathfrak{a}, \frac{1}{2}[, \quad \mathfrak{h} := (1 - \mathfrak{b})/\mathfrak{b}, \quad A \geq 2\mathfrak{b}, \quad B > 0, \quad \beta := \beta_0(\mathfrak{b}, A), \\ x \geq 2, \quad 1/\sqrt{\log x} < \varepsilon \leq \frac{1}{2},$$

and let the multiplicative functions g, r , such that $r \in \mathcal{M}(x; 2A, B)$, $|g| \leq r$, satisfy the conditions

$$(1.12) \quad \sum_{p \leq x} \frac{r(p) - \Re g(p)}{p} \leq \frac{1}{2} \beta \mathfrak{b} \log(1/\varepsilon),$$

$$(1.13) \quad \sum_{x^\varepsilon < p \leq y} \frac{\{r(p) - \Re g(p)\}^\mathfrak{h} \log p}{p} \ll \varepsilon^{\delta \mathfrak{h}} \log y \quad (x^\varepsilon < y \leq x),$$

with $\delta \in]0, \frac{1}{3}\beta\mathfrak{b}]$, and

$$(1.14) \quad \min_{x^\varepsilon < p \leq x} r(p) \geq 4\mathfrak{b}.$$

We then have

$$(1.15) \quad M(x; g) = M(x; r) \prod_p \frac{\sum_{p^\nu \leq x} g(p^\nu)/p^\nu}{\sum_{p^\nu \leq x} r(p^\nu)/p^\nu} + O\left(\frac{x \varepsilon^{w_g \delta} e^{Z(x; r) - \mathfrak{c}Z(x; |g|-g)}}{\log x}\right)$$

where $\mathfrak{c} := \mathfrak{b}/A$. The implicit constant in (1.15) depends at most upon A, B, \mathfrak{a} , and \mathfrak{b} .

2. Proof of Theorem 1.1

Put

$$M(x; \vartheta) := \sum_{n \leq x} \mu(n) e^{2\pi i \vartheta f(n)}.$$

Since f assumes only integer values,⁽²⁾ we plainly have, for any $m \in \mathbb{Z}$,

$$\sum_{\substack{n \leq x \\ f(n)=m}} \mu(n) = \sum_{n \leq x} \mu(n) \int_{-1/2}^{1/2} e^{2\pi i (f(n)-m)\vartheta} d\vartheta = \int_{-1/2}^{1/2} M(x; \vartheta) e^{-2\pi i m \vartheta} d\vartheta,$$

so

$$Q(x; f, \mu) \leq \int_{-1/2}^{1/2} |M(x; \vartheta)| d\vartheta.$$

From Corollary III.4.12 in [8], we infer that, uniformly for $\vartheta \in \mathbb{R}$, $T \geq 1$, $x \geq 1$,

$$(2.1) \quad M(x; \vartheta) \ll \frac{x\{1 + m(x; \vartheta, T)\}}{e^{m(x; \vartheta, T)}} + \frac{x}{T},$$

where we have put

$$m(x; \vartheta, T) := \min_{|\tau| \leq T} \sum_{p \leq x} \frac{1 + \cos(2\pi \vartheta f(p) - \tau \log p)}{p}.$$

We select $T := \log x$, so that the second term on the right of (2.1) is negligible compared to the upper bound in (1.4). Let h_ϑ be defined by

$$h_\vartheta(t) := 1 + \min\{\cos(t), \cos(2\pi \vartheta - t)\} \quad (t \in \mathbb{R}),$$

so that

$$s_\vartheta := \frac{1}{2\pi} \int_{-\pi}^{\pi} h_\vartheta(t) dt = 1 - \frac{2}{\pi} |\sin(\pi \vartheta)| \quad (\vartheta \in [-\frac{1}{2}, \frac{1}{2}]),$$

and, for suitable $\tau \in [-T, T]$,

$$(2.2) \quad m(x; \vartheta, T) \geq \sum_{p \leq x} \frac{h_\vartheta(\tau \log p)}{p}.$$

We note that the constant c appearing in the statement of Theorem 1.1 coincides with the minimal value of $2s_\vartheta/(2 + s_\vartheta)$ as ϑ ranges over $[-\frac{1}{2}, \frac{1}{2}]$.

The right-hand side of (2.2) may be estimated via partial summation as made explicit in lemma III.4.13 of [8]. For any $w \in [2, x]$ and $\vartheta \in [-\frac{1}{2}, \frac{1}{2}]$, we have

$$(2.3) \quad \sum_{w < p \leq x} \frac{h_\vartheta(\tau \log p)}{p} = s_\vartheta \log \left(\frac{\log x}{\log w} \right) + O \left(\frac{1}{|\tau| \log w} + \frac{1 + |\tau|}{e^{\sqrt{\log w}}} \right).$$

If $1 \leq |\tau| \leq T$, we select $w := \exp(\log_2 x)^2$ to obtain

$$m(x; \vartheta, T) \geq s_\vartheta \log_2 x + O(\log_3 x).$$

2. Note, however, that Halász [5] gave a simple argument to show that, without loss of generality, one can assume f is integer-valued for the computation of $Q(x; f)$. In the case of $Q(x; f, \mu)$ a slight modification of his construction is needed to ensure that changing the range of f does not create new coincidences. This is not needed in the present context but could be of use when exploring more general situations.

Next, set

$$\log v := (\log x) \exp \left\{ - \frac{2 \cos^2(\pi\vartheta) E(x) + 2F(x)}{2 + s_\vartheta} \right\}.$$

If $1/\log v < |\tau| \leq 1$, we put $w := v$ in (2.3) and get

$$\sum_{v < p \leq x} \frac{h_\vartheta(\tau \log p)}{p} \geq \frac{2s_\vartheta \cos^2(\pi\vartheta)}{2 + s_\vartheta} E(x) + \frac{2s_\vartheta}{2 + s_\vartheta} F(x) + O(1).$$

And finally, if $|\tau| \leq 1/\log v$, we trivially have

$$\begin{aligned} \sum_{p \leq v} \frac{1 + \cos(2\pi\vartheta f(p) - \tau \log p)}{p} &= \sum_{p \leq v} \frac{1 + \cos(2\pi\vartheta f(p))}{p} + O(1) \\ &= (1 + \cos(2\pi\vartheta)) \sum_{\substack{p \leq v \\ f(p)=1}} \frac{1}{p} + 2 \sum_{\substack{p \leq v \\ f(p)=0}} \frac{1}{p} + O(1) \\ &\geq 2 \cos^2(\pi\vartheta) E(x) + 2F(x) - 2 \log \left(\frac{\log x}{\log v} \right) + O(1) \\ &\geq \frac{2s_\vartheta \cos^2(\pi\vartheta)}{2 + s_\vartheta} E(x) + \frac{2s_\vartheta}{2 + s_\vartheta} F(x) + O(1). \end{aligned}$$

Therefore, we get in all cases

$$(2.4) \quad \begin{aligned} m(x; \vartheta, T) &\geq \frac{2s_\vartheta \cos^2(\pi\vartheta)}{2 + s_\vartheta} E(x) + \frac{2s_\vartheta}{2 + s_\vartheta} F(x) + O(1) \\ &\geq c \cos^2(\pi\vartheta) E(x) + cF(x) + O(1). \end{aligned}$$

Integrating over ϑ immediately yields the result stated. \square

3. Proof of Theorem 1.2

Let us introduce the multiplicative function $g(n) := \mu(n) z^{f(n)}$ with $z := -\varrho e^{2\pi i \vartheta}$, $|\vartheta| \leq \frac{1}{2}$, $\kappa \leq \varrho \leq 1/\kappa$. Put $r(n) := \mu(n)^2 \varrho^{f(n)}$. From (2.4), we see that, with c as in the statement of Theorem 1.1,

$$\sum_{p \leq x} \frac{r(p) - \Re e(g(p)/p^{i\tau})}{p} \geq c\varrho \sin^2(\pi\vartheta) E(x) + c\varrho F(x) + O(1) \quad (|\tau| \leq T := \log x).$$

We may therefore apply Corollary 2.1 of [10] to get

$$(3.1) \quad M(x; g) \ll M(x; r) \left\{ e^{-c\varrho E(x) \sin^2(\pi\vartheta) - c\varrho F(x)} \log_2 x + \frac{1}{(\log x)^\kappa} \right\}.$$

With the aim of applying Cauchy's formula to detect $N_m(x; f, \mu)$, we next seek an estimate for $M(x; g)$ when ϑ is small, namely

$$|\vartheta| \leq \vartheta_0 := K \sqrt{\frac{\log_3 x}{\log_2 x}},$$

where K is a large constant—actually any $K > 1/\sqrt{4\kappa c}$ will do. We have

$$\sum_{p \leq x} \frac{r(p) - \Re e g(p)}{p} = \varrho(1 - \cos 2\pi\vartheta) E(x) + 2\varrho F(x) \leq 2\varrho\pi^2 \vartheta^2 + 2\varrho F(x),$$

hence condition (1.12) is plainly fulfilled with $\varepsilon := |\vartheta|^{2/\delta} + (\log_2 x)^{-c_0/(\mathfrak{h}\delta)}$ provided δ is chosen sufficiently small in terms of \mathfrak{b} , κ and K . Next, for $x^\varepsilon < y \leq x$, we have

$$\begin{aligned} \sum_{x^\varepsilon < p \leq y} \frac{\{r(p) - \Re e g(p)\}^{\mathfrak{h}} \log p}{p} &\ll \varrho^{\mathfrak{h}2} \log y + \varrho \sum_{\substack{x^\varepsilon < p \leq y \\ f(p)=0}} \frac{\log p}{p} \\ &\ll_{\kappa} \{\varepsilon^{\delta \mathfrak{h}} + (\log_2 x)^{-c_0}\} \log y, \end{aligned}$$

so hypothesis (1.13) is also verified. Since (1.14) holds trivially on selecting $\mathfrak{b} := \kappa/4$, and hence $\mathfrak{h} = 4/\kappa - 1$, we conclude that (1.15) is valid. We obtain, with $\mathfrak{c} := \kappa \mathfrak{b}$,

$$(3.2) \quad M(x; g) = M(x; r) \prod_{\substack{p \leq x \\ f(p)=1}} \frac{1 - z/p}{1 + \varrho/p} \prod_{\substack{p \leq x \\ f(p)=0}} \frac{1 - 1/p}{1 + 1/p} + O\left(\frac{x\varepsilon^{\delta/2} e^{Z(x;r) - \mathfrak{c}Z(x;r-g)}}{\log x}\right).$$

Now, appealing for instance to theorem 1.1 of [9], we observe that

$$M(x; r) \asymp \frac{x e^{Z(x;r)}}{\log x}$$

and so we may rewrite (3.2) as

$$M(x; g) = M(x; r) \left\{ \lambda_f e^{-(z+\varrho)E(x) - 2F(x)} + O\left(\left(|\vartheta| + (\log_2 x)^{-c_0 \mathfrak{h}/2}\right) e^{-c_1 \vartheta^2 E(x) - c_1 F(x)}\right) \right\},$$

valid for $|\vartheta| \leq \vartheta_0$ and some constant $c_1 > 0$. Integrating on the circle $|z| = \varrho := m/E(x)$ and taking (3.1) into account, we readily obtain, in the stated range for m ,

$$(3.3) \quad \begin{aligned} N_m(x; f, \mu) &= (-1)^m \int_{-1/2}^{1/2} e^{-2i\pi\vartheta m} \varrho^{-m} M(x; g) d\vartheta \\ &= (-1)^m \lambda_f M(x; r) \frac{E(x)^m}{m! e^m} \left\{ e^{-2F(x)} + O\left(\frac{e^{-c_2 F(x)}}{(\log_2 x)^{\mathfrak{b}}}\right) \right\}, \end{aligned}$$

with $c_2 := \min(c_1, c\kappa)$. Since, by a straightforward variant of corollary 2.4 of [10]⁽³⁾,

$$N_m(x; f) = M(x; r) \frac{E(x)^m}{m! e^m} \left\{ 1 + O\left(\frac{1}{\sqrt{\log_2 x}}\right) \right\},$$

we reach the required conclusion.

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