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# On partial derivatives of some summatory functions

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*To Helmut Maier,*
*as a faithful token*
*of a long-term companionship.*

**Abstract.** Let $f$ be a real arithmetic function and let $g : [1, \infty[\to \mathbb{R}$ be a smooth function. We describe two emblematic instances in which saddle-point estimates may be used to evaluate the frequency, on the set of integers $n \leqslant x$, of the event $\{f(n) \leqslant g(n)\}$ from those relevant to the event $\{f(n) \leqslant y\}$. The first example revisits Dickman's historical contribution to the theory of friable integers. The second is concerned with the distribution of the squarefree kernel of an integer.

**Keywords:** friable integers, largest prime factor, squarefree kernel, saddle-point method, local behaviour, semi-asymptotic formulae.

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## 1. Introduction and statements of results

The study of the distribution of an arithmetic function $f$ naturally leads to estimating the summatory function in two variables

$$\mathcal{F}(x, y) := \sum_{\substack{n \leqslant x \\ f(n) \leqslant y}} 1.$$

However, passing from an asymptotic formula for $\mathcal{F}(x, y)$ to an estimate for

$$\mathcal{H}(x; g) := \sum_{\substack{n \leqslant x \\ f(n) \leqslant g(n)}} 1, \tag{1·1}$$

may not be straightforward, even when $g$ is a smooth function. Formally, this task amounts to integrating the partial derivative $\partial\mathcal{F}(x, g(x))/\partial x$, but, due to inevitable remainder terms in estimates for $\mathcal{F}(x, y)$, this is usually out of reach as it stands.

An obvious path consists in approximating the partial derivative through a discretisation process. This may however turn out to be quite delicate according to available knowledge on the local behaviour of $\mathcal{F}(x, y)$ with respect to the first variable. A favourable case occurs when $\mathcal{F}(x, y)$ has been evaluated by the saddle-point method, which generically provides estimates for the local behaviour—what Erdős used to call "semi-asymptotic formulae". This note is devoted to describe two instances of this situation. We shall see that this approach greatly simplifies the computations.

The first problem under consideration is related to friable integers, i.e. integers free of large prime factors. Initiated by Dickman [4] in 1930, the theory of friable integers gradually gained a privileged place in the development of analytic and probabilistic number theory. A direct link with the Riemann hypothesis is made explicit in [6]. The reader will find a non-exhaustive list of motivations and a description of many fundamental results in the paper [8], in the surveys [9], [10], [5], in the conference proceedings [3], and in the monograph [13].

Let $P^+(n)$ denote the largest prime factor of an integer $n > 1$ and put $P^+(1) := 1$. Sharp estimates for

$$\Psi(x, y) := \sum_{\substack{n \leqslant x \\ P^+(n) \leqslant y}} 1$$

are available in the aforementioned references. However, to the author's knowledge, none of these revisits the original problem considered by Dickman, that is estimating

$$D(x, u) := \sum_{\substack{n \leqslant x \\ P^+(n) \leqslant n^{1/u}}} 1 \quad (x \geqslant 1,\, u \geqslant 1).$$

This question is of type (1·1).

Recall that Dickman's function $\varrho$ is defined as the continuous solution on $\mathbb{R}^+$ to the delay-differential system

$$\begin{cases} \varrho(v) = 1 & (0 \leqslant v \leqslant 1), \\ v\varrho'(v) + \varrho(v-1) = 0 & (v > 1). \end{cases}$$

For $b > 0$, $c \geqslant 0$, let $H_{b,c}$ denote the range

$$(H_{b,c}) \qquad x \geqslant 3, \quad \mathrm{e}^{(\log_2 x)^b} < y \leqslant x/(\log x)^c.$$

Here and throughout, $\log_k$ denotes the $k$th iterated logarithm.

Dickman proved that $D(x,u) \sim x\varrho(u)$ for fixed $u \geqslant 1$ and $x \to \infty$. The best range in which the formula $\Psi(x,y) \sim x\varrho(u)$ $(u := (\log x)/\log y)$ is known to hold is $H_{b,0}$ for any $b > 5/3$: this is due to Hildebrand [7]. Thus, the two quantities agree to the first order. We evaluate their difference.

Our statement involves the logarithmic derivative

$$(1{\cdot}2) \qquad r(v) := -\varrho'(v)/\varrho(v) \quad (v > 0).$$

Writing systematically $u := (\log x)/\log y$, we also use the notation

$$(1{\cdot}3) \qquad \mathfrak{R} = \mathfrak{R}(x,y) := \frac{\sqrt{\log 2u}}{\sqrt{u}(\log y)^{3/2}} \quad (x \geqslant y \geqslant 2).$$

**Theorem 1.1.** *Let $b > 5/3$, $c > 10$. Uniformly for $(x,y) \in H_{b,c}$, we have*

$$(1{\cdot}4) \qquad D(x,u) = \Psi(x,y)\Big\{1 - \frac{r(u)}{\log y} + O\big(\mathfrak{R}\big)\Big\}.$$

*Remarks.* (i) We did not aim at finding the best admissible lower bound for the constant $c$.

(ii) From Saias' theorem—see [12], [13; ch. III.5]—we have in the same range $H_{b,c}$

$$\Psi(x,y) = x\varrho(u) + \frac{(\gamma-1)x\varrho'(u)}{\log y} + O\Big(\frac{x\varrho(u)(\log 2u)^2}{(\log y)^2}\Big),$$

where $\gamma$ denotes Euler's constant. We hence deduce from (1·4) that, still in $H_{b,c}$,

$$(1{\cdot}5) \qquad D(x,u) = x\varrho(u) + \frac{\gamma x\varrho'(u)}{\log y} + O\big(x\varrho(u)\mathfrak{R}_1\big),$$

with $\mathfrak{R}_1 := \mathfrak{R} + (\log 2u)^2/(\log y)^2 \ll (\log 2u)^{7/6}/(\log y)^{3/2}$.

(iii) It is easy to deduce from Buchstab's identity that, for $\sqrt{x} < y \leqslant x$, we have

$$D(x,u) = \Psi(x,y) - \sum_{p \leqslant y}\big\{p^{u-1} + O(1)\big\} = x\varrho(u) + \frac{\gamma x\varrho'(u)}{\log y} + O\bigg(\frac{y}{\log x} + \frac{x}{(\log x)^2}\bigg).$$

The following statement is a straightforward consequence of Theorem 1.1.

**Corollary 1.2.** *We have*

$$(1{\cdot}6) \qquad \sum_{1 < n \leqslant x} \frac{\log n}{\log P^+(n)} = \mathrm{e}^\gamma x - \frac{\gamma \mathrm{e}^\gamma x}{\log x} + O\Big(\frac{x}{(\log x)^{3/2}}\Big) \qquad (x \geqslant 3).$$

If $\log n$ is replaced by $\log x$ in the left-hand side of (1·6), the coefficient $-\gamma\mathrm{e}^\gamma$ changes to $(1-\gamma)\mathrm{e}^\gamma$.

Our second investigation is about the distribution of the squarefree kernel of an integer, i.e.

$$k(n) := \prod_{p|n} p \quad (n \geqslant 1),$$

where, here and in the sequel, $p$ denotes a prime number. In a recent article, Brüdern & Robert [2] obtained an estimate for

$$S(x;\vartheta,\alpha) := \sum_{\substack{n \leqslant x \\ k(n) \leqslant n^\vartheta(\log n)^\alpha}} 1$$

for fixed $\vartheta \in ]0,1[$, $\alpha \in \mathbb{R}$. The following statement significantly improves on their result regarding uniformity ($\vartheta$ and $\alpha$ will not assumed to be fixed) and domain of validity ($\vartheta$ will be allowed to approach 0 or 1). Moreover, replacing $n^\vartheta(\log n)^\alpha$ by another smooth function would easily follow by the same approach. The main ingredients of the proof, which turns out to be very short, are estimates obtained in [11] for the local behaviour of the summatory function

$$N(x,y) := \sum_{\substack{n \leqslant x \\ k(n) \leqslant y}} 1.$$

These estimates are all established by the saddle-point method.

We need the following notation. Let $\psi(n) := \prod_{p|n}(p+1)$ $(n \geqslant 1)$, and define

$$F(t) := \frac{6}{\pi^2}\sum_{n\geqslant 1}\frac{\min(1, \mathrm{e}^t/n)}{n\psi(n)} \qquad (t > 0).$$

Furthermore, consider $g(\sigma) := \sum_p \log\{1 + (1 - p^{\sigma-1})/p(p^\sigma - 1)\}$ and, for $t \geqslant 1$, let $\sigma_t$ denote the solution to the equation $g'(\sigma) + t = 0$. From [11; (2.9)] we have

$$\sigma_t = \sqrt{\frac{2}{t\log t}}\Big\{1 + \sum_{1\leqslant k\leqslant K}\frac{P_k(\log_2 t)}{(\log t)^k} + O_K\Big(\frac{(\log_2 t)^{K+1}}{(\log t)^{K+1}}\Big)\Big\} \quad (t \to \infty), \tag{1·7}$$

where $P_k$ is a polynomial of degree at most $k$. In particular,

$$P_1(z) = \tfrac{1}{2}(z - \log 2), \quad P_2(z) = \tfrac{3}{8}z^2 - (\tfrac{3}{4}\log 2 + \tfrac{1}{2})z + \tfrac{1}{2}\log 2 + \tfrac{3}{8}(\log 2)^2 + \tfrac{2}{3}\pi^2. \tag{1·8}$$

**Theorem 1.3.** *Let $A > 0$, $c \in ]0, \frac{1}{2}[$. Uniformly for $x \geqslant 3$, $1/(\log x)^c \leqslant \vartheta \leqslant 1 - 1/(\log x)^c$, $|\alpha| \leqslant A$, $y = x^\vartheta(\log x)^\alpha$, $v := \log(x/y)$, we have*

$$\begin{aligned} S(x;\vartheta,\alpha) &= \frac{yF(v)\sigma_v}{\vartheta}\Big\{1 - \frac{\sigma_v}{\vartheta} + O\Big(\frac{\sigma_v^2}{\vartheta^2} + \sqrt{\frac{\log v}{v}}\Big)\Big\} \\ &= \frac{N(x,y)\sigma_v}{\vartheta}\Big\{1 - \frac{\sigma_v}{\vartheta} + O\Big(\frac{\sigma_v^2}{\vartheta^2} + \sqrt{\frac{\log v}{v}}\Big)\Big\}. \end{aligned} \tag{1·9}$$

Since $\sigma_v/\vartheta \ll (\log x)^{c-1/2}$, we see that carrying (1·7) back into (1·9) yields an asymptotic expansion of the left-hand side with general term $c_{\ell k}(\log_2 v)^\ell/(\log v)^k$, $0 \leqslant \ell \leqslant k$, $c_{\ell k} \in \mathbb{R}$.

## 2. Proof of Theorem 1.1

Let us assume throughout that $x$ is arbitrarily large since the stated estimates are otherwise trivial. Recall the notation $u := (\log x)/\log y$ and the definition of the function $r$ in (1·2). Put $Z(s) := (s-1)\zeta(s)/s$ $(s \in \mathbb{C})$, where $\zeta(s)$ is the Riemann zeta function. From a slight modification of [1; prop. 4.1(i)], we derive that, for $(x, y) \in (H_{b,c})$, $\beta := 1 - r(u)/\log y$, $b > 5/3$, $c > 8$, we have

$$\Psi(x, y) = x\varrho(u)Z(\beta)\Big\{1 + O\Big(\frac{u}{(\log x)^2}\Big)\Big\}. \tag{2·1}$$

The alteration mentioned above solely concerns the lower bound on $c$, which is anyway not optimal. It stems from the fact that no coprimality is required here, which slightly simplifies the argument.

Estimate (2·1), which is proved through a saddle-point analysis, is the main tool in the proof of (1·4), inasmuch it provides an estimate to the second order.

As mentioned in the introduction, we mimic partial derivatives $\partial\Psi(x, y)/\partial x$ at $y = x^{1/u}$. Thus, given a small parameter $\varepsilon = \varepsilon_x > 0$ to be chosen later, we put $x_k := x\mathrm{e}^{-k\varepsilon}$, $y_k := x_k^{1/u}$ $(k \geqslant 0)$ and write, for some integer $K \geqslant 1$,

$$\begin{aligned} D^-(x,u) &:= \sum_{k<K}\big\{\Psi\big(x_k, y_{k+1}\big) - \Psi\big(x_{k+1}, y_{k+1}\big)\big\}, \\ D^+(x,u) &:= \sum_{k<K}\big\{\Psi\big(x_k, y_k\big) - \Psi\big(x_{k+1}, y_k\big)\big\} + \Psi\big(x_K, y_K\big), \end{aligned}$$

$$D^-(x,u) \leqslant D(x,u) \leqslant D^+(x,u). \tag{2·2}$$

Select $K := \lfloor 2(\log_2 x)/\varepsilon\rfloor$, so that $\Psi\big(x_K, y_K\big) \ll x\varrho(u)/(\log x)^2$, a quantity that may be absorbed by the remainder term of (1·4). This choice implies in particular that $\log x_k \asymp \log x$ $(0 \leqslant k \leqslant K)$.

Observe that $(x, y) \in H_{b,c}$ with $b > 5/3$, $c > 10$ implies $(x_k, y_k) \in H_{b',c'}$ for some $b' > 5/3$, $c' > 8$. Hence our hypotheses enable applying (2·1) for $(x_k, y_k)$, viz.

$$\Psi\big(x_k, y_k\big) = x_k Z(\beta_k)\varrho(u)\Big\{1 + O\Big(\frac{1}{u(\log y)^2}\Big)\Big\} \quad (0 \leqslant k < K), \tag{2·3}$$

with $\beta_k := 1 - r(u)/\log y_k$. By the same device, we have

$$\Psi\big(x_{k+1}, y_k\big) = x_{k+1} Z(\gamma_k)\varrho(v_k)\Big\{1 + O\Big(\frac{1}{u(\log y)^2}\Big)\Big\} \quad (0 \leqslant k < K), \tag{2·4}$$

with

$$v_k := \frac{\log x_{k+1}}{\log y_k} = u - \frac{\varepsilon}{\log y_k}, \quad \gamma_k := 1 - \frac{r(v_k)}{\log y_k}.$$

If $\xi(v)$ denotes the solution to the equation $\mathrm{e}^{\xi} = 1 + v\xi$ for $v \neq 1$ and $\xi(1) := 0$, we have [1], for $v \geqslant 3$,

$$\begin{aligned} &\xi(v) = \log(v \log v) + O\Big(\frac{\log_2 v}{\log v}\Big), \quad \xi'(v) = \frac{1}{v} + \frac{1}{v \log v} + O\Big(\frac{\log_2 v}{v(\log v)^2}\Big), \\ &r(v) - \xi(v) \ll \frac{1}{v}, \quad r'(v) - \xi'(v) \ll \frac{1}{v^2}. \end{aligned} \tag{2·5}$$

Therefore $r(v_k) - r(u) \ll \varepsilon/\log x$, and $\gamma_k - \beta_k \ll \varepsilon/\{u(\log y)^2\}$. Applying the Taylor-Lagrange formula [13; (5.115)] for $\varrho(v)$ at order 1, we get

$$\varrho(v_k) = \varrho(u) - \frac{\varepsilon\varrho'(u)}{\log y_k} + O\Big(\frac{\varepsilon^2\varrho(u)(\log 2u)^2}{(\log y)^2}\Big), \quad Z(\gamma_k) = Z(\beta_k) + O\Big(\frac{\varepsilon}{u(\log y)^2}\Big),$$

under the extra condition $u > 1 + 2\varepsilon/\log y_k$, which is actually implied by the hypothesis $H_{b,c}$.

Carrying back into (2·4) and assuming $\varepsilon \ll 1/\{\sqrt{u}\log 2u\}$, we derive in turn

$$\begin{aligned} \Psi\big(x_{k+1}, y_k\big) &= x_{k+1} Z(\beta_k)\varrho(u)\Big\{1 + \frac{\varepsilon r(u)}{\log y_k} + O\Big(\frac{1}{u(\log y)^2}\Big)\Big\}, \\ \Psi\big(x_k, y_k\big) - \Psi\big(x_{k+1}, y_k\big) &= x_k(1 - \mathrm{e}^{-\varepsilon})Z(\beta_k)\varrho(u) + \frac{\varepsilon \mathrm{e}^{-\varepsilon} x_k Z(\beta_k)\varrho'(u)}{\log y_k} + O\Big(\frac{x_k\varrho(u)}{u(\log y)^2}\Big). \end{aligned}$$

Now

$$\log y_k = \log y - k\varepsilon/u, \quad Z(\beta_k) = Z(\beta) + O\Big(\frac{k\varepsilon \log 2u}{u(\log y)^2}\Big),$$

whence

$$\begin{aligned} &\Psi\big(x_k, y_k\big) - \Psi\big(x_{k+1}, y_k\big) \\ &\qquad = x_k(1 - \mathrm{e}^{-\varepsilon})Z(\beta)\varrho(u) + \frac{\varepsilon x_k Z(\beta)\mathrm{e}^{-\varepsilon}\varrho'(u)}{\log y} + O\bigg(\frac{x_k\varrho(u)\{1 + k\varepsilon^2(\log 2u)^2\}}{u(\log y)^2}\bigg). \end{aligned}$$

Summing over $k$ yields

$$D^+(x,u) = x\varrho(u)Z(\beta) + \frac{Z(\beta)x\varrho'(u)}{\log y} + O\bigg(\frac{\varepsilon x\varrho(u)\log(2u)}{\log y} + \frac{x\varrho(u)}{\varepsilon u(\log y)^2} + \frac{x\varrho(u)(\log 2u)^2}{u(\log y)^2}\bigg).$$

For the quasi-optimal choice $\varepsilon := 1/\sqrt{(\log x)\log 2u}$, we get, with notation (1·3),

$$D^+(x,u) = x\varrho(u)Z(\beta) + \frac{Z(\beta)x\varrho'(u)}{\log y} + O\big(x\varrho(u)\mathfrak{R}\big) = x\varrho(u)Z(\beta)\Big\{1 - \frac{r(u)}{\log y} + O(\mathfrak{R})\Big\},$$

which is compatible with (1·4). Parallel computations provide the same estimate for $D^-(x,u)$.

## 3. Proof of Corollary 1.2

Let $D(x)$ denote the left-hand side of (1·6). We plainly have

$$D(x) = \int_0^\infty \{D(x,u) - 1\} \,\mathrm{d}u. \tag{3·1}$$

From the bound $1 \leqslant D(x,u) \leqslant \Psi\big(x, x^{1/u}\big) \ll x\mathrm{e}^{-u/2}$ $(u \geqslant 1,\ x \geqslant 1)$ [13; th. III.5.1], we see that the contribution of the range $u > 4\log_2 x$ may be encompassed by the stated remainder.

Let $\varepsilon_x := 11(\log_2 x)/\log x$. For large $x$ and $1 + \varepsilon_x < u \leqslant 4\log_2 x$, formula (1·5) is applicable. It is also trivially valid for $0 \leqslant u \leqslant 1$ at the cost of replacing the error term by $O(1)$. When $1 < u \leqslant 1 + \varepsilon_x$, we have, writing $X := \{x/\log x\}^{1/(1+\varepsilon_x)}$,

$$0 \leqslant x - D(x,u) = x\varrho(u) - D(x,u) \leqslant \sum_{X<p\leqslant x} \Big\lfloor \frac{x}{p} \Big\rfloor + \frac{x}{\log x} \ll \frac{x\log_2 x}{\log x}\cdot$$

Therefore

$$\int_0^{1+\varepsilon_x} \Big\{D(x,u) - 1 - x\varrho(u) - \frac{\gamma x u\varrho'(u)}{\log x}\Big\} \,\mathrm{d}u \ll \frac{x(\log_2 x)^2}{(\log x)^2}\cdot$$

Applying (1·5) to evaluate the contribution of the range $1 + \varepsilon_x \leqslant u \leqslant 4\log_2 x$ to the integral in (3·1), we get

$$\int_0^{4\log_2 x} \Big\{D(x,u) - 1 - x\varrho(u) + \frac{\gamma x\varrho(u-1)}{\log x}\Big\} \,\mathrm{d}u \ll \frac{x}{(\log x)^{3/2}}\cdot$$

Now (1·6) follows by extending the last integrals to infinity and appealing to the formula $\int_0^\infty \varrho(u)\,\mathrm{d}u = \mathrm{e}^\gamma$—see e.g. [13; (III.5.45)].

## 4. Proof of Theorem 1.3

As previously we assume $x$ arbitrarily large and mimic partial derivatives $\partial N(x,y)/\partial x$ at $y := x^\vartheta(\log x)^\alpha$. Our main tool will be the following estimate [11; (3.25)], valid for fixed $b > \frac{1}{2}$ and uniformly for $x \geqslant 3$, $\mathrm{e}^{(\log x)^b} < y \leqslant x$, $v := \log(x/y)$, $\eta_x := \sqrt{2/(\log x)\log_2 x}$:

$$N(x,y) = yF(v)\big\{1 + O\big(y^{-\eta_x}\big)\big\}. \tag{4·1}$$

Given $\varepsilon = \varepsilon_x \in ]0,1[$ to be determined later, we put $x_k := x\mathrm{e}^{-k\varepsilon}$, $y_k := x_k^\vartheta(\log x_k)^\alpha$ $(k \geqslant 0)$, and, given an integer $K \in [1, (\log x)/2\varepsilon]$, write

$$\begin{aligned} S^-(x;\vartheta,\alpha) &:= \sum_{k<K} \big\{N\big(x_k, y_{k+1}\big) - N\big(x_{k+1}, y_{k+1}\big)\big\}, \\ S^+(x;\vartheta,\alpha) &:= \sum_{k<K} \big\{N\big(x_k, y_k\big) - N\big(x_{k+1}, y_k\big)\big\} + N\big(x_K, y_K\big), \end{aligned}$$

$$S^-(x;\vartheta,\alpha) \leqslant S(x;\vartheta,\alpha) \leqslant S^+(x;\vartheta,\alpha). \tag{4·2}$$

We note right-away that, by (4·1) and [11; (8.13)] in the form

$$F(v+h) \ll \mathrm{e}^{h\sigma_v}F(v) \quad \big(v \geqslant 2,\ h + v \geqslant 0\big),$$

and since $\log(x_K/y_K) = v - (1-\vartheta)K\varepsilon + O(1)$, we have

$$N(x_K, y_K) \ll y_K\mathrm{e}^{-(1-\vartheta)\varepsilon K\sigma_v}F(v) \ll \mathrm{e}^{-\varepsilon\vartheta K}yF(v).$$

Therefore, selecting $K := \lfloor 2(\log v)/\varepsilon\vartheta \rfloor$ ensures that the last term in the upper bound for $S^+(x;\vartheta,\alpha)$ is absorbable by the error term of (1·9). Note that this implies $v_k := \log(x_k/y_k) \asymp v$ $(0 \leqslant k \leqslant K)$.

Next, we apply (4·1) to $(x_k, y_k)$ and $(x_{k+1}, y_k)$. This yields, with, say, $R := \mathrm{e}^{-(\log x)^{1/4-c/2}}$,

$$\begin{aligned} N(x_k, y_k) &= y_k F(v_k)\{1 + O(R)\},\\ N(x_{k+1}, y_k) &= y_k F(v_k - \varepsilon)\{1 + O(R)\}. \end{aligned}$$

Now, [11; (8.16)] furnishes

$$F(w) - F(w-h) = h\sigma_w F(w)\Big\{1 + O\Big(\frac{|h| + \log w}{\sqrt{w\log w}}\Big)\Big\} \qquad \big(h \ll \sqrt{w\log w}\big), \tag{4·3}$$

from which, specializing $w := v_k$, $h := \varepsilon$, we derive

$$N(x_k, y_k) - N(x_{k+1}, y_k) = y_k\varepsilon\sigma_{v_k}F(v_k)\Big\{1 + O\Big(\sqrt{\frac{\log v}{v}} + \frac{R}{\varepsilon\sigma_v}\Big)\Big\}.$$

The penultimate step consists in substituting $v$ to $v_k$ in the above formula. By [11; (8.4)], we know that $\mathrm{d}\sigma_v/\mathrm{d}v \ll v^{-3/2}(\log v)^{-1/2}$, hence

$$\sigma_{v_k} - \sigma_v \ll \frac{k\varepsilon}{v\sqrt{v\log v}} \ll \frac{k\varepsilon\sigma_v}{v}, \quad F(v_k) = F(v)\Big\{1 - k\varepsilon\sigma_v + O\Big(k^2\varepsilon^2\sigma_v^2 + \sqrt{\frac{\log v}{v}}\Big)\Big\},$$

where the last estimate follows from (4·3) with $w := v$, $h := (1-\vartheta)\varepsilon k - \alpha\log(1 - \varepsilon k/\log x)$, observing that $\vartheta\varepsilon k\sigma_v \ll \sqrt{(\log v)/v}$. Finally, we arrive at

$$N(x_k, y_k) - N(x_{k+1}, y_k) = y_k\varepsilon\sigma_v F(v)\Big\{1 - k\varepsilon\sigma_v + O\Big(k^2\varepsilon^2\sigma_v^2 + \sqrt{\frac{\log v}{v}} + \frac{R}{\varepsilon\sigma_v}\Big)\Big\}.$$

Selecting $\varepsilon = \sqrt{(\log v)/v}$ and summing over $k$ provides the required estimate for $S^+(x;\vartheta,\alpha)$. The corresponding formula for $S^-(x;\vartheta,\alpha)$ is proved by parallel computations.

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