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Gérald Tenenbaum

Paul Erdős and the Divisors

Paul Erdős and I collaborated from March 1977 until his death (he called this a leave) in October 1996. During these nineteen years, most of our mathematical discussions were on the distribution of divisors of integers.

Analytic and probabilistic number theory is mainly concerned with understanding how the multiplicative and the additive structure of integers combine together or ignore each other. The twin primes conjecture and the Goldbach conjecture are enlightening examples: since nothing trivially forbids two primes from having difference two, this should happen with statistical frequency; similarly, if $2n$ is an arbitrary integer, the sequence $2n - p$, where p runs through all primes up to n, should contain a random quota of primes.

The famous *abc* conjecture of Masser and Oesterlé, dating from the early 1970s, is even more representative of this class of problems: addition should destroy multiplicative structure. Let us discuss this in more detail. The squarefree kernel, say $k(n)$, of an integer n is the product of all primes appearing in the canonical decomposition of n, ignoring exponents. A normal integer (in other words an integer belonging to a set that will almost surely show up when one picks an integer at random) has a "large" kernel (see [31] for recent progress on this question), whereas small kernels occur only for integers with a very special structure such as perfect powers or integers divisible by a large perfect power. In qualitative form, the *abc* conjecture states that when a and b are coprime and $c = a + b$, the three integers cannot be simultaneously abnormal. A thorough quantitative discussion of this question may be found in [30].

The distribution of divisors is a problem of similar type to those described above: a divisor has a very special multiplicative structure, since its prime factors, and even their exponents, are restricted in a drastic manner. Thus it is a basic number theoretic challenge to try to understand how the sequence of divisors is distributed by size, that is to say, with respect to additive structure.

Erdős was interested in all aspects of this question: usual behavior of the sequence of divisors of a random, or normal, integer; extremal properties involving divisors, that is, small and large values of arithmetic functions defined in terms of divisors; structure of sequences defined by constraints on the divisors of their elements; stochastic variations of divisor functions; etc.

It is clear that the divisors are made up from the prime factors. So the first step in the problem is to describe the growth of the sequence of prime factors. The step Erdős made was that of a giant. Before anyone else, he understood that basic results in probabilistic number theory, such as the Turán-Kubilius inequality,[1] yield a very strong and very surprising fact: in first approximation, the size of the jth prime factor of a normal integer n does not depend on n but only on j; more precisely, if we let $\{p_j(n)\}_{j=1}^{\omega(n)}$ denote the increasing sequence of the distinct prime factors of n, then we have

$$\log_2 p_j(n) \sim j \qquad (j \to \infty) \tag{1}$$

for almost all integers n.[2] The short proof may be retrieved in Erdős's paper [7]. It will not be reproduced here: we content ourselves with saying that the Turán-Kubilius inequality is the arithmetical analogue of the theorem from probability theory stating that the variance of a sum of independent random variables is the sum of the variances of its terms and that Erdős merely uses a Bienaymé-Chebyshev-type inequality to deduce his result. A much sharper result, actually exhibiting a Gaussian behavior of the prime factors around their means, is obtained in [5]. The reader may consult [18] (Chapter 1) for a detailed proof and [39] (Theorem III.3.10) for a simpler proof of a slightly weaker result.

From the above statement on prime factors, one could guess that the jth divisor of a normal integer is somewhat close to $\exp j^c$ with $c = 1/\log 2$. It was one of Erdős's outstanding qualities to devise a question that would precisely test how adequate the model is to the arithmetic nature of things. Since $c > 1$, we may deduce from the previous heuristics that the minimal ratio $E(n)$ between consecutive divisors, say d'/d, with $d < d'$, does not tend to 1 for almost all integers. However, as early as 1948 and probably much before, Erdős

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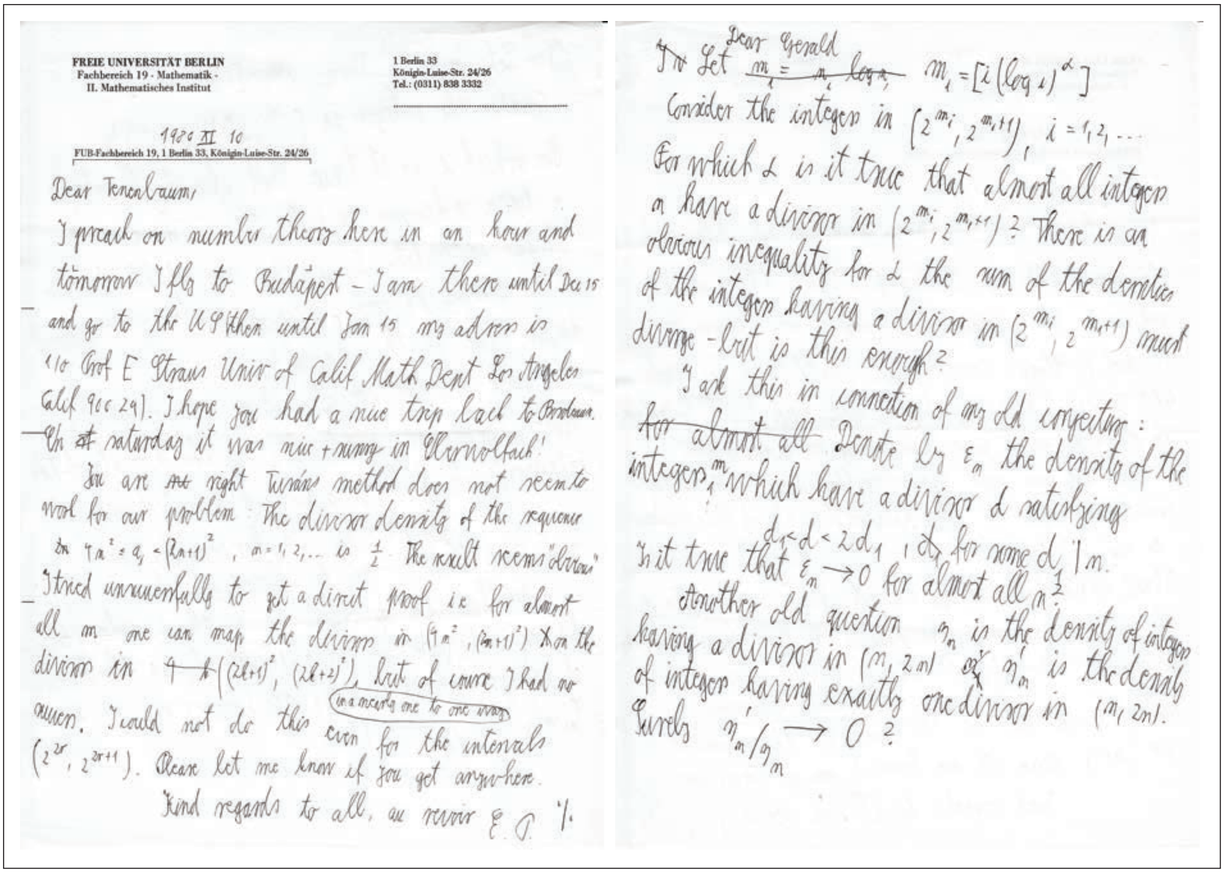
[1] *See, e.g., Elliott* [4] *or Tenenbaum* [39].
[2] *Here and in the sequel we denote by* $\log_k$ *the k-fold iterated logarithm.*



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Dear Tenenbaum,

I preach on number theory here in an hour and tomorrow I fly to Budapest – I am there until Dec 15 and go to the US then until Jan 15 my adress is c/o Prof E Straus Univ of Calif Math Dept Los Angeles Calif 90024. I hope you had a nice trip back to Bordeaux. On saturday it was nice + sunny in Oberwolfach!

You are right Turán's method does not seem to work for our problem. The divisor density of the sequence $4n^2 < a_i < (2n+1)^2$, $n = 1, 2, \ldots$ is $\frac{1}{2}$. The result seems "obvious". I tried unsuccessfully to get a direct proof i.e. for almost all m one can map the divisors in $(4n^2, (2n+1)^2)$ (in a nearly one to one way) on the divisors in $((2k+1)^2, (2k+2)^2)$, but of course I had no success. I could not do this even for the intervals $(2^{2r}, 2^{2r+1})$. Please let me know if you get anywhere.

Kind regards to all, au revoir E. P.

Dear Gerald
Let $m_i = [i(\log i)^\alpha]$
Consider the integers in $(2^{m_i}, 2^{m_i+1})$, $i = 1, 2, \ldots$
For which α is it true that almost all integers n have a divisor in $(2^{m_i}, 2^{m_i+1})$? There is an obvious inequality for α the sum of the densities of the integers having a divisor in $(2^{m_i}, 2^{m_i+1})$ must diverge – but is this enough?

I ask this in connection of my old conjecture: Denote by ε_n the density of the integers m which have a divisor d satisfying $d_1 < d < 2d_1$, d_1 for some $d_1 | n$. Is it true that $\varepsilon_n \to 0$ for almost all n?

Another old question η_n is the density of integers having a divisor in $(n, 2n)$ & η'_n is the density of integers having exactly one divisor in $(n, 2n)$. Surely $\eta'_n/\eta_n \to 0$?

Letter from Erdős to Tenenbaum stating a problem on divisor density.

conjectured the exact opposite: for almost all n, one should have

(2) $E(n) := \min_{dd'|n,\, d<d'} d'/d = 1 + (\log n)^{1-\log 3+o(1)}.$

The reason for this is precisely that the distribution of the prime factors fluctuates significantly around the mean and hence the quantities $\log d'/d$ should be fairly evenly distributed in the interval $[-\log n, \log n]$. Since these quantities are at least $3^{\omega(n)}$ in number and, by the Turán-Kubilius inequality or indeed from (1), we have $\omega(n) \sim \log_2 n$ for almost all n, we naturally arrive at (2). This conjecture was proved by Erdős-Hall [8] for the lower bound and by Maier-Tenenbaum [24] for the upper bound. A further refinement [29] comes even closer to the heuristics: for almost all n, we have

(3) $$E(n) = 1 + \frac{\log n}{3^{\omega(n)}}(\log_2 n)^{\vartheta_n}$$

where $-5 \leqslant \vartheta_n \leqslant 10$.

One of the most challenging problems remaining on the so-called subject of propinquity of divisors concerns the functions

$$E_r(n) := \min_{1\leqslant j\leqslant \tau(n)-r} \log\{d_{j+r}(n)/d_j(n)\} \qquad (r \geqslant 2),$$

where $\{d_j(n)\}_{j=1}^{\tau(n)}$ denotes the increasing sequence of the divisors of n. The precise normal behavior remains still unknown for all $r \geqslant 2$. Using techniques similar to that of the proof of Theorem 3 of [12], it can be shown that, on a sequence of natural density 1, we have

$$E_2(n) > (\log n)^{-\gamma_2+o(1)}$$

for some $\gamma_2 < \log 3 - 1$. Moreover, the methods and results of [25] yield, still normally,

$$E_r(n) \leqslant (\log n)^{-\beta_r+o(1)},$$

with

$$\beta_r := \frac{(\log 3 - 1)^m}{(3\log 3 - 1)^{m-1}}, \quad 2^{m-1} < r + 1 \leqslant 2^m.$$

Thus, we have

$$\beta_1 = \log 3 - 1 \approx 0.09861, \quad \beta_2 = \beta_3 \approx 0.00423,$$
$$\beta_r \approx 0.00018 \quad (4 \leqslant r \leqslant 7).$$

Also, it is proved in [25] (Thm. 1.1) that $E_r(n) > \tau(n)^{-1/r+o(1)}$ holds for almost all integers, uniformly in $r \geqslant 1$, and thus, on a sequence of density 1,

$$E_r(n) = 1/(\log n)^{o(1)} \qquad (r = r(n) \to \infty),$$

a result which might look surprising at first sight.

Maier and I conjecture the existence of a strictly decreasing sequence $\{\alpha_r\}_{r=1}^{\infty}$ such that we have

$$E_r(n) = (\log n)^{-\alpha_r+o(1)}$$

on a sequence of density 1. It is particularly irritating, for instance, to be unable to find a better normal upper bound for $E_2(n)$ than for $E_3(n)$.

I refer the reader to the recent survey [40] for a further account of Erdős's motivations for the conjecture (2), in particular related to the concept of set of multiples. The link is particularly apparent in Erdős's letter dated November 10, 1980, reproduced herein.

On page 1, Erdős mentions *divisor density*. The definition, due to R. R. Hall [15], is as follows. Let $\tau(n, \mathcal{A})$ designate the number of divisors of an integer n belonging to a sequence $\mathcal{A}$, and write $\tau(n) = \tau(n, \mathbb{Z}^+)$. We say that the integer sequence $\mathcal{A}$ has divisor density z, and we write $D\mathcal{A} = z$ if we have $\tau(n, \mathcal{A}) = \{z + o(1)\}\tau(n)$ as n tends to infinity on a sequence of natural density 1.[3]

Divisor density is a fruitful and surprising notion. For instance, Hall proved in [15] that, for any pair $(z, w) \in [0, 1]^2$, there is an integer sequence $\mathcal{A}$ with divisor density z and logarithmic density w.[4] A criterion for $D\mathcal{A} = z$ is given in [33].

With the above definition, Erdős's question on page 1 may be stated as follows: define $\mathcal{A} := \bigcup_{m\geqslant 1}[4m^2, (2m+1)^2[\cap\mathbb{Z}^+$; is it true that $\mathcal{D}\mathcal{A} = \frac{1}{2}$? It needed a lot of work and the appeal to many deep results from analytic number theory, such as estimates of Karatsuba on exponential sums [22], to answer, positively, Erdős's question; see Hall-Tenenbaum [17] and Tenenbaum [38] (Theorem 11). To be slightly more precise, we observe that the sequence $\mathcal{A}$ may alternatively be defined by the condition $\left\langle \frac{1}{2}\sqrt{n} \right\rangle \leqslant \frac{1}{2}$ where $\langle x \rangle$ denotes the fractional part of the real number x. The two theorems above actually imply that, for any real number c, any $z \in [0, 1]$, and any nonintegral positive number α, the sequence $\{n \geqslant 1 : \langle cn^{\alpha} \rangle \leqslant z\}$ has divisor density z.

On page 2 of the reproduced letter, Erdős asks a slightly different question: given a sequence $m_j := \lfloor j(\log j)^{\alpha} \rfloor$, and setting $\mathcal{A} := \bigcup_{j\geqslant 1}[2^{m_j}, 2^{m_j+1}[$, for which α do we have $\tau(n, \mathcal{A}) \geqslant 1$ on a sequence of asymptotic density 1? Here it is clear, and Erdős explicitly notes the fact that the problem is linked with conjecture (2): it deals with the distribution of divisors in dyadic intervals.

Once again, it took years of struggle to solve the problem: as proved in [37], the answer is positive for all α. As it turns out, one needs to take faster growing sequences to see a threshold: if we now define $m_j = j^{\beta}$, then Hall-Tenenbaum proved in 1992 [19] that the answer to Erdős's question is affirmative if, and only if, $\beta \leqslant 1/(1 - \log 2)$.

Aside from his interest in the normal behavior of the set of divisors, Erdős was also intrigued by extremal properties. Out of many of his problems, I extract two.

To describe the first, let us put, for $\alpha > 0$,

$$F_\alpha(n) := \sum_{1\leqslant j<\tau(n)} \Big(\frac{d_{j+1}(n)}{d_j(n)} - 1\Big)^{\alpha}.$$

The conjecture asserted that, for all $\alpha > 1$,

$$\liminf_{n\to\infty} F_\alpha(n) < \infty. \tag{4}$$

Since $F_1(n) \geqslant \log n$, it is clear that the condition $\alpha > 1$ cannot be weakened. This conjecture may be seen as dual to another conjecture of Erdős, related to the sequence $\{a_j\}_{j=1}^{\varphi(n)}$ of integers in $[1, n]$ and coprime to n. The problem here was to show that, for all $\gamma > 0$, we have

$$\limsup_{n\to\infty} \frac{\varphi(n)^{\gamma-1}}{n^{\gamma}} \sum_{1\leqslant j<\varphi(n)} (a_{j+1}(n) - a_j(n))^{\gamma} < \infty.$$

Improving a result of Hooley [20] disposing of the case $\gamma < 2$, Montgomery and Vaughan confirmed this conjecture in 1986 [27].

These two problems are specific to Erdős's particular way of thinking. He manufactured innocent-looking questions whose solution actually requires a deep understanding of the structure of integers defined by multiplicative constraints.

Conjecture (4) was solved by Vose in 1984. Since, from Holder's inequality, we have

$$(\log n)^{\alpha} \leqslant F_\alpha(n)\tau(n)^{\alpha-1},$$

candidates to bounded values of $F_\alpha(n)$ must have a large number of divisors.[5] This led Erdős to the further conjecture that F_α should be bounded by natural sequences with many divisors, such as $n!$, l.c.m.$\{1, 2, \ldots, n\}$, or $\prod_{p\leqslant n} p$. I could establish this in [35] as a consequence of a more general result proved by the saddle-point method.

The second extremal problem was asked by Erdős on numerous occasions and is referred to as Problem 23 in the appendix of Montgomery's book [28]. It states that, for suitable constant C, the inequality

$$\sum_{d|n,\, t|n,\, d<t} \frac{1}{t-d} \leqslant C\tau(n) \tag{5}$$

---

[3] *The natural density of an integer sequence is, when it exists, the limit, as $N \to \infty$, of the frequency of $\mathcal{A}$ among the N first integers.*

[4] *The logarithmic density of an integer sequence $\mathcal{A}$ is, when it exists, the value of the limit*

$$\delta(\mathcal{A}) := \lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leqslant N,\, n\in\mathcal{A}} \frac{1}{n}.$$

[5] *Recall that a normal integer n has about $(\log n)^{\log 2+o(1)}$ divisors.*

should hold for all positive integers n. Here again, the aim is to test the lacunarity of the sequence of the divisors of an integer: despite the fact that d and t in the above sum may get fairly close, we expect that this happens sufficiently rarely so that (5) remains true. Thus we are really facing a sieve problem of a delicate nature. I could establish (5) in a strong form, improved by La Bretèche [1].

Erdős's interest in divisors was so constant and so intense that a whole book would be necessary to describe his problems, attempts at solutions, and original methods on this topic—and the two already written books, [18] and [16], largely dominated by the work of Erdős, would only be a small part of the story. The references of this short survey constitute an incomplete and partial list of articles related to the subject, either by Erdős himself and his collaborators or inspired by his appealing way of thinking of mathematical problems.

Acknowledgments

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AN INTERPOLATION PROBLEM ASSOCIATED WITH THE CONTINUUM HYPOTHESIS

P. Erdös

In the Ann Arbor Problem Book, Wetzel asked (under the date December, 1962) the following question: Let $\{f_\alpha\}$ be a family of analytic functions such that for each z the set of values $f_\alpha(z)$ is countable (we shall call this property P_0). Does it then follow that the family itself is countable?

An unsigned comment points out that if "countable" is replaced with "finite" both in the hypothesis and in the conclusion, then the result follows easily. R. C. Lyndon has remarked that if "analytic" is replaced with "infinitely differentiable," one can easily give $\mathfrak{c}$ functions f_α $(1 \leq \alpha < \Omega_{\mathfrak{c}})$ such that, for each z, the set $\{f_\alpha(z)\}$ contains only two values.

We shall show that the answer to Wetzel's question depends on the continuum hypothesis.

THEOREM. *If* $\mathfrak{c} > \aleph_1$, *then every family* $\{f_\alpha\}$ *with property* P_0 *is denumerable. If* $\mathfrak{c} = \aleph_1$, *some family* $\{f_\alpha\}$ *with property* P_0 *has the power* $\mathfrak{c}$. (I have been informed that R. D. Dixon proved the first part of the theorem last year.)

Figure 1. The first few paragraphs of Erdős's paper [6].

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Stephan Ramon Garcia and Amy L. Shoemaker

Wetzel's Problem, Paul Erdős, and the Continuum Hypothesis: A Mathematical Mystery

We are concerned here with the curious history of *Wetzel's problem*: If $\{f_\alpha\}$ is a family of distinct analytic functions (on some fixed domain) such that for each z the set of values $\{f_\alpha(z)\}$ is countable, is the family itself countable?

In September 1963, Paul Erdős submitted to the *Michigan Mathematical Journal* a stunning solution to Wetzel's problem (Figure 1). He proved that an affirmative answer is equivalent to the negation of the continuum hypothesis. Erdős ends in an understated manner: "Paul Cohen's recent proof of the independence of the continuum hypothesis gives this problem some added interest." Together these results render Wetzel's problem undecidable in ZFC.

Erdős had a knack for solving "innocent-looking problems whose solutions shed light on the shape of the mathematical landscape" [7, p. 2]. In this case, the landscape he revealed was one of underground tunnels, surprising links, and glittering mysteries. However, our interest lies not with the solution itself but rather with the story of how Erdős encountered Wetzel's problem in the first place.

Our first exposure to Wetzel's problem was in *Proofs from The Book* by Aigner and Ziegler. "Paul Erdős liked to talk about The Book," they write, "in which God maintains the perfect proofs for mathematical theorems, following the dictum of G. H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book" [1]. Erdős asked Aigner and Ziegler to assemble a moderate approximation of The Book; included in it was Erdős's answer to Wetzel's problem [1, pp. 102–6].

Regarding the origin of the problem, Erdős simply asserted that Wetzel posed the question in the Ann Arbor Problem Book in December 1962. Ziegler suggested several mathematicians who might be the Wetzel in question, eventually putting us in contact with John E. Wetzel (professor emeritus at the University of Illinois Urbana-Champaign), who confirmed that the problem was indeed his.

John Wetzel, born on March 6, 1932, in Hammond, Indiana, earned a B.S. in mathematics and physics from Purdue University in 1954 and went on to study mathematics at Stanford University (see Figure 2). While studying spaces of harmonic functions on Riemann surfaces under Halsey Royden, he posed the following question in his dissertation:

> Let V be a collection of harmonic functions on a Riemann surface R such that for each point p of R the set $V_p = \{v(p) : v \in V\}$ is countable. Must V then be countable? [9, p. 98]

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