Acta Arithmetica **208**
(2023), 279-283.

Note on a conjecture of Hildebrand regarding friable integers*

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Abstract. Hildebrand proved that the smooth approximation for the number $\Psi(x,y)$ of y-friable integers not exceeding x holds for $y > (\log x)^{2+\varepsilon}$ under the Riemann hypothesis and he conjectured that it fails when $y \leqslant (\log x)^{2-\varepsilon}$. This conjecture has been recently confirmed by Gorodetsky by an intricate argument. We propose a short, straight-forward proof.

Keywords: Friable integers, Dickman's function, saddle-point method, Riemann hypothesis.
2020 Mathematics Subject Classification: primary 11N25.

Let $\Psi(x,y)$ denote the number of y-friable integers not exceeding x, i.e. integers $n \leqslant x$ free of prime factors $> y$, and let ϱ denote Dickman's function, defined as the continuous solution to the delay differential equation

$$v\varrho'(v) + \varrho(v-1) = 0 \qquad (v > 1)$$

with initial condition $\varrho(v) = 1$ $(0 \leqslant v \leqslant 1)$. For supplementary material on this function, see, e.g. [8; ch. III.5 & III.6]. In [4], Hildebrand proved that, for any given $\varepsilon > 0$ and with $u := (\log x)/\log y$, we have

$$\Psi(x,y) \sim x\varrho(u) \qquad \big(x \to \infty,\ \exp\{(\log_2 x)^{5/3+\varepsilon}\} \leqslant y \leqslant x\big). \tag{1}$$

Here and in the sequel, $\log_k$ denotes the k-fold iterated logarithm. Moreover, in the same paper Hildebrand showed that, under the Riemann hypothesis, (1) persists for $y > (\log x)^{2+\varepsilon}$ in the form $\Psi(x,y) \asymp x\varrho(u)$ and he expressed the belief that this fails for $y \leqslant (\log x)^{2-\varepsilon}$.

This latter precision has recently been established by Gorodestky [3] through a rather involved approach. In this note we present a very short, straightforward proof.

In the following statement, $\mathcal{D}$ stands for an infinite set of real numbers tending to infinity and, for $x \geqslant y \geqslant 2$, the quantity $\alpha = \alpha(x,y)$ is defined by the equation

$$\sum_{p \leqslant y} \frac{\log p}{p^\alpha - 1} = \log x.$$

From [5; (7.6)], we have

$$\alpha = \frac{\log\big(1 + y/\log x\big)}{\log y}\Big\{1 + O\Big(\frac{1}{\log y}\Big)\Big\} \qquad \big(2 \leqslant y \leqslant (\log x)^2\big), \tag{2}$$

so that $\alpha = 1 - 1/c + o(1)$ if $c > 1$.

Theorem. *Let $c \in]0,2[$. For $y \geqslant 2$, put $x := \exp\big\{\sum_{p \leqslant y}(\log p)/(p^{1-1/c} - 1)\big\}$ if $c > 1$, and $x := \exp y^{1/c}$ otherwise, so that $y = (\log x)^{c+o(1)}$ in all cases. We then have:*

$$\frac{\Psi(x,y)}{x\varrho(u)} \begin{cases} \geqslant \exp\Big\{\big(\tfrac12 + o(1)\big)\dfrac{y^{2/c-1}}{(2/c-1)\log y}\Big\} \quad (y \in \mathcal{D}) & \text{if } 1 < c < 2 \\ = \exp\Big\{\dfrac{(\log 4 - 1)\log x}{\log_2 x} + O\Big(\dfrac{\log x}{(\log_2 x)^2}\Big)\Big\} & \text{if } c = 1, \\ = x^{1/c-1+o(1)} & \text{if } 0 < c < 1. \end{cases} \tag{3}$$

Proof. Consider first the case $1 < c < 2$, so that $\alpha = 1 - 1/c$. Let $\xi = \xi(u)$ denote the unique non-zero solution to the equation $\mathrm{e}^\xi = 1 + u\xi$ when $u \neq 1$ and put $\xi(1) := 0$. From [8; th. III.5.13] and [8; (III.5.110)], we have, as $u \to \infty$,

$$\xi(u) = \log u + \log_2 u + \frac{\log_2 u}{\log u} + O\Big(\frac{(\log_2 u)^2}{(\log u)^2}\Big), \tag{4}$$

$$\varrho(u) \sim \frac{1}{\sqrt{2\pi u}}\mathrm{e}^{\gamma - u\xi(u) + \int_1^u t\xi'(t)\,\mathrm{d}t}, \tag{5}$$

* We are indebted to Ofir Gorodestky for pointing out that some changes to the published version were necessary. These are included here.

and, writing $\beta := 1 - \xi(u)/\log y$, $I(s) := \int_0^s (\mathrm{e}^v - 1)\,\mathrm{d}v/v$, $f(\sigma) := \sigma \log x + I\big((1-\sigma)\log y\big)$,

$$f(\beta) = \log x - u\xi(u) + \int_1^u t\xi'(t)\,\mathrm{d}t.$$

Let $\varepsilon \in]0,1[$, and put $L(y) := \mathrm{e}^{(\log y)^{3/5-\varepsilon}}$. We have $f'(\beta) = 0$ and $f''(\sigma) \geqslant 0$ $(0 \leqslant \sigma \leqslant 1)$, and $\alpha - \beta \ll 1/L(y)$ by [5; (7.9)]. Therefore $f(\alpha) \geqslant f(\beta)$. Now, the saddle-point estimate proved in [5] implies $\Psi(x,y) \sim x^\alpha \zeta(\alpha,y)/\{\alpha(\log y)\sqrt{2\pi u}\}$ in the case under consideration, where $\zeta(s,y) := \prod_{p\leqslant y}(1-1/p^s)^{-1}$. It follows that

$$\frac{\Psi(x,y)}{x\varrho(u)} \geqslant \Big\{\frac{c\mathrm{e}^{-\gamma}}{c-1} + o(1)\Big\}\mathrm{e}^{\alpha\log x + \log\zeta(\alpha,y) - f(\alpha)} = \Big\{\frac{c\mathrm{e}^{-\gamma}}{c-1} + o(1)\Big\}\mathrm{e}^{\log\zeta(\alpha,y) - I((1-\alpha)\log y)}. \tag{6}$$

It remains to estimate the last exponent. Plainly

$$\log\zeta(\alpha,y) \geqslant \sum_{p\leqslant y}\frac{1}{p^\alpha} + \tfrac12\sum_{p\leqslant y}\frac{1}{p^{2\alpha}} =: S(\alpha,y) + \tfrac12 T(\alpha,y).$$

Put $w_\sigma := (y^{1-\sigma}-1)/\{(1-\sigma)\log y\}$. In [1; lemma 3.6], it has been shown that $T(\alpha,y) \sim w_{2\alpha}$ and that the main term for $S(\alpha,y)$ arising from the prime number theorem is

$$\int_2^y \frac{\mathrm{d}\,\mathrm{li}(t)}{t^\alpha} = \log(w_\alpha \log y) + \int_1^{w_\alpha} t\xi'(t)\,\mathrm{d}t + O(1) = I\big((1-\alpha)\log y\big) + O(\log u).$$

We shall establish the oscillation result

$$S(\alpha,y) - I\big((1-\alpha)\log y\big) = \Omega_\pm\Big(\frac{y^{1/2-\alpha}\log_3 y}{\log y}\Big). \tag{7}$$

Since $\alpha < \frac12$, this implies the first estimate in (3).

Let us now prove (7). Put

$$R(t) := \psi(t) - t, \quad \Pi(t) := \sum_{1<n\leqslant t}\frac{\Lambda(n)}{\log n}, \quad Q(t) := \Pi(t) - \mathrm{li}(t) \qquad (t \geqslant 2).$$

Furthermore, let $\Theta := \sup_\varrho \Re\varrho$, where ϱ runs over the non trivial zeros of the Riemann zeta function.

If $\Theta > \frac12$, the argument of [7; th. 15.2] readily yields, for any given $\varepsilon > 0$,

$$\int_1^y \frac{\mathrm{d}Q(t)}{t^\alpha} = \Omega_\pm\big(y^{\Theta-\alpha-\varepsilon}\big),$$

which readily implies (7) in view of the estimate

$$\int_2^y \frac{\mathrm{d}\{\pi(t)-\mathrm{li}(t)\}}{t^\alpha} = \int_2^y \frac{\mathrm{d}Q(t)}{t^\alpha} + O\Big(\frac{y^{1/2-\alpha}}{\log y}\Big) \qquad (y \geqslant 2). \tag{8}$$

If $\Theta = \frac12$, we have

$$\int_1^y \frac{\mathrm{d}Q(t)}{t^\alpha} = \int_1^y \frac{\mathrm{d}R(t)}{t^\alpha \log t} = \frac{R(y)}{y^\alpha \log y} + J(y) \qquad (y \geqslant 2), \tag{9}$$

with

$$\begin{aligned} J(y) &:= \int_{3/2}^y \frac{R(t)}{t^{\alpha+1}\log t}\Big(\alpha + \frac{1}{\log t}\Big)\,\mathrm{d}t \\ &= \int_{3/2}^y \frac{R(t)}{\sqrt t}\int_t^\infty \Big(\alpha(\alpha+\tfrac12) + \frac{2\alpha+1/2}{\log v} + \frac{2}{(\log v)^2}\Big)\frac{\mathrm{d}v\,\mathrm{d}t}{v^{\alpha+3/2}\log v} \\ &= \int_{3/2}^\infty \Big(\alpha(\alpha+\tfrac12) + \frac{2\alpha+1/2}{\log v} + \frac{2}{(\log v)^2}\Big)\frac{\mathrm{d}v}{v^{\alpha+3/2}\log v}\int_{3/2}^{\min(v,y)}\frac{R(t)}{\sqrt t}\,\mathrm{d}t \\ &\ll \int_{3/2}^\infty \frac{\min(v,y)\,\mathrm{d}v}{v^{\alpha+3/2}\log v} \ll \frac{y^{1/2-\alpha}}{\log y}, \end{aligned}$$

where the penultimate bound follows from Cramér's estimate [2] (see also [7; th. 13.5]) conditional to the Riemann hypothesis

$$\int_1^z \frac{R(t)^2}{t}\,\mathrm{d}t \ll z \qquad (z \geqslant 2).$$

Since as, shown by Littlewood [6],

$$R(y) = \Omega_\pm\big(\sqrt y \log_3 y\big) \qquad (y \to \infty),$$

this completes the proof of (7) by carrying back into (9) taking (8) into account.

The case $c \leqslant 1$ may be dealt with by appealing to the formula

$$\log\{x\varrho(u)\} = \frac{c-1}{c}\log x + \frac{(1+\log c)\log x}{c\log_2 x} + \frac{(\log x)(1-\log c)}{c(\log_2 x)^2} + O\Big(\frac{(\log x)(\log_3 x)^2}{(\log_2 x)^3}\Big). \tag{10}$$

This follows from (4) and the estimate, essentially due to de Bruijn,

$$\log\Psi(x,y) = Z(x,y)\Big\{1 + O\Big(\frac{1}{\log y} + \frac{1}{\log_2 x}\Big)\Big\} \qquad (x \geqslant y \geqslant 3), \tag{11}$$

appearing in [8; th. III.5.2] in this form. Note that

$$Z(x,y) = \begin{cases} \dfrac{c-1}{c}\log x + \dfrac{\log x}{c\log_2 x} + \dfrac{(\log x)^{2-c}}{2c\log_2 x} + O\Big(\dfrac{(\log x)^{3-2c}}{\log_2 x}\Big) & \text{if } c > 1 \\ \dfrac{(\log 4)\log x}{\log_2 x} & \text{if } c = 1 \\ \dfrac{(1-c)(\log x)^c}{c} + \dfrac{(\log x)^c}{c\log_2 x} + O\Big(\dfrac{(\log x)^{2c-1}}{\log_2 x}\Big) & \text{if } c < 1. \end{cases}$$

This readily implies the last two estimates in (3). □

References

[1] R. de la Bretèche & G. Tenenbaum, Propriétés statistiques des entiers friables, *Ramanujan J.* **9** (2005), 139–202.
[2] H. Cramér, Über die Herleitung der Riemannschen Primzahlformel, *Arkiv för Mat., Astr. och Fys.* **13** (1918), 24.
[3] O. Gorodetsky, Friable integers and the Dickman ϱ function, preprint, 2022:arXiv:2211.08973v2.
[4] A. Hildebrand, On the number of positive integers $\leqslant x$ and free of prime factors $> y$, *J. Number theory* **22**,n° 3 (1986), 289–307.
[5] A. Hildebrand & G. Tenenbaum, On integers free of large prime factors, *Trans. Amer. Math. Soc.* **296** (1986), 265–290.
[6] J.E. Littlewood, Sur la distribution des nombres premiers, *C. R. Acad. Sci. Paris* **158** (1914), 1869-1872.
[7] H.L. Montgomery & R.C. Vaughan, *Multiplicative number theory, I. Classical theory*, Cambridge studies in advanced mathematics 97, 2007.
[8] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Graduate Studies in Mathematics 163, Amer. Math. Soc. 2015; see also *Introduction à la théorie analytique et probabiliste des nombres*, fifth ed., Dunod, Sciences Sup, 2022.

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