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On a family of arithmetic series related to the Möbius function

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To George Andrews and Bruce Berndt,
as a friendly token of companionship

Abstract. Let $P^-(n)$ denote the smallest prime factor of a natural integer $n > 1$. Furthermore let μ and ω denote respectively the Möbius function and the number of distinct prime factors function. We show that, given any set $\mathcal{P}$ of prime numbers with a natural density, we have $\sum_{P^-(n)\in\mathcal{P}} \mu(n)\omega(n)/n = 0$ and provide a effective estimate for the rate of convergence. This extends a recent result of Alladi and Johnson, who considered the case when $\mathcal{P}$ is an arithmetic progression.

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1. Introduction and statements

Let $P^-(n)$ (resp. $P^+(n)$) denote the smallest (resp. the largest) prime factor of a natural integer $n > 1$ and put $P^-(1) := \infty$ (resp. $P^+(1) := 1$). Furthermore, let μ and ω denote respectively the Möbius function and the number of distinct prime factors function.

In a recent paper [2], Alladi and Johnson proved that, for given integers k, ℓ, such that $(k, \ell) = 1$, we have

$$\sum_{\substack{n\leqslant x\\ P^-(n)\equiv\ell \,(\mathrm{mod}\, k)}} \frac{\mu(n)\omega(n)}{n} \ll \frac{(\log_2 x)^{5/2}}{\sqrt{\log x}} \qquad (x \geqslant 3),$$

and consequently that

$$\sum_{P^-(n)\equiv\ell \,(\mathrm{mod}\, k)} \frac{\mu(n)\omega(n)}{n} = 0. \tag{1·1}$$

Their proof rests significantly on the prime number theorem for arithmetic progressions and on a duality identity due to Alladi [1], connecting small and large prime factors via Möbius inversion. The purpose of this note is to investigate to what extent (1·1) depends on the subset of the primes appearing in the summation condition. We obtain the following result. Here and in the sequel we use the notation $u := (\log x)/\log y$ $(x \geqslant y \geqslant 2)$, and we let $\log_k$ denote the k-fold iterated logarithm.

Theorem 1.1. *Let $\mathcal{P}$ be a set of prime numbers satisfying, for suitable $\delta \in [0, 1]$,*

$$\varepsilon_{\mathcal{P}}(t) := \frac{1}{t}\sum_{\substack{p\leqslant t\\ p\in\mathcal{P}}} \log p - \delta = o(1) \qquad (t \to \infty). \tag{1·2}$$

Then

$$\sum_{P^-(n)\in\mathcal{P}} \frac{\mu(n)\omega(n)}{n} = 0. \tag{1·3}$$

Moreover, for any fixed $b > 5/3$ and uniformly for $\mathrm{e}^{(\log_2 x)^b} \leqslant y \leqslant \sqrt{x}$, we have

$$\sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{P}}} \frac{\mu(n)\omega(n)}{n} \ll \varepsilon^*_{\mathcal{P}}(y)\log u + \frac{1}{u}, \tag{1·4}$$

where $\varepsilon^*_{\mathcal{P}}(y) := \sup_{t>y} |\varepsilon_{\mathcal{P}}(t)|$.

Remark. Quasi-optimal choices for y yield that the upper bound in (1·4) is, with arbitrary constants $\sigma > 0$, $0 < \tau < 3/5$,

$$\ll \begin{cases} \dfrac{\log_3 x}{(\log_2 x)^\sigma} & \text{if } \varepsilon^*_{\mathcal{P}}(y) \ll 1/(\log_2 y)^\sigma \\ \dfrac{(\log_2 x)^{1/(1+\sigma)}}{(\log x)^{\sigma/(1+\sigma)}} & \text{if } \varepsilon^*_{\mathcal{P}}(y) \ll 1/(\log y)^\sigma, \\ \dfrac{(\log_2 x)^{1/\tau}}{\log x} & \text{if } \varepsilon^*_{\mathcal{P}}(y) \ll \mathrm{e}^{-(\log y)^\tau}. \end{cases}$$

The last case covers that of an arithmetic progression.

Let $\mathbb{P}$ denote the set of all prime numbers. We note that (1·3) does not hold for an arbitrary set of primes. As suggested by Alladi in private communication, selecting

$$\mathcal{P} := \mathbb{P} \cap \cup_{j\geqslant 1}]\sqrt{x_j}, x_j] \tag{1·5}$$

for sufficiently rapidly increasing sequence $\{x_j\}_{j=1}^\infty$ furnishes a counter-example. Indeed, a straightforward consequence of (2·4) *infra* is that

$$\liminf_{x\to\infty} \sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{P}}} \frac{\mu(n)\omega(n)}{n} \leqslant -\log 2. \tag{1·6}$$

See Remark 2.3 below.

2. Proof of Theorem 1.1

Let $y \in [2, x]$ be a parameter at our disposal, and put $\mathcal{P}_y := \mathcal{P} \cap [2, y]$. We first aim at estimating the contribution from $\mathcal{P}_y$ to the sum (1·4) when y is sufficiently small in front of x.The following lemma will be useful. We define

$$\chi_{\mathcal{P}}(n) := \mathbf{1}_{\mathcal{P}}\big(P^-(n)\big), \quad g_y(n) := \sum_{m\mid n} \chi_{\mathcal{P}_y}(m)\mu(m)\omega(m) \quad \big(n \geqslant 1,\, y \geqslant 2\big), \tag{2·1}$$

let $\mathcal{S}$ denote the set of prime powers, and write $\mathcal{S}^* := \mathcal{S} \cup \{1\}$

Lemma 2.1. *We have*

$$g_y(n) = -\mathbf{1}_{\mathcal{P}_y}\big(P^+(n)\big) + \sum_{\substack{rs=n\\ P^+(r)\in\mathcal{P}_y\\ P^+(r)<P^-(s)=P^+(s)}} 1 \quad (n \geqslant 1). \tag{2·2}$$

In particular, for all $n \geqslant 1$, *we have* $|g_y(n)| \leqslant 1$ *and*

$$|g_y(n)| \leqslant \sum_{\substack{rs=n\\ P^+(r)\leqslant y,\, s\in\mathcal{S}^*\\ P^-(s)>P^+(r)}} 1. \tag{2·3}$$

Proof. Put $\chi(n, y) := \mathbf{1}_{[1,y]}\big(P^+(n)\big)$ $(n \geqslant 1)$. Using the representation $n = ab$ with $\chi(a, y) = 1$, $P^-(b) > y$, we have

$$\begin{aligned} g_y(n) &= \sum_{d\mid a,\, t\mid b} \chi_{\mathcal{P}_y}(d)\mu(d)\mu(t)\{\omega(d) + \omega(t)\} \\ &= \sum_{d\mid a} \chi_{\mathcal{P}_y}(d)\mu(d)\omega(d) \sum_{t\mid b} \mu(t) + \sum_{d\mid a} \chi_{\mathcal{P}_y}(d)\mu(d) \sum_{t\mid b} \mu(t)\omega(t). \end{aligned}$$

However

$$\sum_{t\mid b}\mu(t)=\chi(n,y),\quad \sum_{t\mid m}\mu(t)\omega(t)=\Big[\frac{\mathrm{d}(1-z)^{\omega(m)}}{\mathrm{d}z}\Big]_{z=1}=-\mathbf{1}_{\mathcal{S}(m)}\quad(m\geqslant 1),$$

$$\begin{aligned}\sum_{d\mid a}\chi_{\mathcal{P}_y}(d)\mu(d)\omega(d)&=-\sum_{p\mid a}\mathbf{1}_{\mathcal{P}_y}(p)\sum_{\substack{d\mid a/p\\ P^-(d)>p}}\mu(d)\{1+\omega(d)\}\\&=-\mathbf{1}_{\mathcal{P}_y}\big(P^+(a)\big)+\sum_{\substack{a=rm\\ P^+(r)\in\mathcal{P}_y\\ P^+(r)<P^-(m)=P^+(m)\leqslant y}}1,\end{aligned}$$

$$\sum_{d\mid a}\chi_{\mathcal{P}_y}(d)\mu(d)=-\sum_{\substack{p\in\mathcal{P}_y\\ p\mid a}}\sum_{\substack{d\mid a/p\\ P^-(d)>p}}\mu(d)=-\mathbf{1}_{\mathcal{P}_y}\big(P^+(a)\big).$$

This plainly implies (2·2). □

We are now in a position to estimate the quantity

$$A^-(x,y):=\sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{P}_y}}\frac{\mu(n)\omega(n)}{n}\cdot$$

Lemma 2.2. *Without any hypothesis on $\mathcal{P}$ and uniformly for $2\leqslant y\leqslant\sqrt{x}$, we have*

$$A^-(x,y)\ll\frac{1}{u}\cdot\tag{2·4}$$

Proof. Interpreting definition (2·1) as the Dirichlet convolution $g_y=\chi_{\mathcal{P}_y}\mu\omega*\mathbf{1}$, we get $\chi_{\mathcal{P}_y}\mu\omega=g_y*\mu$ by Möbius inversion. Appealing to a strong form of the prime number theorem (see e.g. [3; th. 8.17] or [6; ex. 178]), it follows that

$$\begin{aligned}A^-(x,y)&=\sum_{n\leqslant x}\frac{\chi_{\mathcal{P}_y}(n)\mu(n)\omega(n)}{n}\\&=\sum_{d\leqslant x}\frac{g_y(d)}{d}\sum_{m\leqslant x/d}\frac{\mu(m)}{m}\ll\sum_{d\leqslant x}\frac{|g_y(d)|}{d}\mathrm{e}^{-c\sqrt{\log x/d}}\end{aligned}$$

for a suitable absolute constant $c>0$. Now by (2·3) we have, for $2\leqslant y\leqslant D^{2/3}$,

$$\sum_{d\leqslant D}|g_y(d)|\ll\sum_{\substack{r\leqslant D\\ P^+(r)\leqslant y}}1+\sum_{\substack{rP^+(r)\leqslant D\\ P^+(r)\leqslant y}}\frac{D}{r\log(DP^+(r)/r)}\ll\frac{D\log y}{\log D},\tag{2·5}$$

where the r-sums have been estimated using the estimate [6; th. III.5.1]

$$\sum_{\substack{r\leqslant t\\ P^+(r)\leqslant y}}1\ll t\mathrm{e}^{-(\log t)/2\log y}\quad(t\geqslant 1,\ y\geqslant 2).$$

Recalling notation $u:=(\log x)/\log y$, we get

$$\sum_{d\leqslant x}\frac{|g_y(d)|}{d}\mathrm{e}^{-c\sqrt{\log x/d}}\ll\frac{1}{u}\qquad\big(2\leqslant y\leqslant\sqrt{x}\big),$$

where the estimate is obtained by splitting the summation at $x^{3/4}$, appealing to the inequality $|g_y(d)|\leqslant 1$ for the lower range and performing partial summation resting on (2·5) for the upper range.

This implies (2·4), as required. □

Remark 2.3. To prove (1·6), observe that if $\mathcal{P}$ is defined by (1·5) then

$$A^-\big(x_j,\sqrt{x_j}\big)\ll\frac{\log x_{j-1}}{\log x_j}$$

by Lemma 2.2 and

$$\sum_{\substack{n\leqslant x_j\\ P^-(n)\in\mathcal{P}\\ P^-(n)>\sqrt{x_j}}}\frac{\mu(n)\omega(n)}{n}=\sum_{\sqrt{x_j}<p\leqslant x_j}\frac{-1}{p}=-\log 2+o(1)\quad(j\to\infty).$$

This implies (1·6) on selecting $\{x_j\}_{j=1}^\infty$ tending to infinity sufficiently fast.

To complete the proof of Theorem 1.1, it remains to estimate the contribution from $\mathcal{Q}_y:=\mathcal{P}\smallsetminus\mathcal{P}_y$ to the sum (1·4), viz.

$$A^+(x,y):=\sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{Q}_y}}\frac{\mu(n)\omega(n)}{n}\cdot$$

This is the purpose of the following statement. Here and throughout, we let γ denote Euler's constant.

Lemma 2.4. *Let $b>5/3$. Unformly for $x\geqslant 3$, $\mathrm{e}^{(\log_2 x)^b}\leqslant y\leqslant\sqrt{x}$, we have*

$$A^+(x,y)=\frac{\delta\mathrm{e}^\gamma}{u}+O\Big(\varepsilon_{\mathcal{P}}^*(y)\log u+\frac{1}{u^{9/5}}+\frac{u^{6/5}}{\log x}\Big),\tag{2·6}$$

and

$$A^+(x,y)=\frac{\delta\mathrm{e}^\gamma}{u}+O\Big(\varepsilon_{\mathcal{P}}^*(y)\log_2 x+\frac{1}{u^{9/5}}+\frac{1}{\log x}\Big).\tag{2·7}$$

Proof. Both estimates will be proved by interpreting $A^+(x,y)$ as the derivative at $z=1$ of the polynomial

$$A(x,y;z):=\sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{Q}_y}}\frac{\mu(n)z^{\omega(n)}}{n}$$

and estimating this quantity by the Selberg-Delange method.

We first consider (2·6), which turns out to be the more delicate of the two.

Let the letters p and q denote prime numbers. For $|z-1|\leqslant 1/5$, $w\geqslant 1$, $\Re s>1$, define

$$G(s;w,z):=\prod_{q>w}\Big(1-\frac{1}{q^s}\Big)^{-z}\Big(1-\frac{z}{q^s}\Big),$$

$$F(s;w,z):=\sum_{P^-(n)>w}\frac{z^{\omega(n)}\mu(n)}{n^s}=\prod_{q>w}\Big(1-\frac{z}{q^s}\Big)=\prod_{q\leqslant w}\Big(1-\frac{1}{q^s}\Big)^{-z}\frac{G(s;w,z)}{\zeta(s)^z}\cdot$$

Then

$$\mathcal{H}(s;y,z):=\sum_{P^-(n)\in\mathcal{Q}_y}\frac{z^{\omega(n)}\mu(n)}{n^s}=-z\sum_{p\in\mathcal{Q}_y}\frac{F(s;p,z)}{p^s}\cdot\tag{2·8}$$

By a variant of Perron's formula [6; lemma II.2.6], there exist two constants α and β such that, writing

$$k(s):=\frac{1}{s}+\frac{\alpha}{s+1}+\frac{\beta}{s+2}\big(s\in\mathbb{C}\smallsetminus\{-2,-1,0\}\big),\quad g(t):=\mathbf{1}_{[1,\infty]}(t)\Big\{1+\frac{\alpha}{t}+\frac{\beta}{t^2}\Big\}\quad(t>0),$$

we have, uniformly for $v>0$, $\kappa>0$,

$$\frac{1}{2\pi i}\int_{\kappa-i}^{\kappa+i}k(s)v^s\,\mathrm{d}s=g(v)+O\Big(\frac{v^\kappa}{1+(\log v)^2}+\kappa v^\kappa\Big).$$

We infer that, for $|z| = r$, $\kappa := 1/\log x$,

$$A(x,y;z) := \sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{Q}_y}} \frac{\mu(n)z^{\omega(n)}}{n} = \frac{1}{2\pi i}\int_{\kappa-i}^{\kappa+i} \mathcal{H}(s+1,y;z)k(s)x^s\,\mathrm{d}s + O\Big(\sum_{1\leqslant j\leqslant 4} R_j\Big),$$

with

$$R_1 := \sum_{P^-(n)\in\mathcal{Q}_y} \frac{\mu(n)^2r^{\omega(n)}}{n^{\kappa+1}\{1+\log(x/n)^2\}},\qquad R_2 := \kappa\sum_{P^-(n)\in\mathcal{Q}_y}\frac{\mu(n)^2r^{\omega(n)}}{n^{\kappa+1}},$$

$$R_3 := \frac{1}{x}\sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{Q}_y}} \mu(n)z^{\omega(n)},\qquad R_4 := \frac{1}{x^2}\sum_{\substack{n\leqslant x\\ P^-(n)\in\mathcal{Q}_y}} n\mu(n)z^{\omega(n)}.$$

We readily have $R_2 \ll u^r/\log x$. To estimate R_1, first consider the contribution, say R_{11}, of those integers n such that $|\log(x/n)| > 1$. Summing over dyadic intervals and appealing to standard bounds for averages of non-negative arithmetic functions, e.g. [6; th. III.3.5], we see that $R_{11} \ll u^r/\log x$. The complementary contribution R_{12} is a sum over $[x/\mathrm{e}, \mathrm{e}x]$. It is readily evaluated by applying the same standard bounds. This yields again $R_{12} \ll u^r/\log x$. The terms R_3 and R_4 may be estimated trivially by bounding $\mu(n)$ by $\mu(n)^2$ and $z^{\omega(n)}$ by $r^{\omega(n)}$. This still furnishes $R_3 + R_4 \ll u^r/\log x$.

We may finally state that

$$A(x,y;z) = \frac{1}{2\pi i}\int_{\kappa-i}^{\kappa+i} \mathcal{H}(s+1,y;z)k(s)x^s\,\mathrm{d}s + O\Big(\frac{u^r}{\log x}\Big). \tag{2·9}$$

Define

$$J(s) := \int_0^\infty \mathrm{e}^{-s-t}\frac{\mathrm{d}t}{s+t}\quad\big(\Re s > 0\big),\qquad L_\varepsilon(t) := \mathrm{e}^{(\log t)^{3/5-\varepsilon}}\quad(\varepsilon > 0,\ t \geqslant 2).$$

When $\Re s > 0$, $p > y$, $s_p := s\log p$, [6; lemma III.5.16] yields, for any fixed $\varepsilon > 0$,

$$F(s+1;p,z) = \mathrm{e}^{-zJ(s_p)}\Big\{1 + O\Big(\frac{1}{L_\varepsilon(y)}\Big)\Big\}\qquad\Big(|\Im s| \leqslant L_\varepsilon(y)\Big). \tag{2·10}$$

(This is proved using the Korobov-Vinogradov zero free region and accounts for the exponent 3/5 in the definition of $L_\varepsilon(y)$.)

Let us insert (2·10) into (2·8) and then into (2·9) keeping in mind the hypothesis $\log y \geqslant (\log_2 x)^b$. Using the estimate $|\mathrm{e}^{-J(s)}| \asymp \min(|s|,1)$ $(\Re s \geqslant -1)$ proved in [4; lemma 2] in the form

$$|\mathrm{e}^{-zJ(s)}| \asymp \min(|s|^{\Re z},1)\qquad(\Re s \geqslant -1,\ \Re z > 0), \tag{2·11}$$

we obtain

$$A(x,y;z) = \frac{1}{2\pi i}\int_{\kappa-i}^{\kappa+i} B(s;y,z)k(s)x^s\,\mathrm{d}s + O\Big(\frac{u^r}{\log x}\Big), \tag{2·12}$$

with

$$B(s;y,z) := \sum_{p\in\mathcal{Q}_y}\frac{-z\mathrm{e}^{-zJ(s_p)}}{p^{s+1}}\cdot$$

Now we introduce the remainder

$$R(t) := t\varepsilon_{\mathcal{P}}(t) = \sum_{\substack{p\leqslant t\\ p\in\mathcal{P}}}\log p - \delta t = o(t)\qquad(t > 1).$$

Taking into account that $J'(s) = -\mathrm{e}^{-s}/s$, we get

$$B(s;y,z) = D(s;y,z) - z\int_y^\infty \frac{\mathrm{e}^{-zJ(s_t)}}{t^{s+1}\log t}\,\mathrm{d}R(t), \tag{2·13}$$

with

$$D(s;y,z) := -\delta z\int_y^\infty \frac{\mathrm{e}^{-zJ(s_t)}}{t^{s+1}\log t}\,\mathrm{d}t = \delta\int_{s_y}^{s\infty}\frac{-z\mathrm{e}^{-zJ(v)}}{v\mathrm{e}^v}\,\mathrm{d}v = \delta\big\{1-\mathrm{e}^{-zJ(s_y)}\big\}.$$

Carrying back into (2·12), we obtain

$$A(x,y;z) = \frac{\delta}{2\pi i}\int_{\kappa-i}^{\kappa+i}\big\{1-\mathrm{e}^{-zJ(s_y)}\big\}k(s)x^s\,\mathrm{d}s + O\Big(\mathfrak{R}_{\mathcal{P}}(x,y;z)+\frac{u^r}{\log x}\Big), \tag{2·14}$$

with

$$\mathfrak{R}_{\mathcal{P}}(x,y;z) := \int_y^\infty \frac{\lambda_x(t)}{t\log t}\,\mathrm{d}R(t),\quad \lambda_x(t) := \int_{\kappa-i}^{\kappa+i}\mathrm{e}^{-zJ(s_t)}\Big(\frac{x}{t}\Big)^s k(s)\,\mathrm{d}s.$$

By (2·11), we have, for $t > y$,

$$\begin{aligned}
\lambda_x(t) &\ll \int_{\kappa-i}^{\kappa+i}\Big(\frac{x}{t}\Big)^s\min(|s|\log t,1)^{\Re z}\frac{|\mathrm{d}s|}{|s|} \ll \Big(\frac{x}{t}\Big)^\kappa \log_2 t,\\
\lambda_x'(t) &= \int_{\kappa-i}^{\kappa+i} x^s k(s)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{e}^{-zJ(s_t)}}{t^s}\Big)\,\mathrm{d}s = \int_{\kappa-i}^{\kappa+i}\frac{x^s sk(s)\mathrm{e}^{-zJ(s_t)}}{t^{s+1}}\Big\{-1+\frac{z}{st^s\log t}\Big\}\,\mathrm{d}s\\
&\ll \frac{x^\kappa}{t^{\kappa+1}}\Big\{1+\int_{\kappa-i}^{\kappa+i}\frac{\min(|s|\log t,1)^{\Re z}}{|s|\log t}|\mathrm{d}s|\Big\} \ll \frac{x^\kappa}{t^{\kappa+1}}\cdot
\end{aligned}$$

Partial integration hence furnishes

$$\mathfrak{R}_{\mathcal{P}}(x,y;z) \ll \varepsilon_{\mathcal{P}}^*(y)\log u. \tag{2·15}$$

Now we know [6; (III.5.41)] that $\mathrm{e}^{-J(s)} = s\widehat{\varrho}(s)$, where

$$\widehat{\varrho}(s) := \int_0^\infty \varrho(v)\mathrm{e}^{-sv}\,\mathrm{d}v,$$

an entire function, is the Laplace transform of the Dickman function $\varrho(v)$. Therefore, assuming with no loss of generality that $x \in \frac12 + \mathbb{N}$, the main term, say M, in (2·14) satisfies

$$\begin{aligned}
M &:= \delta - \frac{\delta}{2\pi i}\int_{\kappa-i}^{\kappa+i}\{s\log y\}^z\widehat{\varrho}(s\log y)^z k(s)x^s\,\mathrm{d}s + O\Big(\frac{1}{\log x}\Big)\\
&= \delta - \frac{\delta}{2\pi i}\int_{1/u-i\log y}^{1/u+i\log y} w^{z-1}\widehat{\varrho}(w)^z k_y(w)\mathrm{e}^{uw}\,\mathrm{d}w + O\Big(\frac{1}{\log x}\Big),
\end{aligned} \tag{2·16}$$

where we have put

$$k_y(w) := 1 + \frac{\alpha w}{w+\log y} + \frac{\beta w}{w+2\log y}\cdot$$

The last integral may be evaluated on replacing the integration segment by a truncated Hankel contour around $\big[-\frac12, 1/u\big]$,(1) concatenated with two vertical segments $\big[-\frac12, -\frac12 \pm i\log y\big]$ and two horizontal segments $\big[-\frac12 \pm i\log y, 1/u \pm i\log y\big]$. Appealing for instance to [6; cor. II.0.18] to take care of the truncation, and noting that $\widehat{\varrho}(w) = \mathrm{e}^\gamma + O(w)$ for $w \ll 1$, we see that the contribution of the Hankel contour is

$$\frac{\mathrm{e}^{\gamma z}}{\Gamma(1-z)u^z} + O\Big(\frac{1}{u^{\Re z+1}}\Big).$$

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1. That is the path consisting of the circle $|w| = 1/u$ excluding the point $w = -1/u$ and the segment $[-1/2, -1/u]$ covered twice with respective arguments $+\pi$ and $-\pi$.

On the vertical parts of the contour, we have $w^z\widehat{\varrho}(w)^z \ll 1$ if $|\Im w| \leqslant 1$ and $w^z\widehat{\varrho}(w)^z = 1 + O(1/w)$ if $|\Im w| > 1$. The first range contributes $\ll \mathrm{e}^{-u/2}$ to the integral. The contribution of the second may be estimated on noting that, writing $I := [-\log y, -1] \cup [1, \log y]$, we have

$$\int_I \frac{\mathrm{e}^{-u/2+i\tau u}}{-1/2+i\tau}\,\mathrm{d}\tau = \int_1^{\log y} \frac{-\mathrm{e}^{-u/2}\{\cos(\tau u) + 2\tau\sin(\tau u)\}}{1/4+\tau^2}\,\mathrm{d}\tau \ll \mathrm{e}^{-u/2}$$

by the second mean-value theorem. Finally, the contribution of the horizontal parts is trivially

$$\ll \int_{-1/2}^{1/u} \frac{\mathrm{e}^{\sigma u}}{\log y}\,\mathrm{d}\sigma \ll \frac{1}{\log x}\cdot$$

Thus

$$M = \delta - \frac{\delta\mathrm{e}^{\gamma z}}{\Gamma(1-z)u^z} + O\Big(\frac{1}{u^{\Re z+1}} + \frac{1}{\log x}\Big).$$

Gathering our estimates, we arrive at

$$A(x,y;z) = \delta - \frac{\delta\mathrm{e}^{\gamma z}}{\Gamma(1-z)u^z} + O\Big(\varepsilon_{\mathcal{P}}^*(y)\log u + \frac{1}{u^{\Re z+1}} + \frac{u^r}{\log x}\Big).$$

Differentiating at $z = 1$ using Cauchy's integral formula, we get (2·6) as required since $\Re z \geqslant 4/5$, $r \leqslant 6/5$.

We now turn our attention to proving (2·7). To this end, we appeal to a standard Perron formula [6; th. II.2.3], viz.

$$A(x,y;z) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \mathcal{H}(s+1,y;z)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\sum_{P^-(n)\in\mathcal{Q}_y} \frac{\mu(n)^2 r^{\omega(n)}}{n^{\kappa+1}(1+T|\log(x/n)|)}\Big),$$

where $r = |z| \in [4/5, 6/5]$, $|z-1| \leqslant 1/5$. Those integers n such that $|\log(x/n)| > 1$ contribute $\ll u^r/(T\log x) \ll (\log x)^{r-1}/T$ to the error term. Splitting the summation range of the complementary sum into intervals $]x + hx/T, x + (h+1)x/T]$ $(|h| \leqslant T)$ and applying Shiu's theorem [5] for short sums of multiplicative functions, we obtain that it is $\ll (u^r\log T)/(T\log x) \ll (\log x)^{r-1}(\log T)/T$ provided, say, $2 \leqslant T \leqslant \sqrt{x}$.

Select $T := (\log x)^{r+1}$, so that, in view of hypothesis $\log y \geqslant (\log_2 x)^b$, we have $T \leqslant L_\varepsilon(y)$ for suitable $\varepsilon > 0$. This yields

$$\begin{aligned} A(x,y;z) &= \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \mathcal{H}(s+1,y;z)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{(\log x)^r}{T}\Big) \\ &= \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} B(s;y,z)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{1}{\log x}\Big), \end{aligned} \tag{2·17}$$

by (2·10).

From (2·13), we get

$$A(x,y;z) = \frac{\delta}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \big\{1 - \mathrm{e}^{-zJ(s_y)}\big\}\frac{x^s}{s}\,\mathrm{d}s + O\Big(\mathfrak{R}_{\mathcal{P}}^+(x,y;z) + \frac{1}{\log x}\Big), \tag{2·18}$$

with

$$\mathfrak{R}_{\mathcal{P}}^+(x,y;z) := \int_y^\infty \frac{\nu_x(t)}{t\log t}\,\mathrm{d}R(t), \quad \nu_x(t) := \int_{\kappa-iT}^{\kappa+iT} \mathrm{e}^{-zJ(s_t)}\Big(\frac{x}{t}\Big)^s\frac{\mathrm{d}s}{s}\cdot$$

Appealing to the estimates

$$\big|\mathrm{e}^{-zJ(s)}\big| \asymp \min\big(|s|^{\Re z}, 1\big), \quad \mathrm{e}^{-zJ(s)} = 1 + O(1/s) \quad (\Re z > 0,\ \Re s > 0),$$

we get, for $t > y$, keeping in mind the hypothesis $\log y \geqslant (\log_2 x)^b$,

$$\begin{aligned}\nu_x(t) &\ll \Big(\frac{x}{t}\Big)^\kappa \log_2 x,\\ \nu_x'(t) &= \int_{\kappa-iT}^{\kappa+iT} \mathrm{e}^{-zJ(s_t)}\frac{x^s}{t^{s+1}}\Big\{\frac{z}{t^s s\log t}-1\Big\}\,\mathrm{d}s\\ &= -\int_{\kappa-iT}^{\kappa+iT} \mathrm{e}^{-zJ(s_t)}\frac{x^s}{t^{s+1}}\,\mathrm{d}s + O\Big(\frac{x^\kappa\log_2 x}{t^{\kappa+1}\log y}\Big)\\ &= -\int_{\kappa-iT}^{\kappa+iT} \frac{x^s}{t^{s+1}}\,\mathrm{d}s + O\Big(\int_{\kappa-iT}^{\kappa+iT}\Big|\frac{x^s}{t^{s+1}s\log t}\Big|\,|\mathrm{d}s| + \frac{x^\kappa}{t^{\kappa+1}}\Big)\\ &\ll \frac{x^\kappa T}{t^{\kappa+1}(1+T|\log(x/t)|)}+\frac{x^\kappa}{t^{\kappa+1}},\end{aligned}$$

where we used the bound

$$\int_{\kappa-iT}^{\kappa+iT} w^s\,\mathrm{d}s \ll \frac{w^\kappa T}{1+T|\log w|}\qquad (w>0,\ \kappa>0,\ T\geqslant 1).$$

From the above estimates, we infer that $\mathfrak{R}^+_{\mathcal{P}}(x,y;z) \ll \varepsilon^*_{\mathcal{P}}(y)\log_2 x$.

Assuming as before that $x \in \frac12 + \mathbb{N}$, the main term M^+ in (2·18) satisfies

$$\begin{aligned}M^+ &= \delta - \frac{\delta}{2\pi i}\int_{1/u-iT\log y}^{1/u+iT\log y} w^{z-1}\widehat{\varrho}(w)^z\mathrm{e}^{uw}\,\mathrm{d}w + O\Big(\frac{1}{\log x}\Big)\\ &= \delta - \frac{\delta\mathrm{e}^{\gamma z}}{\Gamma(1-z)u^z} + O\Big(\frac{1}{u^{\Re z+1}}+\frac{1}{\log x}\Big),\end{aligned}$$

after deforming the integration segment and exploiting the relevant Hankel contour.

Finally, we may state that, for $|z-1|\leqslant 1/5$, we have

$$A(x,y;z) = \delta - \frac{\delta\mathrm{e}^{\gamma z}}{\Gamma(1-z)u^z} + O\Big(\varepsilon^*_{\mathcal{P}}(y)\log_2 x + \frac{1}{u^{9/5}}+\frac{1}{\log x}\Big).$$

Differentiating the above formula at $z = 1$ furnishes (2·7). □

We are now able to complete the proof of the effective estimate (1·4). This amounts to showing, that, in the stated range for y, we have

$$A^-(x,y) + A^+(x,y) \ll \varepsilon^*_{\mathcal{P}}(y)\log u + 1/u. \tag{2·19}$$

If $\log y \gg (\log x)^{6/11}$, then $u^{6/5}/\log x = (\log x)^{6/5}/\{u(\log y)^{11/5}\} \ll 1/u$, so (2·4) and (2·6) imply (2·19). If $\log y \ll (\log x)^{6/11}$, then $u \gg (\log x)^{5/11}$, so $\log_2 x \ll \log u$, and (2·19) follows from (2·4) and (2·7).

3. Special cases

We provide asymptotic formulae when $\mathcal{P}$ is either the set of all primes or a singleton. In these special cases, direct computations can be performed via standard applications of the Selberg-Delange method, furnishing improved estimates.(2) Consequently we only sketch the main lines.

Proposition 3.1. *We have*

$$V_1(x) := \sum_{n\leqslant x}\frac{\mu(n)\omega(n)}{n} \sim \frac{-1}{\log x}\qquad (x\to\infty). \tag{3·1}$$

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2. This is due in particular to the fact that estimates like (2·10), requiring information on the zeros of the zeta function, may be avoided.

Proof. Observe that, for $z \in \mathbb{C}$, $|z| \leqslant \frac{3}{2}$,

$$F_1(s,z) := \sum_{n \geqslant 1} \frac{\mu(n) z^{\omega(n)}}{n^s} = \prod_q \Big(1 - \frac{z}{q^s}\Big) = \frac{G_1(s,z)}{\zeta(s)^z},$$

with

$$G_1(s,z) := \prod_q \Big(1 - \frac{z}{q^s}\Big)\Big(1 - \frac{1}{q^s}\Big)^{-z}.$$

Hence

$$V_1(x;z) := \sum_{n \leqslant x} \frac{z^{\omega(n)}\mu(n)}{n} = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{G_1(s+1,z)}{\{s\zeta(s+1)\}^z} \frac{x^s}{s^{1-z}} \, \mathrm{d}s.$$

The main contribution arises from a Hankel contour around $[-c, 0]$ for arbitrary constant $c > 0$. By Hankel's formula, we get, as $x \to \infty$,

$$V_1(x;z) \sim \frac{G_1(1,z)}{\Gamma(1-z)(\log x)^z} = \frac{(1-z)G_1(1,z)}{\Gamma(2-z)(\log x)^z}\cdot$$

Hence

$$V_1(x) = \Big[\frac{\mathrm{d}V_1(x;z)}{\mathrm{d}z}\Big]_{z=1} \sim \frac{-G_1(1,1)}{\log x} = \frac{-1}{\log x},$$

as wanted. □

Next consider the case of $\mathcal{P}$ being reduced to a single element. Write

$$\zeta(s,y) := \prod_{q \leqslant y} \Big(1 - \frac{1}{q^s}\Big)^{-1} \qquad \big(\Re s > 0,\, y \geqslant 2\big).$$

Proposition 3.2. *Let $p \in \mathbb{P}$. We have*

$$V_p(x) := \sum_{\substack{n \leqslant x \\ P^-(n) = p}} \frac{\mu(n)\omega(n)}{n} \sim \frac{\zeta(1,p)}{p \log x} \qquad (x \to \infty). \tag{3·2}$$

Proof. Consider

$$F_p(s,z) := \sum_{\substack{n \geqslant 1 \\ P^-(n) = p}} \frac{\mu(n) z^{\omega(n)}}{n^s} = \frac{-z}{p^s} \prod_{q > p} \Big(1 - \frac{z}{q^s}\Big) = \frac{-z G_p(s,z)}{p^s \zeta(s)^z},$$

with now

$$G_p(s,z) := \prod_{q \leqslant p} \Big(1 - \frac{z}{q^s}\Big)^{-1} G_1(s,z).$$

It follows that, as $x \to \infty$,

$$\begin{aligned} V_p(x;z) :&= \sum_{\substack{n \leqslant x \\ P^-(n) = p}} \frac{\mu(n) z^{\omega(n)}}{n} = \frac{-z}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{G_p(s+1,z)}{\{s\zeta(s+1)\}^z} \frac{x^s}{p^s s^{1-z}} \, \mathrm{d}s \\ &\sim \frac{-z(1-z)G_p(1,z)}{\Gamma(2-z)(\log x/p)^z}\cdot \end{aligned}$$

Differentiating at $z = 1$ taking the zero of the numerator into account, we obtain (3·2). □

From the two propositions above, it follows that one cannot heuristically reconstruct (3·1) from (3·2). This phenomenon is similar to that arising from the formulae

$$\sum_{n\geqslant 1}\frac{\mu(n)\log n}{n}=-1,\qquad \sum_{\substack{n\geqslant 1\\ P^-(n)=p}}\frac{\mu(n)\log n}{n}=\frac{\zeta(1,p)}{p}\cdot \tag{3·3}$$

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