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Remarks on the Selberg–Delange method*

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Abstract. Let ϱ be a complex number and let f be a multiplicative arithmetic function whose Dirichlet series takes the form $\zeta(s)^\varrho G(s)$, where G is associated to a multiplicative function g. The classical Selberg-Delange method furnishes asymptotic estimates for the averages of f under assumptions of either analytic continuation for G, or absolute convergence of a finite number of derivatives of $G(s)$ at $s = 1$. We consider different set of hypotheses, not directly comparable to the previous ones, and investigate how they can yield sharp asymptotic estimates for the averages of f.

Keywords: Averages of multiplicative functions, Selberg-Delange method, Dirichlet series, powers of the Riemann zeta function.

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1. Introduction and statement of results

In a series of papers published in 1953 and 1954, [12], [13], L. G. Sathe studied the local laws of the arithmetic functions counting the number of prime factors, with or without multiplicity, a problem previously considered by Hardy and Ramanujan. Sathe's results provided asymptotic formulae while only upper and lower bounds were previously known. However Sathe's method, based on induction formulae, involved very long and technical estimates. In the same year 1954, Selberg devised a fruitful method based on the idea that the Dirichlet series of the search for probabilities may be expressed through Taylor coefficients of powers of the Riemann zeta function. This idea was then systematically developed by Delange [4], [5]. In the second author's book [18] (latest edition, first in 1990), the results were generalized and made effective regarding various parameters—the method being there named after Selberg and Delange.

This theory provides estimates for counting functions associated to Dirichlet series of the form

$$F(s) = \zeta(s)^\varrho G(s), \tag{1·1}$$

where ϱ is a complex number, $\zeta(s)$ is Riemann's zeta function, and $G(s)$ satisfies suitable regularity conditions. Usually, G is associated with a multiplicative arithmetic function g, but this need not be so — see, e.g., [11]. In the sequel, we shall however concentrate on the case when g is indeed multiplicative. Then $F(s)$ is the Dirichlet series of a multiplicative function f.

In [18], two types of assumptions on $G(s)$ are considered : (a) analytic continuation at the left of the vertical line $\sigma = \Re e\, s = 1$; (b) absolute convergence at $s = 1$ for a finite number of derivatives.

In a recent work [8], Granville and Koukoulopoulos, propose a third type of condition, implying mere convergence, instead of absolute convergence, for a finite number of right-derivatives of $G(s)$ at $s = 1$. However, their analysis actually rests upon a much stronger assumption, viz., for some constant $A > 0$, not necessarily an integer,

$$\sum_{p\leqslant x} g(p)\log p = \sum_{n\leqslant x}\{f(p) - \varrho\}\log p \ll x/(\log x)^A \quad (x \geqslant 2). \tag{1·2}$$

Some extra, secondary hypotheses are also needed for the values $f(p^\nu)$ at prime powers. Here and in the sequel, the letter p denotes a prime number. For the sake of comparison, it is worthwhile to note that hypothesis (b) above essentially amounts to $\sum_p |g(p)|(\log p)^j/p < \infty$ for a finite number of exponents j.

Similar, but weaker, conditions have been considered by Wirsing [21], in the frame of comparison theorems, evaluating the ratio of averages of f and of a non-negative majorant. Via a further weakening of the hypotheses, such results were improved in [22]. In [19], the second author considered generalizations, and obtained effective forms of the results. In the present work, the

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* Some minor corrections with respect to the published version are included here.

viewpoint is quite different: taking advantage of the strength of assumptions like (1·2), one aims at directly deriving an asymptotic estimate for the averages of f. As in the classical Selberg-Delange approach, this also enters in the frame of comparison theorems, but the average of $f(n)$ is now compared with that of $\tau_\varrho(n)$—the nth coefficient in the Dirichlet series expansion of $\zeta(s)^\varrho$— instead of being compared with that of a majorant.

Assuming (1·1), (1·2), and $|f| \leqslant \tau_r$ for some parameter $r > 0$, the main result in [8] states that, with $J := \lceil A - 1 \rceil$ and suitable coefficients $\{\lambda_j(f)\}_{j=0}^J$, we have, for $x \geqslant 3$,

$$M(x; f) := \sum_{n \leqslant x} f(n) = x(\log x)^{\varrho-1} \Big\{ \sum_{0 \leqslant j \leqslant J} \frac{\lambda_j(f)}{(\log x)^j} \Big\} + O\Big(\frac{x(\log_2 x)^{\delta_{A,J+1}}}{(\log x)^{A+1-r}} \Big), \tag{1·3}$$

with Kronecker's δ-notation. Here and henceforth, $\log_k$ denotes the k-fold iterated logarithm.

Formula (1·3) is specially interesting when A is small, for less is then required on g. However, it furnishes no more than an upper bound when $A \leqslant r - \Re e\,\varrho$. Incidentally, under the weaker assumption that the series

$$\sum_p \frac{g(p)}{p}$$

converges, and arguing as in [10] (see also [18; th. III.4.14]), theorem 1.1 of [19] readily yields, for real f,

$$M(x; f) \ll x(\log x)^{r-1-\min\{1, K(r-\varrho)\}} \qquad (x \geqslant 2), \tag{1·4}$$

where $K \approx 0.32867$ is optimal. Moreover, in the case $\varrho = 0$, the same technique furnishes

$$M(x; f) \ll x(\log x)^{r-1-\min\{1,(1-2/\pi)r\}} \qquad (x \geqslant 2). \tag{1·5}$$

In particular, (1·4) supersedes (1·3) as soon as $A < K(r - \varrho) \leqslant 1$. Analogous upper bounds are available for complex f, under suitable hypotheses upon $f(p)$: see e.g. [9].

The purpose of the present work is two-fold: (a) to investigate refinements of (1·2) enabling an improvement of the error term in (1·3) by replacing the exponent r of $\log x$ by $\Re e\,\varrho$, as expected in view of standard estimates in the theory; (b) to propose a simpler and more natural (i.e. relying on a direct convolution argument) proof of (1·3), with possibly weaker hypotheses.

In the first direction, we meet the set goal when $\Re e\,\varrho \geqslant 0$, where ϱ is the exponent appearing in the generic assumption (1·1). This latter restriction is actually necessary for achieving target (a): we construct a family of counter-examples in Section 5.1.

In the following statement, hypothesis (1·2) is replaced by a short interval version, with the same value of A. The other assumptions concern $|f|$: we use those now classical introduced by Shiu [15], although they could be somewhat weakened if needed. Accordingly, we define the class $\mathcal{S}(B)$ of those multiplicative functions f such that:

(i) $|f(p^\nu)| \leqslant B^\nu \quad (\nu \geqslant 0)$,

(ii) $\forall \varepsilon > 0\ \exists C = C_\varepsilon : |f(n)| \leqslant C_\varepsilon n^\varepsilon \ (n \geqslant 1)$,

and, for $r > 0$, we consider the subclass $\mathcal{S}(B, r)$ of those f satisfying the extra assumption:

(iii) $\displaystyle\sum_{p \leqslant x} \frac{|f(p)|}{p} \leqslant r \log_2 x + O(1) \quad (x \geqslant 3)$.

Theorem 1.1. *Let $A > 0$, $B > 0$, $0 < \alpha < 1$, $r > 0$, $\varrho \in \mathbb{C}$, $J := \lceil A - 1 \rceil$, and let $f \in \mathcal{S}(B, r)$ verify*

$$\sum_{x < p \leqslant x+z} f(p) \log p = \varrho z + O\Big(\frac{z}{(\log x)^A} \Big) \quad (x \geqslant 2,\ x^{1-\alpha} \leqslant z \leqslant x). \tag{1·6}$$

Then, for suitable constants $\{\lambda_j(f)\}_{0 \leqslant j \leqslant J}$ and with $\vartheta := B + |\varrho| + 1$, we have

$$M(x; f) = x(\log x)^{\varrho-1} \sum_{0 \leqslant j \leqslant J} \frac{\lambda_j(f)}{(\log x)^j} + O\Big(\frac{x(\log_2 x)^\vartheta}{(\log x)^{A+1-\max(0,\Re e\,\varrho)}} \Big) \qquad (x \geqslant 3). \tag{1·7}$$

The coefficients $\lambda_j(f)$ may be described as follows. Representing $f=\tau_\varrho * g$ consistently with (1·1), we prove the convergence of the series

$$\gamma_j(g):=\sum_{n\geqslant 1}\frac{g(n)(\log n)^j}{n}\qquad(0\leqslant j\leqslant J),\tag{1·8}$$

and derive

$$\lambda_j(f)=\frac{1}{\Gamma(\varrho-j)}\sum_{\ell+h=j}\frac{\alpha_\ell(\varrho)\gamma_h(g)}{\ell!h!}\qquad(0\leqslant j\leqslant J),\tag{1·9}$$

where Γ is Euler's function and $\alpha_\ell(\varrho)/\ell!$ is the ℓ-th Taylor coefficient at the origin of

$$\{s\zeta(s+1)\}^\varrho/(s+1).$$

Alternatively, we also have

$$\lambda_j(f)=\frac{\mathrm{d}^j\{s^\varrho F(s+1)/(s+1)\}}{j!\Gamma(\varrho-j)\,\mathrm{d}s^j}(0+)\qquad(0\leqslant j\leqslant J).\tag{1·10}$$

In particular, $\lambda_j(f)=0$ for all j if ϱ is an integer $\leqslant 0$.

As for our second aim, namely goal (b) described above, we apply *friable convergence* (see below) to show the following result, essentially equivalent, in the intersection of the respective validity domains, to [8; th. 1]. For $r>0$, $\sigma\in]0,1[$, we define the class $\mathcal{F}(r,\sigma)$ comprising those complex multiplicative functions f such that

$$\begin{aligned}&\sum_{v<p\leqslant w}\frac{|f(p)|}{p}\leqslant r\log\Big(\frac{\log w}{\log v}\Big)+O(1)\quad(w\geqslant v\geqslant 2),\\ &\sum_p\Big\{\frac{|f(p)|^2}{p^{2\sigma}}+\sum_{\nu\geqslant 2}\frac{|f(p^\nu)|}{p^{\nu\sigma}}\Big\}<\infty.\end{aligned}\tag{1·11}$$

These conditions are weaker than those of [8; th. 1] and not directly comparable to those described in [8; § 7]. For instance, letting p_k denote the kth prime number, conditions (1·11) allow $f(p_{k^2})\asymp k/\log 2k$, whereas the latter do not.

Contrary to that of [8], our analysis does not ascribe a special role to the case when A is an integer.

Theorem 1.2. *Let $A>0$, $J:=\lceil A-1\rceil$, $r>0$, $\varrho\in\mathbb{C}$, $\sigma\in]0,1[$, and let $f\in\mathcal{F}(r,\sigma)$ be a multiplicative function such that $p\mapsto g(p):=f(p)-\varrho$ verifies (1·2). Then, for suitable constants $\beta>0$ and $\{\lambda_j(f)\}_{0\leqslant j\leqslant J}$, we have*

$$M(x;f)=x(\log x)^{\varrho-1}\sum_{0\leqslant j\leqslant J}\frac{\lambda_j(f)}{(\log x)^j}+O\Big(\frac{x(\log_3 x)^\beta}{(\log x)^{A+1-r}}\Big).\tag{1·12}$$

We shall see that the $\lambda_j(f)$ are still given by (1·10).

In Section 5.2, we show that, even when $\Re e\,\varrho\geqslant 0$, the remainder term of (1·12) cannot be sharpened so as to meet that of (1·7) —although a possibility of some improvement remains open if f is real. To this end, we construct a family of counterexamples $f\in\mathcal{F}(\sigma,r)$ satisfying (1·2) with $\varrho=0$, $0<A<r$ (resp. $0<A<2r/\pi$ in the real case), but contravening the short interval condition (1·6), and for which the exponent r in (1·12) cannot be replaced by cr if $c<1$ (resp. $c<2/\pi$).

As a final remark, we note the following: as that of [8], our analysis heavily depends on the asymptotic expansion for

$$T_\varrho(x):=M(x;\tau_\varrho)=\sum_{n\leqslant x}\tau_\varrho(n)\tag{1·13}$$

provided by the Selberg-Delange method; since the arithmetical functions under consideration are of the form $f=g*\tau_\varrho$, refinements regarding hypotheses on g or on its generating series $G(s)$ may hence be regarded as genuine parts of this theory.

2. On the case $\varrho = r$

The bound (1·4) shows that the error term of (1·3) is not optimal in the general case. Moreover, Theorem 1.1 provides fairly standard assumptions under which an essentially optimal remainder is achieved. However, inasmuch the power of the logarithm is concerned, (1·3) is expected to be sharp when $\varrho = r$. This latter case is discussed in [8], where it is confirmed that the error term may be replaced by a quantity $\asymp x(\log x)^{r-A-1}$ when $\varrho = r \geqslant 1$, $A \notin \mathbb{Z}$.

Assuming $\varrho = r$ essentially amounts to considering $f \geqslant 0$. In this latter circumstance, an asymptotic *formula* with optimal remainder may be obtained in a very simple way under lighter hypotheses. We present the details below.

Theorem 2.1. *Let $A > 0$, $\sigma \in]0,1[$, $r > 0$, and assume f is a non-negative multiplicative function such that*

$$\text{(i)}\qquad \sum_{p\leqslant x} f(p)\log p = rx + O\Big(\frac{x}{(\log x)^A}\Big) \quad (x \geqslant 2),$$

$$\text{(ii)}\qquad \sum_p \Big\{\frac{f(p)^2}{p^{2\sigma}} + \sum_{\nu\geqslant 2}\frac{f(p^\nu)}{p^{\nu\sigma}}\Big\} < \infty.$$

We then have

$$M(x;f) = \lambda_0(f)x(\log x)^{r-1}\Big\{1 + O\Big(\frac{(\log_2 x)^{\delta_{1,A}}}{(\log x)^{\min(1,A)}}\Big)\Big\} \quad (x \geqslant 2), \tag{2·1}$$

with $\lambda_0(f) := \dfrac{1}{\Gamma(r)}\displaystyle\prod_p \Big(1 - \frac{1}{p}\Big)^r \sum_{\nu\geqslant 0}\frac{f(p^\nu)}{p^\nu}$.

Proof. We have

$$\begin{aligned} M(x;f)\log x &= \sum_{n\leqslant x} f(n)\log n + \sum_{n\leqslant x} f(n)\log(x/n) \\ &= \sum_{m\leqslant x} f(m) \sum_{\substack{p^\nu\leqslant x/m\\ p\nmid m}} f(p^\nu)\log p^\nu + \int_1^x \frac{M(t;f)}{t}\,\mathrm{d}t \\ &= rx\sum_{m\leqslant x}\frac{f(m)}{m} + O\Big(R + S + x(\log x)^{r-1}\Big), \end{aligned} \tag{2·2}$$

with

$$R := \sum_{\substack{mp\leqslant x\\ p\,|\,m}} f(m)f(p)\log p + \sum_{\substack{p^\nu m\leqslant x\\ \nu\geqslant 2}} f(m)f(p^\nu)\log p^\nu, \qquad S := \sum_{m\leqslant x}\frac{xf(m)}{m(\log 2x/m)^A},$$

and where the last integral has been estimated by an Halberstam-Richert type bound—see, e.g., [18; th. III.3.5].

Now, with $\alpha := (\sigma+1)/2 < 1$, we have

$$\begin{aligned} R &\leqslant \sum_{m\leqslant x} f(m) \sum_{\substack{p^\nu m\leqslant x\\ \nu\geqslant 2}} \{f(p)f(p^{\nu-1}) + f(p^\nu)\}\log p^\nu \\ &\ll \sum_{m\leqslant x}\frac{f(m)x^\alpha}{m^\alpha}\sum_p\sum_{\nu\geqslant 2}\frac{f(p)f(p^{\nu-1}) + f(p^\nu)}{p^{\nu\sigma}} \ll \sum_{m\leqslant x}\frac{f(m)x^\alpha}{m^\alpha} \ll x(\log x)^{r-1}, \end{aligned}$$

where we used assumption (ii). Moreover, partial summation furnishes

$$S \ll x(\log x)^{r-\min(1,A)}(\log_2 x)^{\delta_{1,A}}.$$

It remains to note that, by [20; lemma 8.2] (with $\kappa = r$), we have

$$\sum_{m\leqslant x}\frac{f(m)}{m} = \Big\{1 + O\Big(\frac{1}{(\log x)^{\min(1,A)}}\Big)\Big\}\frac{\lambda_0(f)}{r}(\log x)^r \qquad (x \geqslant 2),$$

and carry back into (2·2). □

3. Proof of Theorem 1.1

3·1. Preparation

Given $\varrho \in \mathbb{C}$, let $\mathcal{M}(\varrho)$ denote the class of those multiplicative functions f satisfying

$$f(p^\nu) = \tau_\varrho(p^\nu) + \tau_\varrho(p^{\nu-1})\{f(p) - \varrho\} \qquad (p \geqslant 2,\, \nu \geqslant 1). \tag{3·1}$$

When $f \in \mathcal{S}(B, r)$, we can write $f = \mathfrak{f} * h$ where $\mathfrak{f} \in \mathcal{M}(\varrho)$, h is multiplicative, supported on the set of squareful integers, and such that the series

$$\sum_{n \geqslant 1} \frac{h(n)(\log n)^j}{n} \qquad (j \geqslant 0)$$

are absolutely convergent and uniformly bounded in terms of B for bounded j. Assuming (1·7) for $\mathfrak{f}$, this is enough to yield the required estimate, arguing as in [18; th. II.5.4]. Therefore, it will be sufficient to prove that (1·7) holds when $f \in \mathcal{M}(\varrho)$, $|f(p)| \leqslant B$, and f satisfies (1·6).

As a consequence of this reduction, we can assume that $f = \tau_\varrho * g$, where g is supported on the set of squarefree integers. Moreover, $g \in \mathcal{S}(\mathfrak{B}, r + |\varrho|)$ with $\mathfrak{B} := B + |\varrho|$, and g satisfies (1·2).

3·2. Main estimates

The main part of the argument consists in providing effective estimates for averages of the function g defined above.

Lemma 3.1. *For a suitable constant c_0 and uniformly for $x \geqslant 3$, $0 \leqslant j \leqslant J$, $k \ll \log_2 x$, we have*

$$G_{j,k}(x) := \sum_{\substack{n \leqslant x \\ \omega(n) = k}} g(n)(\log n)^j \ll \frac{x(\mathfrak{B} \log_3 x + c_0)^{k-1}}{(k-1)!(\log x)^{A+1-j}}. \tag{3·2}$$

For further reference we note right away that a consequence of (3·2) is that

$$G_j(x) := \sum_{n \leqslant x} g(n)(\log n)^j \ll_j \frac{x(\log_2 x)^{\mathfrak{B}}}{(\log x)^{A+1-j}} \qquad (j \geqslant 0,\, x \geqslant 3), \tag{3·3}$$

$$\mathfrak{g}_j(x) := \sum_{n \leqslant x} \frac{g(n)(\log n)^j}{n} = \gamma_j(g) + O\Big(\frac{(\log_2 x)^{\mathfrak{B}}}{(\log x)^{A-j}}\Big) \quad (0 \leqslant j \leqslant J,\, x \geqslant 3). \tag{3·4}$$

Indeed, it suffices to apply, e.g., lemma 1 from [17] in order to note that, for large, constant D, the contribution of those n with $\omega(n) > D \log_2 x$ is negligible and then appeal to (3·2) for $k \leqslant D \log_2 x$.

In the following proof and henceforth, we let $P^+(n)$—resp. $P^-(n)$—denote the largest—resp. the smallest—prime factor of an integer $n > 1$, and make the standard convention that $P^+(1) = 1$, $P^-(1) = \infty$.

Proof of Lemma 3.1. By partial summation, it is enough to consider $j = 0$. We may also plainly restrict to bounding the subsum over $n \in]\sqrt{x}, x]$.

Given a suitably large parameter $\mathcal{K}$, set $\mathcal{X} := (\log x)^{\mathcal{K}}$ and represent each n arising in (3·2) as $n = md$ with $P^+(m) \leqslant \mathcal{X}$, $P^-(d) > \mathcal{X}$. By, for instance, [16; lemma 2] (a Rankin-type bound), we see that the contribution to $G_{0,k}(x)$ of $d \leqslant x^{1/4}$ is, for suitable positive constants c_j,

$$\ll \sum_{d \leqslant x^{1/4}} |g(d)| \sum_{\substack{x^{1/4} < m \leqslant x/d \\ P^+(m) \leqslant \mathcal{X}}} |g(m)| \ll \sum_{d \leqslant x^{1/4}} \frac{|g(d)|}{d} x^{1 - c_1/\log \mathcal{X}} \ll x^{1 - c_2/\log \mathcal{X}}.$$

Whence

$$G_{0,k}(x) = \sum_{\substack{s+t=k \\ t \geqslant 1}} \sum_{\substack{m \leqslant x^{3/4} \\ \omega(m) = s \\ P^+(m) \leqslant \mathcal{X}}} g(m) \mathscr{G}_t\Big(\frac{x}{m}; \mathcal{X}\Big) + O\Big(x^{1 - c_3/\log_2 x}\Big), \tag{3·5}$$

with

$$\mathscr{G}_t(w;\mathcal{X}) := \sum_{\substack{x^{1/4}<d\leqslant w\\ \omega(d)=t\\ P^-(d)>\mathcal{X}}} g(d) \qquad \big(1\leqslant t\leqslant k,\ x^{1/4}<w\leqslant x\big). \tag{3·6}$$

Let $\mathcal{H} := \alpha\mathcal{K}$, where $\alpha > 0$ is the constant appearing in (1·6), and let $\delta := 1/(\log x)^{\mathcal{H}}$. Put $I(\ell) :=]\mathrm{e}^{\delta\ell}, \mathrm{e}^{\delta(\ell+1)}]$ $(\ell > L := (\mathcal{K}\log_2 x)/\delta)$. The contribution to (3·6) from those integers d having at least two prime factors in a same interval $I(\ell)$ is

$$\ll \mathfrak{B}^t \sum_{1\leqslant j<t} \sum_{\substack{\mathcal{X}<p_1<\cdots<p_t\\ p_1\cdots p_t\leqslant w\\ p_{j+1}<p_j\mathrm{e}^{\delta}}} \frac{w}{p_1\cdots p_t} \ll \delta w(\log w)^{\mathfrak{B}} \ll \frac{w}{(\log x)^{A+2}},$$

for a suitable choice of $\mathcal{K}$ and hence of $\mathcal{H}$.

For the remaining integers, split the prime factors of the summation variable d in (3·6) among the various $I(\ell)$ and consider the multiple sum over all hypercubes that are hit. If $\ell_1 < \ell_2 < \ldots < \ell_t$ is a sequence of admissible indexes, then $\delta(\ell_1 + \ldots + \ell_t) \leqslant \log w$ since $d \leqslant w$ in (3·6). If some product $v := \prod_{j=1}^t p_j$ happens to be $> w$, we must have $w < v \leqslant w\mathrm{e}^{t\delta}$. Therefore the total contribution of those admissible hypercubes containing at least one product $> w$ is

$$\ll \sum_{w<v\leqslant w\mathrm{e}^{t\delta}} \mathfrak{B}^{\omega(v)}\mu(v)^2 \ll w(\log w)^{\mathfrak{B}-1}\delta t \ll \frac{w}{(\log x)^{A+2}}.$$

We have shown so far that, for relevant values of t and w,

$$\mathscr{G}_t(w;\mathcal{X}) = S_t + O\bigg(\frac{w}{(\log x)^{A+2}}\bigg),$$

with, for some absolute constants C_j $(j = 0, 1)$,

$$\begin{aligned}
S_t &:= \sum_{\substack{L<\ell_1<\cdots<\ell_t\\ (\log x)/4\delta<\ell_1+\ldots+\ell_t\leqslant(\log w)/\delta}} \prod_{1\leqslant j\leqslant t} \sum_{p\in I(\ell_j)} g(p)\\
&\ll \sum_{\substack{L<\ell_1<\cdots<\ell_t\\ (\log x)/4\delta<\ell_1+\ldots+\ell_t\leqslant(\log w)/\delta}} \prod_{1\leqslant j\leqslant t} \frac{C_0\mathrm{e}^{\ell_j\delta}}{\delta^A\ell_j^{A+1}}\\
&\ll \frac{C_1^t t^{A+1}}{(t-1)!(L\delta)^{A(t-1)}} \sum_{(\log x)/4\delta<\ell\leqslant(\log w)/\delta} \frac{\mathrm{e}^{\delta\ell}}{\delta^A\ell^{A+1}}\\
&\ll \frac{C_1^t t^{A+1} w}{(t-1)!(\delta L)^{A(t-1)}(\log x)^{A+1}} \ll \frac{w}{(t-1)!(\kappa\log_2 x)^{t-1}(\log x)^{A+1}},
\end{aligned}$$

where $\kappa = \kappa(\mathcal{K})$ may be chosen as large as we wish. Here, we made use of the short-interval assumption (1·6) at the very first step. In the third line, we set $\ell := \ell_1 + \ldots + \ell_t$ and bounded $1/\ell_t^{A+1}$ by t^{A+1}/ℓ^{A+1}. Carrying back into (3·5), we obtain

$$G_{0,k}(x) \ll \sum_{\substack{s+t=k\\ t\geqslant 1}} \frac{x(\mathfrak{B}\log_3 x + c_3)^s}{s!(t-1)!(\kappa\log_2 x)^{t-1}(\log x)^{A+1}} \ll \frac{x(\mathfrak{B}\log_3 x + c_4)^{k-1}}{(k-1)!(\log x)^{A+1}},$$

provided $\mathcal{K}$, and therefore κ, is suitably chosen. This is the required estimate. □

3·3. Completion of the argument

It is now a simple matter to derive (1·7). Recalling definition (1·13), we have

$$M(x;f)=\sum_{n\leqslant x} g(n)T_\varrho\Big(\frac{x}{n}\Big).$$

By (3·4) and partial summation, we have

$$\mathfrak{g}_j(y)=\sum_{n\leqslant y}\frac{g(n)(\log n)^j}{n}\ll j(\log y)^{j-A}(\log_2 y)^{\mathfrak{B}+\delta_{A,J+1}}\qquad (y\geqslant 3,\ j>J). \tag{3·7}$$

Recall the definition of the coefficients $\alpha_h(\varrho)$ appearing in (1·9) and put

$$\nu_h:=\alpha_h(\varrho)/h!\Gamma(\varrho-h)\quad (h\geqslant 0).$$

By [18; th. II.5.2], for a suitable constant b, we have $\nu_h\ll(bh+1)^h$ and

$$T_\varrho(x)=x(\log x)^{\varrho-1}\bigg\{\sum_{0\leqslant h\leqslant H}\frac{\nu_h}{(\log x)^h}+O\bigg(\frac{(bH+1)^{H+1}}{(\log x)^{H+1}}\bigg)\bigg\}\quad (x\geqslant 2), \tag{3·8}$$

uniformly for $H\ll(\log x)^{1/3}$, say.

Now Dirichlet's hyperbola formula provides

$$M(x;f)=U+V+O\Big(x(\log x)^{\varrho-1-A}\Big),$$

with

$$U:=\sum_{n\leqslant\sqrt{x}} g(n)T_\varrho\Big(\frac{x}{n}\Big),\quad V:=\sum_{d\leqslant\sqrt{x}}\tau_\varrho(d)\sum_{\sqrt{x}<n\leqslant x/d} g(n).$$

The sum V may be treated as an error term: for $d\leqslant\sqrt{x}$, we have, by (3·3),

$$\sum_{\sqrt{x}<n\leqslant x/d} g(n)\ll\frac{x(\log_2 x)^{\mathfrak{B}}}{d(\log x)^{A+1}}.$$

By distributing the variable d into intervals $]\mathrm{e}^{-\delta(j+1)}\sqrt{x},\mathrm{e}^{-\delta j}\sqrt{x}]$ and approximating the corresponding n-range by $[\sqrt{x},\sqrt{x}\mathrm{e}^{\delta j}]$ with $\delta:=1/(\log x)^{\mathcal{K}}$ for suitably large $\mathcal{K}$, we obtain

$$\begin{aligned}V&\ll\sum_{\substack{j\geqslant 0\\ \mathrm{e}^{j\delta}\leqslant\sqrt{x}}}\bigg|\Big\{G_0(\mathrm{e}^{\delta j}\sqrt{x})-G_0(\sqrt{x})\Big\}\sum_{\mathrm{e}^{-\delta(j+1)}\sqrt{x}<d\leqslant\mathrm{e}^{-\delta j}\sqrt{x}}\tau_\varrho(d)\bigg|+\frac{x}{(\log x)^{A+1}}\\ &\ll\frac{x\delta(\log_2 x)^{\mathfrak{B}}}{(\log x)^{A+1}}\sum_{\substack{j\geqslant 0\\ \mathrm{e}^{j\delta}\leqslant\sqrt{x}}}\{(\log x)-2j\delta+1\}^{\Re e\,\varrho-1}\\ &\ll\frac{x(\log_2 x)^{\mathfrak{B}}}{(\log x)^{A+1}}\Big\{1+(\log_2 x)^{\delta_{0,\Re e\,\varrho}}(\log x)^{\Re e\,\varrho}\Big\},\end{aligned}$$

with Kronecker's notation.

Finally, we apply (3·8) with $H=\lceil A+2r\rceil$ to get

$$U=x\sum_{n\leqslant\sqrt{x}}\frac{g(n)}{n}\bigg\{\sum_{0\leqslant h\leqslant H}\nu_j\Big(\log\frac{x}{n}\Big)^{\varrho-h-1}+O\Big((\log x)^{\Re e\,\varrho-H-2}\Big)\bigg\}.$$

From the expansion

$$\Big(\frac{\log x/n}{\log x}\Big)^{\varrho-h-1}=\sum_{0\leqslant k\leqslant K}c_{kh}\Big(\frac{\log n}{\log x}\Big)^k+O\Big(K^{|\varrho-1-j|}\Big(\frac{\log n}{\log x}\Big)^K\Big)\quad(1\leqslant n\leqslant\sqrt{x})$$

with $c_{kh}\ll k^{|\varrho-1-h|}$, we get, for each $h\in[0,H]$,

$$\sum_{n\leqslant\sqrt{x}}\frac{g(n)}{n}\Big(\frac{\log x/n}{\log x}\Big)^{\varrho-h-1}=\sum_{0\leqslant k\leqslant K}c_{kh}\frac{\mathfrak{g}_k\big(\sqrt{x}\big)}{(\log x)^k}+O\Big(\frac{K^{|\varrho-1-h|}}{2^K}(\log_2 x)^{\mathfrak{B}}\Big),\tag{3·9}$$

where we used the trivial estimate $|g|\leqslant\mathfrak{B}^\omega$ to bound the error term.

Select $K:=\lfloor 2(A+\mathfrak{B}+1)\log_2 x\rfloor$. Applying (3·4) for $j\leqslant J$, and (3·7) when $J<j\leqslant H$, we get

$$U=x(\log x)^{\varrho-1}\bigg\{\sum_{0\leqslant j\leqslant J}\frac{\lambda_j(f)}{(\log x)^j}+O\Big(\frac{(\log_2 x)^{\mathfrak{B}+\delta_{A,J+1}}}{(\log x)^A}\Big)\bigg\},$$

the exponent $\delta_{A,J+1}$ arising from that in (3·7). This completes the proof.

4. Proof of Theorem 1.2—friable summation

4·1. Setting

An integer n such that $P^+(n)\leqslant y$ is said to be y-friable. Friable summability of series, was defined in [6], [7], and has been employed systematically in [1], [2], [3]. A series $\sum_{n\geqslant1}a_n$ is said to be *friably summable* to a (or is said to have friable sum a) if the subseries

$$\sum_{\substack{n\geqslant1\\P^+(n)\leqslant y}}a_n$$

converges for each $y\geqslant2$ and tends to a as y tends to infinity. We then write

$$\sum_{n\geqslant1}a_n=a\quad(P).$$

Letting $\zeta(s)$ denote Riemann's zeta function, it is well known that, for any given real number $\tau\neq0$, the series $\sum_{n\geqslant1}1/n^{1+i\tau}$ has friable sum $\zeta(1+i\tau)$ (by the convergence of the Eulerian product on the pointed line $1+i\tau$, $\tau\neq0$), while being divergent in the ordinary meaning.

We shall show that, representing f as a Dirichlet convolution $f=\tau_\varrho*g$, then, for $0\leqslant j\leqslant J$, we have

$$\gamma_j(g):=\sum_{n\geqslant1}\frac{g(n)(\log n)^j}{n}\qquad(P).\tag{4·1}$$

With this notation, the coefficients $\lambda_j(f)$ appearing in (1·12) are given by

$$\lambda_j(f)=\frac{1}{\Gamma(\varrho-j)}\sum_{\ell+h=j}\frac{\alpha_\ell(\varrho)\gamma_h(g)}{\ell!h!}\qquad(0\leqslant j\leqslant J),\tag{4·2}$$

while (1·10) remains valid.

4·2. Reduction

Let g be exponentially multiplicative, i.e. such that $g(p^\nu)=g(p)^\nu/\nu!$ for all primes p and all integers $\nu\geqslant0$,[1] and define $g(p):=f(p)-\varrho$. Put $\mathfrak{f}:=\tau_\varrho*g$. Then $f=\mathfrak{f}*h$ where h is

1. This concept has been extensively used in the literature, in particular by Wirsing [22].

multiplicative, supported on the set of squareful integers, and, for each p, the values $h(p^\nu)$ are given by the power series expansion

$$\sum_{\nu\geqslant 0} h(p^\nu)\xi^\nu = (1-\xi)^\varrho \mathrm{e}^{-\xi g(p)} \sum_{\nu\geqslant 0} f(p^\nu)\xi^\nu \qquad (|\xi|<1/p^\sigma).$$

Thus, $\sum_{n\geqslant 1}|h(n)|/n^\tau<\infty$ for all $\tau>\sigma$ and so, by a standard convolution argument left to the reader, we may restrict to proving (1·12) for $\mathfrak{f}$.

We note right away that $g\in\mathcal{F}(2r,\sigma)$ and that (1·2) holds. Introducing the series

$$\sum_{P^+(n)\leqslant y}\frac{g(n)}{n^s} = \exp\Big\{\sum_{p\leqslant y}\frac{g(p)}{p^s}\Big\} \qquad (\Re e\, s\geqslant 1),$$

we readily see by partial summation that the right-derivatives of any order $j\leqslant J$ at $s=1$ of the left-hand side converge to a limit as $y\to\infty$, in other words that (4·1) holds. Moreover partial summation yields that, for integer j and $y\geqslant 2$,

$$\gamma_j(y;g) := \sum_{P^+(n)\leqslant y}\frac{g(n)(\log n)^j}{n} = \begin{cases}\gamma_j(g)+O\big((\log y)^{j-A}\big) & (0\leqslant j\leqslant J),\\ O\Big(j!(\log y)^{j-A}(\log_2 y)^{\delta_{j,A}}\Big) & (j>J).\end{cases} \tag{4·3}$$

To prove this, observe that, for $|w|\leqslant 1/\log y$, we have

$$\begin{aligned}\sum_{p\leqslant y}\frac{g(p)}{p^{1-w}} &= \sum_{k\geqslant 0}\sum_{p\leqslant y}\frac{g(p)w^k(\log p)^k}{k!p} = \sum_{k\leqslant J}\Big\{\frac{\mu_k w^k}{k!}+O\Big(\frac{1}{k!(\log y)^A}\Big)\Big\}+O\Big(\frac{(\log_2 y)^{\delta_{A,J+1}}}{(\log y)^A}\Big)\\ &= \sum_{k\leqslant J}\frac{\mu_k w^k}{k!}+O\Big(\frac{(\log_2 y)^{\delta_{A,J+1}}}{(\log y)^A}\Big),\end{aligned}$$

with $\mu_k:=\sum_p g(p)(\log p)^k/p$ $(0\leqslant k\leqslant J)$. Estimate (4·3) then follows from Cauchy's formula.

4·3. An auxiliary estimate

For $f=\tau_\varrho * g$ and g as above, let us write $f=\tau_\varrho * g_y * h_y$, where g_y and h_y are the multiplicative functions defined by

$$g_y(p^\nu) := \mathbf{1}_{\{p\leqslant y\}}g(p^\nu),\quad h_y(p^\nu)=\mathbf{1}_{\{p>y\}}g(p^\nu) \qquad (p\geqslant 2,\ \nu\geqslant 1).$$

Our first goal is to estimate

$$H_y(x) := \sum_{y<n\leqslant x} h_y(n) \qquad (2\leqslant y\leqslant x).$$

Lemma 4.1. *Uniformly for $x\geqslant y\geqslant 2$, and with $u:=(\log x)/\log y$, we have*

$$H_y(x) \ll \frac{xu^{2r}}{(\log x)^{A+1}}. \tag{4·4}$$

Proof. We have

$$H_y(x) = \sum_{\substack{y<p\leqslant x\\ p^\nu\leqslant x}} g(p^\nu) + \sum_{y<m\leqslant x/y} h_y(m) \sum_{\substack{p^\nu\leqslant x/m\\ p>P^+(m)}} g(p^\nu) \ll \frac{x}{(\log x)^{A+1}} + S,$$

with

$$S := \sum_{\substack{y<m\leqslant x/y\\ mP^+(m)\leqslant x}} \frac{|h_y(m)|x}{m\{\log(x/m)\}^{A+1}}.$$

Writing $m = nq^\nu$ with $P^+(n) < q$, where, here and in the sequel of this proof, q denotes a prime number, we have

$$S = \sum_{\substack{q>y\\ \nu\geqslant 1}} \frac{|g(q)|^\nu}{\nu! q^\nu} \sum_{\substack{n\leqslant x/q^{\nu+1}\\ P^+(n)<q}} \frac{|h_y(n)|x}{n\{\log(x/nq^\nu)\}^{A+1}},$$

.

Let S^- denote the contribution to S of $n=1$, and S^+ that of $n>y$. We plainly have

$$\begin{aligned} S^- &\ll \sum_{\nu\geqslant 1} \sum_{y<q\leqslant x^{1/(\nu+1)}} \frac{|g(q)|^\nu x}{\nu! q^\nu (\log x/q^\nu)^{A+1}} \\ &\ll \sum_{\nu\geqslant 1} \sum_{y<q\leqslant x^{1/(\nu+1)}} \frac{x|g(q)|^\nu(\nu+1)^{A+1}}{\nu! q^\nu (\log x)^{A+1}} \ll \frac{x\log(2u)}{(\log x)^{A+1}}. \end{aligned}$$

In order to majorize S^+, we appeal to the Rankin-type bound

$$\sum_{\substack{y<n\leqslant t\\ P^+(n)\leqslant q}} |h_y(n)| \ll t^{1-c/\log q}\frac{(\log t)^{2r-1}}{(\log y)^{2r}} \qquad (t, q > y),$$

where $c>0$ is an abolute constant. It follows that

$$\begin{aligned} S^+ &\ll \sum_{\nu\geqslant 1} \sum_{y<q\leqslant x^{1/(\nu+1)}} \frac{x|g(q)|^\nu}{\nu! q^\nu} \int_y^{x/q^{\nu+1}} \frac{1}{t(\log x/tq^\nu)^{A+1}}\, \mathrm{d}O\Big(\frac{t^{1-c/\log q}(\log t)^{2r-1}}{(\log y)^{2r}}\Big) \\ &\ll \sum_{\nu\geqslant 1} \sum_{y<q\leqslant x^{1/(\nu+1)}} \frac{x|g(q)|^\nu}{\nu! q^\nu(\log y)^{2r}} \int_y^{x/q^{\nu+1}} \frac{(\log t)^{2r-1}\,\mathrm{d}t}{t^{1+c/\log q}(\log x/tq^\nu)^{A+1}} \\ &\ll \sum_{\nu\geqslant 1} \sum_{y<q\leqslant x^{1/(\nu+1)}} \frac{x|g(q)|^\nu(\log q)^{2r-A-1}}{\nu! q^\nu(\log y)^{2r}} \int_0^{(\log x)/(\log q)-\nu-1} \frac{\mathrm{e}^{-cw}w^{2r-1}\,\mathrm{d}w}{\{(\log x)/(\log q)-\nu-w\}^{A+1}} \\ &\ll \sum_{\nu\geqslant 1} \sum_{y<q\leqslant x^{1/(\nu+1)}} \frac{x|g(q)|^\nu(\log q^\nu)^{2r}}{\nu! q^\nu(\log y)^{2r}(\log x)^{A+1}} \ll \frac{xu^{2r}}{(\log x)^{A+1}}. \end{aligned}$$

□

4·4. Completion of the argument

Put $f_y := \tau_\varrho * g_y$ and let $y := x^{c/\log_2 x}$, where c is a sufficiently small constant. We have

$$M(x; f_y) = V_y(x) + E_y(x)$$

with

$$V_y(x) := \sum_{n\leqslant\sqrt{x}} g_y(n) T_\varrho\Big(\frac{x}{n}\Big), \quad E_y(x) := \sum_{d\leqslant\sqrt{x}} \tau_\varrho(d) \sum_{\sqrt{x}<n\leqslant x/d} g_y(n).$$

We immediately note that a Rankin-type estimate such as [16; lemma 2] provides

$$E_y(x) \ll x^{1-1/\log y} \sum_{d\leqslant x} \frac{\tau_r(d)}{d} \ll x(\log x)^{\varrho-1-N}, \tag{4·5}$$

for any given integer N.

From (3·8), we deduce, for any fixed integer H, that

$$V_y(x) = x \sum_{n \leqslant \sqrt{x}} \frac{g_y(n)(\log x/n)^{\varrho-1}}{n} \bigg\{ \sum_{0 \leqslant h \leqslant H} \frac{\nu_h}{(\log x/n)^h} + O_H\bigg(\frac{1}{(\log x)^{H+1}}\bigg)\bigg\}.$$

Using the bound provided by (1·11) and selecting H sufficiently large in terms of r, we get that the overall contribution of the above remainder term is

$$\ll x(\log x)^{\varrho+2r-H-2} \ll x(\log x)^{\varrho-1-N},$$

for any given integer N.

Appealing to (3·9) for g_y, we may write

$$\sum_{n \leqslant \sqrt{x}} \frac{g_y(n)}{n}\Big(\frac{\log x/n}{\log x}\Big)^{\varrho-h-1} = \sum_{0 \leqslant k \leqslant K} c_{kh} \sum_{n \leqslant \sqrt{x}} \frac{g_y(n)}{n}\Big(\frac{\log n}{\log x}\Big)^k + O\Big(\frac{1}{(\log x)^{N+r+1}}\Big), \tag{4·6}$$

for sufficiently large K. Let $v := 1/\log y$. For all $0 \leqslant k \leqslant K$, we have, by Rankin's device again, provided c is suitably chosen,

$$\sum_{n > \sqrt{x}} \frac{|g_y(n)|(\log n)^k}{n} \leqslant \frac{k!}{v^k x^{v/4}} \exp\bigg\{\sum_{p \leqslant y} \frac{|g_y(p)|}{p^{1-v/2}}\bigg\} \ll \frac{k!}{x^{v/4}}(\log y)^{k+2r} \ll \frac{1}{(\log x)^{N+r+1}}.$$

We may therefore extend, in (4·6), the inner n-sum to all y-friable integers without perturbing the error. Taking (4·5) into account, we thus obtain

$$M(x; f_y) = x(\log x)^{\varrho-1}\bigg\{\sum_{0 \leqslant h \leqslant H} \frac{\nu_h}{(\log x)^h} \sum_{0 \leqslant k \leqslant K} c_{kh} \frac{\gamma_k(y; g)}{(\log x)^k} + O\bigg(\frac{1}{(\log x)^N}\bigg)\bigg\},$$

which, by rearranging the double sum, yields

$$M(x; f_y) = x(\log x)^{\varrho-1}\bigg\{\sum_{0 \leqslant j \leqslant K+H} \frac{\lambda_j(y; f)}{(\log x)^j} + O\bigg(\frac{1}{(\log x)^N}\bigg)\bigg\} \tag{4·7}$$

where, in view of (4·3),

$$\lambda_j(y; f) = \lambda_j(f) + O\Big(j^j(\log y)^{j-A}(\log_2 y)^{\delta_{A,j}}\Big) \qquad (0 \leqslant j \leqslant K + H), \tag{4·8}$$

with the convention that, say, $\lambda_j(f) = 0$ for $j > J$.

By Dirichlet's hyperbola formula, we have

$$M(x; f - f_y) = M(x; f_y * h_y - f_y) = S + U - W, \tag{4·9}$$

with

$$S := \sum_{n \leqslant \sqrt{x}} f_y(n) H_y\Big(\frac{x}{n}\Big), \quad U := \sum_{y < m \leqslant \sqrt{x}} h_y(m) M\Big(\frac{x}{m}; f_y\Big), \qquad W := M\big(\sqrt{x}; f_y\big) H_y\big(\sqrt{x}\big).$$

Now

$$S \ll \sum_{n \leqslant \sqrt{x}} \frac{|f_y(n)| x(\log_2 x)^{2r}}{n(\log x)^{A+1}} \ll \frac{x(\log_2 x)^{2r}}{(\log x)^{A+1-r}},$$

and similarly

$$W \ll x(\log x)^{\varrho-A-2}(\log_2 x)^{2r} \ll \frac{x(\log_2 x)^{2r}}{(\log x)^{A+1-r}}.$$

To estimate the sum U, we insert (4·7) and evaluate the contribution of the main terms by partial summation from (4·4). Taking (4·8) into acount, we obtain

$$U \ll xu^{2r}(\log x)^{\varrho-A-1}.$$

Recalling (4·7), (4·8) and (4·9), we have thus almost reached (1·12), but with a remainder term

$$\mathcal{R}(x;f) \ll \frac{x(\log_2 x)^{2r}}{(\log x)^{A+1-r}}\cdot$$

We now show that an inductive argument enables us to replace $(\log_2 x)^{2r}$ by a power of $\log_3 x$. Indeed, under assumption (1·11), consider the exponentially multiplicative function φ defined by

$$\varphi(p^\nu) = f(p)^\nu/2^\nu \nu! \quad (p \geqslant 2,\, \nu \geqslant 1).$$

Then $\varphi \in \mathcal{F}(r/2,\sigma)$. Moreover, we may write $f = \varphi * \varphi * \psi$ with

$$\psi(p^\nu) = \sum_{0\leqslant j\leqslant \nu} \frac{(-1)^j f(p)^j f(p^{\nu-j})}{j!} \quad (\nu \geqslant 1).$$

Thus $\psi(p) = 0$ and, writing $\varepsilon_p := |f(p)|^2/p^{2\sigma} + \sum_{\nu\geqslant 2} |f(p^\nu)|/p^{\nu\sigma}$, we have

$$\begin{aligned}\sum_{\nu\geqslant 2} \frac{|\psi(p^\nu)|}{p^{\sigma\nu}} &\leqslant \sum_{j+k\geqslant 2} \frac{|f(p)|^j|f(p^k)|}{j!p^{\sigma(k+j)}}\\ &\leqslant \varepsilon_p + \frac{|f(p)|}{p^\sigma}\varepsilon_p + \varepsilon_p \mathrm{e}^{|f(p)|/p^\sigma}\Big\{1 + \frac{|f(p)|}{p^\sigma} + \varepsilon_p\Big\} \ll \varepsilon_p.\end{aligned}$$

Applying the estimate already proved for φ and writing the hyperbola formula for f furnishes

$$\mathcal{R}(x;f) \ll \frac{x(\log_2 x)^{r}}{(\log x)^{A+1-r}}\cdot$$

After $k+1$ iterations, the exponent of $\log_2 x$ becomes $r/2^k$, while the implicit constant in the upper bound for $\mathcal{R}(x;f)$ gets multiplied by C^k for a suitable constant C. Selecting $k \asymp \log_4 x$ yields (1·12).

5. Limitations

5·1. Optimatility of Theorem 1.1

We show here that it is not possible, in general, to replace $\max(0, \Re e\, \varrho)$ by $\Re e\, \varrho$ in the error term of (1·7). To this end, we initially consider the completely multiplicative function g_0 defined by

$$g_0(p^\nu) := 1/(\log p)^{A\nu} \quad (p \geqslant 2,\, \nu \geqslant 1)$$

with $0 < A < 1$—note however that, at the cost of slightly more complicated computations, our approach would work for any positive A. As a preliminary step we show that, for a suitable constant C, we have

$$M(x;g_0) = \frac{Cx}{(\log x)^{A+1}}\Big\{1 + O\Big(\frac{1}{(\log x)^A}\Big)\Big\} \quad (x \geqslant 2). \tag{5·1}$$

Indeed, on the one hand, since $\sum g_0(n)/n < \infty$, we have

$$M(x;g_0) \ll x/\log x \quad (x \geqslant 2), \tag{5·2}$$

and, on the other hand,

$$(5\cdot3)\qquad \begin{aligned} M(x;g_0)\log x-\int_1^x \frac{M(t;g_0)}{t}\,\mathrm{d}t &= \sum_{m\leqslant x} g_0(m)\sum_{d\leqslant x/m} g_0(d)\Lambda(d)\\ &= x\sum_{m\leqslant x}\frac{g_0(m)}{m(\log 2x/m)^A}+O\Big(x\sum_{m\leqslant x}\frac{g_0(m)}{m(\log 2x/m)^{A+1}}\Big). \end{aligned}$$

From this and (5·2), we deduce that $M(x;g_0)\ll x/(\log x)^{A+1}$ $(x\geqslant 2)$. Carrying this estimate back into (5·3) yields

$$\begin{aligned} M(x;g_0) &= \frac{x}{\log x}\sum_{m\leqslant x}\frac{g_0(m)}{m(\log 2x/m)^A}+O\Big(\frac{x}{(\log x)^{A+2}}\Big).\\ &= \frac{x}{\log x}\sum_{m\leqslant\sqrt{x}}\frac{g_0(m)}{m(\log x/m)^A}+O\Big(\frac{x}{(\log x)^{2A+1}}\Big)\\ &= \frac{x}{(\log x)^{A+1}}\sum_{m\leqslant\sqrt{x}}\frac{g_0(m)}{m}\Big\{1+O\Big(\frac{\log m}{\log x}\Big)\Big\}+O\Big(\frac{x}{(\log x)^{2A+1}}\Big)\\ &= \frac{x}{(\log x)^{A+1}}\sum_{m\leqslant\sqrt{x}}\frac{g_0(m)}{m}+O\Big(\frac{x}{(\log x)^{2A+1}}\Big), \end{aligned}$$

which implies (5·1) with $C:=\sum_{m\geqslant 1} g_0(m)/m$.

Next, define $g(n):=g_0(n)n^i$ $(n\geqslant 1)$. By partial summation, we have

$$(5\cdot4)\qquad M(x;g)=\frac{Cx^{1+i}}{(1+i)(\log x)^{A+1}}\Big\{1+O\Big(\frac{1}{(\log x)^A}\Big)\Big\}\quad (x\geqslant 2).$$

Now, select $\varrho=-r$, with $A<r<1$ and define $f=g*\tau_\varrho$. This function satisfies the hypotheses of Theorem 1.1. By the hyperbola formula, we have, for $1\leqslant y\leqslant x$,

$$(5\cdot5)\qquad M(x;f)=U+V-W$$

with

$$U:=\sum_{n\leqslant x/y} g(n)T_\varrho\Big(\frac{x}{n}\Big),\quad V:=\sum_{n\leqslant y}\tau_\varrho(n)M\Big(\frac{x}{n};g\Big),\quad W:=T_\varrho(y)M\Big(\frac{x}{y};g\Big).$$

We choose $y:=\mathrm{e}^{a(\log_2 x)^2}$, where a is a sufficiently large constant. We plainly have

$$W\ll\frac{x}{(\log y)^{r+1}(\log x)^{A+1}}.$$

The sum U may be evaluated by the Selberg-Delange method in the form given in [18; § II.5.4, Notes]. For suitable positive constants b, c, we have

$$T_\varrho(x)=\int_0^b \alpha(t)x^{1-t}t^r\,\mathrm{d}t+O\Big(x\mathrm{e}^{-c\sqrt{\log x}}\Big)\qquad (x\geqslant 2),$$

where α is continuous on $[0,b]$. Since we may deduce from (5·4) that

$$Z(t):=\sum_{n\leqslant x/y}\frac{g(n)}{n^{1-t}}\ll\frac{(x/y)^t}{(\log x)^{A+1}}\qquad (0\leqslant t\leqslant b),$$

we get, with an appropriate choice of a,

$$U=\int_0^b \alpha(t)x^{1-t}Z(t)t^r\,\mathrm{d}t+O\Big(\frac{x}{(\log x)^{A+2}}\Big)\ll\frac{x}{(\log x)^{A+1}(\log y)^{r+1}}.$$

Finally

$$V = \frac{Cx^{1+i}}{(1+i)(\log x)^{A+1}} \sum_{n \leqslant y} \frac{\tau_\varrho(n)}{n^{1+i}} \Big\{1 + O\Big(\frac{\log n}{\log x}\Big)\Big\} + O\Big(\frac{x}{(\log x)^{2A+1}} \sum_{n \leqslant y} \frac{\tau_r(n)}{n}\Big)$$
$$= \frac{\{1+o(1)\}Cx^{1+i}}{(1+i)\zeta(1+i)^r(\log x)^{A+1}}.$$

Carrying back into (5·5), we obtain

$$M(x;f) \gg \frac{x}{(\log x)^{A+1}},$$

which implies the stated property.

5·2. *Optimality of Theorem 1.2*

We show here that the exponent r appearing in the remainder term of (1·12) cannot in general be replaced by $\Re e\, \varrho$, even when $\Re e\, \varrho \geqslant 0$. To this end we exhibit a function f belonging to $\mathcal{F}(\sigma, r)$ for $1/2 < \sigma < 1$ and for which: (i) condition (1·6) fails, (ii) $\varrho = 0$, $0 < A < r$, and (iii) given any $c < 1$, one cannot replace r by cr in (1·12). If f happens to be real, the conditions become $0 < A < 2r/\pi$ and $c < 2/\pi$. Our construction relies on exploiting possible resonance between $f(p)$ and p^{it} for certain values of the real parameter t.

Let $A > 0$, $r > 0$, $C > 0$ (large), and let $x_k := \exp\exp C^k$, $t_k := (\log x_{k+1})^A$ $(k \geqslant 1)$. Put $\varphi(v) := \mathrm{e}^{iv}$ and define f as the exponentially multiplicative function such that $f(p) := r\varphi(t_k \log p)$ whenever $x_k < p \leqslant x_{k+1}$ and, say, $f(p) = 0$ when $p \leqslant x_1$. Observe that $\int_0^y \varphi(tw)\,\mathrm{d}w \ll 1/t$ $(t \geqslant 1,\ y > 0)$. Due to the rapid increase of x_k, partial summation hence yields

$$\sum_{p \leqslant x} f(p) \log p \ll \frac{x}{(\log x)^A} \qquad (x \geqslant 2).$$

Now, for $x := x_{k+1}$, $t := t_k$, $\sigma := 1 + 1/\log x$, we have

$$\Big|\sum_{n \geqslant 1} \frac{f(n)}{n^{\sigma+it}}\Big| \asymp \prod_{h \leqslant k} \mathrm{e}^{S_h} \ll t \int_1^x \frac{|M(y;f)|}{y^2}\,\mathrm{d}y \tag{5·6}$$

with

$$S_h := r \sum_{x_h < p \leqslant x_{h+1}} \frac{\Re e\,\{f(p)/p^{it}\}}{p} \qquad (1 \leqslant h \leqslant k).$$

Of course

$$S_k := r\log\Big(\frac{\log x_{k+1}}{\log x_k}\Big) + O(1) = rC^{k+1}(1 - 1/C) + O(1). \tag{5·7}$$

Next, for $2\log k < h \leqslant k-1$, we have, by the prime number theorem in a strong form,

$$S_h = r\int_{x_h}^{x_{h+1}} \frac{\cos\{(t_k - t_h)\log v\}}{v \log v}\,\mathrm{d}v + O(1)$$
$$= r\int_{\log x_h}^{\log x_{h+1}} \frac{\cos\{(t_k - t_h)w\}}{w}\,\mathrm{d}w + O(1) \ll \frac{1}{t_k \log x_h} + 1 \ll 1.$$

Bounding S_h trivially for $h \leqslant 2\log k$, we finally get

$$\sum_{h \leqslant k} S_h \geqslant rC^{k+1}(1 - 1/C) + O\big(k^{2\log C}\big) = r(1 - 1/C)\log_2 x + O\Big((\log_3 x)^{2\log C}\Big).$$

Carrying back into (5·6) yields

$$\int_1^x \frac{|M(y;f)|}{y^2}\,\mathrm{d}y \gg (\log x)^{r(1-1/C)-A+o(1)}.$$

and so, as $x\to\infty$, provided $A < r(1-1/C)$,

$$M(x;f) = \Omega\Big(\frac{x}{(\log x)^{A+1-r(1-1/C)+o(1)}}\Big). \tag{5·8}$$

Since C may be chosen arbitrarily large, this furnishes the required result.

If we require f to be real, we define $\varphi(v) := \mathrm{sgn}(\cos v)$ $(v\in\mathbb{R})$. Since $\varphi(v)\cos v$ has mean value $2/\pi$, we obtain $S_k = (2r/\pi)C^{k+1}(1-1/C)+O(1)$ instead of (5·7), appealling, e.g., to [18; Lemma III.4.13]. The remainder of the analysis is essentially identical and so we get (5·8) with $2r/\pi$ in place of r. It is noticeable that, provided A is suitably chosen, the implied lower bound and (1·5) may agree to an arbitrary small power of $\log x$.

Remark. The above construction shows that assumptions $f\in\mathcal{F}(\sigma,r)$ $(\sigma<1,\ r>1)$ and (1·2) with $\varrho=0$, $0<A<r-1$, are insufficient to imply the convergence of the series $\sum_{n\geqslant 1} f(n)/n$.

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