Mean values of arithmetic functions and application to sums of powers

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Abstract. We provide new upper bounds for sums of certain arithmetic functions in many variables at polynomial arguments and, exploiting recent progress on the mean-value of the Erdős-Hooley Δ-function, we derive lower bounds for the cardinality of those integers not exceeding a given limit that are expressible as certain sums of powers.

Keywords: Erdős-Hooley Δ-function, sieve, sums of powers, arithmetic functions of many variables, mean-values of arithmetic functions.

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1. Introduction

The Erdős-Hooley Δ-function is defined by the formulae

$$\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \quad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u) \qquad (n \in \mathbb{N}^*).$$

Recent, spectacular progress on the mean-value of the Δ-function has been obtained in [17], [9], and refined in [4]. This invites one to revisit some of the applications linked to this quantity. Put

$$S(x) := \sum_{n\leqslant x} \Delta(n), \quad \mathfrak{S}(x) := \frac{1}{\log x}\sum_{n\leqslant x}\frac{\Delta(n)}{n},$$

so that (see [12; th. 61])

$$S(x) \ll x\mathfrak{S}(x) \qquad (x \geqslant 2).^{(1)}$$

After the works of Erdős [8], Hall-Tenenbaum [11], [12], Hooley [15], Tenenbaum [28], Koukoulopoulos-Tao [17], Ford-Koukoulopoulos-Tao [9], and La Bretèche-Tenenbaum [4], we are now equipped with the bounds

$$(\log_2 x)^{1+\eta+o(1)} \ll \mathfrak{S}(x) \ll (\log_2 x)^{5/2} \qquad (x \geqslant 3),^{(2)} \tag{1·1}$$

where $\eta \approx 0.353327$ is the exponent appearing in the new lower bound for the normal order of $\Delta(n)$ by Ford, Green and Koukoulopoulos [7]. A further description of successive advances is available in the article [4].

The fact that the asymptotic order of magnitude of $\mathfrak{S}(x)$ is a power of $\log_2 x$ makes it pertinent to refine the estimates relating to the various applications, as the number of representations of an integer as a sum of powers, the Diophantine approximation of an irrational number by a rational with square denominator, or Waring's problem.

Two other applications are also worth mentioning. First note that [19; cor. 8] readily yields that, for any given $\varepsilon > 0$, and uniformly for $x \geqslant 3$, $1 \leqslant |a| \leqslant x$, $x^\varepsilon < y \leqslant x$, we have

$$\sum_{x<p\leqslant x+y} \Delta(|p+a|) \ll \frac{|a|y\mathfrak{S}(x)}{\varphi(|a|)\log x} \ll \frac{|a|y(\log_2 x)^{5/2}}{\varphi(|a|)\log x},$$

where the letter p denotes a prime number. Second, observe that the upper bound of (1·1) furnishes, for any fixed $\varepsilon > 0$ and uniformly for $1 \leqslant a \leqslant q \leqslant x^{2-\varepsilon}$, $(a,q) = 1$, $x \geqslant 3$, the estimate

$$\sum_{\substack{n,m\leqslant x\\ mn\equiv a \bmod q}} 1 \ll \frac{x^2}{q}(\log_2 x)^{5/2}. \tag{1·2}$$

Indeed, if $N \leqslant x^{1-\varepsilon/2}$, a trivial bound, using $\tau(n) = n^{o(1)}$ suffices, and if $x^{1-\varepsilon/2} < N \leqslant x$, we have

$$\sum_{\substack{m\leqslant x\\ N<n\leqslant 2N\\ mn\equiv a \bmod q}} 1 \leqslant \sum_{0\leqslant k\leqslant 2Nx/q} \Delta(kq+a) \ll \frac{Nx}{q}\mathfrak{S}(x),$$

where the latter bound results from a special case of [19; th. 1]. This implies (1·2) by summing over N.

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1. It is actually easy to show that $S(x) \asymp x\mathfrak{S}(x)$ $(x \geqslant 2)$, but we shall not need the lower bound here.
2. Here and throughout, we let $\log_k$ denote the kth iterated of the logarithm.

We intend here to return to the problem of sums of powers, also addressed in [17].

Given $\boldsymbol{c} := \{c_j\}_{0\leqslant j\leqslant t}$, $\boldsymbol{\ell} := \{\ell_j\}_{0\leqslant j\leqslant t} \in (\mathbb{N}^*)^{t+1}$, consider the number

$$r(n) = r(n; \boldsymbol{c}, \boldsymbol{\ell})$$

of representations of a natural integer n in the form

$$n = \sum_{0\leqslant j\leqslant t} c_j m_j^{\ell_j} \qquad (\boldsymbol{m} \in \mathbb{N}^{t+1}).$$

In all the sequel, we shall assume without loss of generality that $\{\ell_j\}_{0\leqslant j\leqslant t}$ is non-decreasing.

Under the condition $\sum_{0\leqslant j\leqslant t} 1/\ell_j = 1$, we trivially have

$$V_0(x; \boldsymbol{c}, \boldsymbol{\ell}) := \{n \leqslant x : r(n) \geqslant 1\} \leqslant x \qquad (x \geqslant 1).$$

The exact order of magnitude of $V_0(x) = V_0(x; \boldsymbol{c}, \boldsymbol{\ell})$ is only known in the case $\boldsymbol{\ell} = (2, 2)$. A famous conjecture (see, for instance, [1], or [31]) is that $V_0(x) \asymp x$ for $\boldsymbol{\ell} = (3, 3, 3)$, $\boldsymbol{c} = (1, 1, 1)$. The case $t = 2$, $\boldsymbol{c} = (1, 1, 1)$, $\boldsymbol{\ell} = (2, 4, 4)$ was studied by Hooley [15], Tenenbaum [29], Robert [21], and Koukoulopoulos–Tao [17]. Robert [21] also considered the case $\boldsymbol{\ell} = (2, 3, 6)$.

We obtain the following result.

Theorem 1.1. *Let $t \geqslant 2$, $\boldsymbol{c} \in (\mathbb{N}^*)^{t+1}$, $\boldsymbol{\ell} = \{\ell_j\}_{0\leqslant j\leqslant t}$ with $\ell_0 = 2$, $\ell_1 \in \{3, 4\}$ and $\sum_{j=1}^{t} 1/\ell_j = \frac{1}{2}$. Let $s := \max\{j \in [1, t-1] : \ell_{t-j+1} = \ldots = \ell_t\}$. Assume:*

(i) $1 \leqslant s \leqslant 3$,

(ii) $\ell_t \geqslant 26$ *if* $s = 3$,

(iii) $\sum_{j\geqslant r} 1/\ell_j \leqslant 1/\ell_{r-1}$ $(1 \leqslant r \leqslant t - s + 1)$.

Then, we have

$$V_0(x; \boldsymbol{c}, \boldsymbol{\ell}) \gg x/\mathfrak{S}(x) \gg x/(\log_2 x)^{5/2}. \tag{1·3}$$

We have not sought to specify uniformity in respect to the coefficients c_j in this statement.

Observe that the following sets fulfill the assumptions:

$t = 2$, $\boldsymbol{\ell} \in \{(2, 3, 6), (2, 4, 4)\}$,

$t = 3$, $\boldsymbol{\ell} \in \{(2, 3, 7, 42), (2, 3, 8, 24), (2, 3, 9, 18), (2, 3, 10, 15), (2, 3, 12, 12), (2, 4, 8, 8)\}$,

$t = 4$, $\boldsymbol{\ell} \in \{(2, 3, 7, 84, 84), (2, 3, 8, 48, 48), (2, 4, 8, 16, 16), (2, 3, 12, 24, 24)\}$,

$t = 5$, $\boldsymbol{\ell} \in \{(2, 3, 12, 36, 36, 36), (2, 3, 10, 45, 45, 45), (2, 4, 8, 16, 32, 32)\}$,

this list being non exhaustive. New sets can indeed be constructed by increasing the cardinality t. For instance, when $\boldsymbol{\ell}$ belongs to the above list, we can define $\boldsymbol{\ell}^+ \in (\mathbb{N}^*)^{t+2}$ by setting $\ell_j^+ := \ell_j$ for $0 \leqslant j \leqslant t - 1$, $\ell_t^+ := 2\ell_t$, $\ell_{t+1}^+ = 2\ell_t$. Yet another possibility is to define $\boldsymbol{\ell}^* \in (\mathbb{N}^*)^{t+3}$ with $\ell_t^* = \ell_{t+1}^* = \ell_{t+2}^* := 3\ell_t$ provided $\ell_t \geqslant 9$. Thus, $\ell = (2, 3, 6)$ induces $(2, 3, 12, 12)$ and $(2, 3, 18, 18, 18)$.

The article [17] comes back on the case $\boldsymbol{\ell} = (2, 4, 4)$—see, however, the remark following the statement of Proposition 5.1 *infra.*

The cases with $t \geqslant 3$ in Theorem 1.1 are new. The corresponding results rely on the bound

$$\left|\left\{(m_{t-2}, m_{t-1}, m_t, n_{t-2}, n_{t-1}, n_t) : \sum_{t-2\leqslant j\leqslant t} c_j m_j^{\ell_t} = \sum_{t-2\leqslant j\leqslant t} c_j n_j^{\ell_t} \leqslant x\right\}\right| \ll x^{3/\ell_t}, \tag{1·4}$$

established by Salberger provided $\ell_t \geqslant 26$—see the proof of Proposition 5.2 *infra* for precise references. In private communication [24], Salberger informed us that, as a direct consequence of his works [23] and [22], the bound (1·4) persists under the weaker condition $\ell_t \geqslant 16$. Assuming this, we get that (1·3) also holds for $\boldsymbol{\ell} = (2, 3, 18, 18, 18)$ and $\boldsymbol{\ell} = (2, 4, 8, 24, 24, 24)$.

The starting point of the proof of Theorem 1.1 is classically the Cauchy-Schwarz inequality

$$V_1(x)^2 \leqslant V_0(x)V_2(x) \tag{1·5}$$

where we have set

$$V_j(x) = V_j(x; \boldsymbol{c}, \boldsymbol{\ell}) := \sum_{n\leqslant x} r(n)^j \qquad (j = 1, 2). \tag{1·6}$$

We shall deduce Theorem 1.1 from a general estimate regarding $j = 2$.

2. Notation

Some notation must be introduced in order to state our results.

Given $k \in \mathbb{N}^*$, $\boldsymbol{a} = (a_j)_{1\leqslant j\leqslant k}, \boldsymbol{b} = (b_j)_{1\leqslant j\leqslant k} \in \mathbb{N}^{*k}$, we put

$$\mathfrak{s}\boldsymbol{a} := \sum_{1\leqslant j\leqslant k} a_j, \qquad \wp\boldsymbol{a} := a_1\cdots a_k, \quad \boldsymbol{ab} = (a_1b_1,\ldots,a_kb_k).$$

For each value of the parameters $A \geqslant 1$, $B \geqslant 1$, $\varepsilon > 0$, we designate by $\mathcal{M}_k(A,B,\varepsilon)$ the class of those non-negative arithmetic functions $F : \mathbb{N}^{*k} :\to \mathbb{R}^+$, satisfying the condition

$$F(\boldsymbol{ab}) \leqslant \min\big\{A^{\Omega(\wp\boldsymbol{a})}, B(\wp\boldsymbol{a})^{\varepsilon}\big\}F(\boldsymbol{b}) \tag{2·1}$$

for all $\boldsymbol{a},\boldsymbol{b} \in \mathbb{N}^{*k}$ such that $(\wp\boldsymbol{a},\wp\boldsymbol{b}) = 1$. By convention, the functions $F \in \mathcal{M}_k(A,B,\varepsilon)$ are extended to $\mathbb{N}^k$ by setting $F(\boldsymbol{a}) = 0$ if $\min_j a_j = 0$. When $F \neq 0$, we define

$$G(\boldsymbol{a}) = G_F(\boldsymbol{a}) := \max_{\substack{\boldsymbol{b}\in\mathbb{N}^{*k}\\ (\wp\boldsymbol{a},\wp\boldsymbol{b})=1\\ F(\boldsymbol{b})\neq 0}} \frac{F(\boldsymbol{ab})}{F(\boldsymbol{b})} \qquad \big(\boldsymbol{a}\in\mathbb{N}^{*k}\big). \tag{2·2}$$

Given a family $\{Q_j\}_{j=1}^k \in \mathbb{Z}[X_1,\ldots,X_t]^k$ of polynomials in t variables, we put

$$Q = \prod_{1\leqslant j\leqslant k} Q_j = \prod_{1\leqslant h\leqslant r} R_h^{\gamma_h} \in \mathbb{Z}[X_1,\ldots,X_t], \quad g := \deg Q, \tag{2·3}$$

where the R_h are irreducible in $\mathbb{Z}[X_1,\ldots,X_t]$. We may then write, canonically,

$$Q_j = \prod_{1\leqslant h\leqslant r} R_h^{\gamma_{jh}} \qquad (1\leqslant j\leqslant k)$$

where $\gamma_{jh} \geqslant 0$ for all j,h, so that

$$\gamma_h = \sum_{1\leqslant j\leqslant k}\gamma_{jh} \qquad (1\leqslant h\leqslant r), \qquad \sum_{1\leqslant h\leqslant r}\gamma_h \deg R_h = g.$$

Recall that a polynomial is said to be primitive if the greatest common divisor of its coefficients is 1. In all the sequel we assume the form Q to be primitive, which implies that the same holds for all the Q_j.

For $T \in \mathbb{Z}[X_1,\ldots,X_t]$, we write

$$\varrho_T^+(s) := \sum_{\substack{\boldsymbol{\xi}\in[1,s]^t\\ T(\boldsymbol{\xi})\equiv 0 \,(\mathrm{mod}\, s)}} 1 \qquad (s\geqslant 1). \tag{2·4}$$

Letting $\kappa(s) := \prod_{p|s} p$ denote the squarefree kernel of a natural integer s, we further put

$$\mathcal{K}(\boldsymbol{s}) := \mathrm{lcm}(s_j\kappa(s_j))_{1\leqslant j\leqslant r} = [s_1\kappa(s_1),\ldots,s_r\kappa(s_r)] \qquad \big(\boldsymbol{s}=(s_1,\ldots,s_r)\in\mathbb{N}^{*r}\big). \tag{2·5}$$

Given $\boldsymbol{R} \in \mathbb{Z}[X_1,\ldots,X_t]^r$, we may then define two arithmetic functions of $\boldsymbol{s} = (s_1,\ldots,s_r) \in \mathbb{N}^{*r}$ by

$$\varrho_{\boldsymbol{R}}^+(\boldsymbol{s}) := \sum_{\substack{\boldsymbol{\xi}\in[1,\wp\boldsymbol{s}]^t\\ R_h(\boldsymbol{\xi})\equiv 0\,(\mathrm{mod}\, s_h)\ (1\leqslant h\leqslant r)}} 1, \tag{2·6}$$

$$\varrho_{\boldsymbol{R}}^{\#}(\boldsymbol{s}) := \sum_{\substack{\boldsymbol{\xi}\in[1,\mathcal{K}(\boldsymbol{s})]^t\\ s_h\| R_h(\boldsymbol{\xi})\ (1\leqslant h\leqslant r)\\ (R_h(\boldsymbol{\xi})/s_h,\wp\boldsymbol{s})=1}} 1, \tag{2·7}$$

where the symbol $a\|b$ means that the conditions $a|b$ and $(a,b/a) = 1$ hold simultaneously. It is worthwhile to note that the above definition of $\varrho_{\boldsymbol{R}}^{\#}$ is more restrictive than that of [3]. The grounds for this alteration are described in [14].

For any $\boldsymbol{s} = (s_1,\ldots,s_r) \in \mathbb{N}^{*r}$, let us put

$$s_j' := \prod_{1\leqslant h\leqslant r} s_h^{\gamma_{jh}} \quad (1\leqslant j\leqslant k), \quad \boldsymbol{s}' := (s_1',\ldots,s_k'), \qquad s'' := \wp\boldsymbol{s}' = \prod_{1\leqslant j\leqslant k} s_j' = \prod_{1\leqslant h\leqslant r} s_h^{\gamma_h}.$$

Thus, given $\boldsymbol{m} = (m_1,\ldots,m_t) \in \mathbb{N}^{*t}$, and writing $s_h = R_h(\boldsymbol{m})$ $(1\leqslant h\leqslant r)$, we have

$$s_j' = Q_j(\boldsymbol{m}) \quad (1\leqslant j\leqslant k), \quad s'' = Q(\boldsymbol{m}).$$

With these pieces of notation, to any function F in $\mathcal{M}_k(A,B,\varepsilon)$ we may associate a further function $\widehat{F}$ in $\mathcal{M}_r(A^g,B,g\varepsilon)$ defined by

$$\widehat{F}(\boldsymbol{s}) = F(\boldsymbol{s}').$$

We then designate by $\widehat{G} := G_{\widehat{F}}$ the function associated to $\widehat{F}$ by (2·2) with $k = r$ and, for $1 \leqslant h \leqslant r$, we let $\widehat{G}_h : \mathbb{N}^* \to \mathbb{R}^+$ denote the composition of $\widehat{G}$ with the hth coordinate.

3. Statement of results

The key tool of this work consists in an estimate of the quantity

$$S := \sum_{x_j < n_j \leqslant x_j + y_j \ (1 \leqslant j \leqslant t)} F\big(|Q_1(\boldsymbol{n})|, \ldots, |Q_k(\boldsymbol{n})|\big) = \sum_{x_j < n_j \leqslant x_j + y_j \ (1 \leqslant j \leqslant t)} \widehat{F}\big(R_1(\boldsymbol{n}), \ldots, R_r(\boldsymbol{n})\big).$$

The following result is obtained. It is the t-dimensional counterpart of that of [13]. We let $\|Q\|$ denote the maximum of the absolute values of the coefficients of a real form Q.

Theorem 3.1. *Let $k, t \in \mathbb{N}^*$, and let $\{Q_j\}_{j=1}^k \in \mathbb{Z}[X_1, \ldots, X_t]^k$ be a family of primitive polynomials. Define $Q \in \mathbb{Z}[X_1, \ldots, X_t]$, $\{R_h\}_{1 \leqslant h \leqslant r} \in \mathbb{Z}[X_1, \ldots, X_t]^r$, $g \in \mathbb{N}$ by (2·3), ϱ_Q^+ by (2·4), and $\varrho_{\boldsymbol{R}}^\#$ by (2·7). For all*

$$\alpha \in]0,1], \quad \beta \in]0,1[, \quad A \geqslant 1, \quad B \geqslant 1, \quad 0 < \varepsilon \leqslant \alpha\beta/\{50g^2(\beta g + 1)\}, \tag{3·1}$$

and uniformly under the conditions

$$\begin{aligned} &F \in \mathcal{M}_k(A, B, \varepsilon), \quad \boldsymbol{x} = (x_1, \ldots, x_t) \in \mathbb{N}^{*t}, \quad \boldsymbol{y} = (y_1, \ldots, y_t) \in \mathbb{N}^{*t}, \\ &x := \min_{1 \leqslant j \leqslant t} x_j \geqslant c\big\{\max_{1 \leqslant j \leqslant t} x_j + \|Q\|\big\}^\beta, \quad x_j^\alpha \leqslant y_j \leqslant x_j \quad (1 \leqslant j \leqslant t), \end{aligned} \tag{3·2}$$

we have

$$\sum_{\substack{\boldsymbol{n} \in \mathbb{N}^{*t} \\ x_j - y_j < n_j \leqslant x_j (1 \leqslant j \leqslant t)}} F\big(|Q_1(\boldsymbol{n})|, \ldots, |Q_k(\boldsymbol{n})|\big) \ll \wp\boldsymbol{y} E_{\boldsymbol{R}}(\mathfrak{s}\boldsymbol{x}) \prod_{g < p \leqslant x} \Big(1 - \frac{\varrho_Q^+(p)}{p^t}\Big) \tag{3·3}$$

where c and the implicit constant depend at most upon g, α, β, A, B and where we have set

$$E_{\boldsymbol{R}}(v) := \sum_{\substack{\boldsymbol{s} \in \mathbb{N}^{*r} \\ \wp\boldsymbol{s} \leqslant v}} \widehat{F}(\boldsymbol{s}) \frac{\varrho_{\boldsymbol{R}}^\#(\boldsymbol{s})}{\mathcal{K}(\boldsymbol{s})^t} \qquad (v \geqslant 1). \tag{3·4}$$

Remarks. (i) We have

$$\frac{\varrho_{\boldsymbol{R}}^\#(\boldsymbol{s})}{\mathcal{K}(\boldsymbol{s})^t} \leqslant \frac{\varrho_{\boldsymbol{R}}^+(\boldsymbol{s})}{(\wp\boldsymbol{s})^t} \qquad \big(\boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{N}^{*r}\big), \tag{3·5}$$

however the sum $E_{\boldsymbol{R}}(x)$ may turn out to be smaller than that which follows from this upper bound. For instance, if $k = 1$, $t = 1$, $Q(X) = Q_1(X) = X(X + \ell)$, we have $\varrho_{\boldsymbol{R}}^\#(p) = 0$ as soon as $p \mid \ell$, while $\varrho_{\boldsymbol{R}}^+(p) = 1$. As noted in [13] and [14], this can be significant when it comes to obtaining results uniform with respect to ℓ.

(ii) A weaker version of (3·3) for $k = 1$ has recently been established [6; th. 1.15], where, in definition (3·4), the left-hand side of (3·5) is replaced by its majorant. Note that the dependence of our bounds with respect to the coefficients of Q is effective.

(iii) In the case $t = 1$, Theorem 3.1 is due to Henriot [13], [14], extending [26], then [18] and [19]. When the Q_j are binary forms in $\mathbb{Z}[X_1, X_2]$, Theorem 3.1 has been proved in [3] with a slightly different definition of $\varrho^\#$, refining and extending the estimates of [2].

(iv) When F is multiplicative and satisfies an assumption such that $F(\boldsymbol{n}) \geqslant \eta^{\Omega(\wp\boldsymbol{n})}$ for some $\eta > 0$, it is possible to establish lower bounds for the the left-hand side of (3·3) of the same order of magnitude as the right-hand side. We refer to [13] for this aspect of the question.

4. Proof of Theorem 3.1

Recall a consequence of the Schwartz-Zippel lemma — see [27; lemma 1], or [32; th. 1].

Lemma 4.1. *Let $t \geqslant 1$ and let $Q \in \mathbb{Z}[X_1, \ldots, X_t]$ be a primitive polynomial of degree g. For all prime numbers p, we have*

$$\varrho_Q^+(p) \leqslant gp^{t-1}. \tag{4·1}$$

In particular, for all integers $\nu \geqslant 1$, we have $\varrho_Q^+(p^\nu) \leqslant gp^{\nu t - 1} < p^{\nu t}$ provided $p > g$.

Let $c(Q)$ denote the content of the polynomial Q of t variables and let $v_p(n)$ denote the p-adic valuation of an integer n. The following lemma is an effective version of [6; lemma 2.8]—see also [20; lemma 4.10].

Lemma 4.2. *Let $t \geqslant 1$ and let $Q \in \mathbb{Z}[X_1, \ldots, X_t]$ be a polynomial of degree g. For all prime numbers p and all integers $\nu \geqslant 1$, we have*

$$\varrho_Q^+(p^\nu) \leqslant g^t(\nu+1)^{t-1} p^{\nu(t-1/g)+v_p(c(Q))/g}. \tag{4·2}$$

Proof. Consider first the case $t = 1$. When $(c(Q), p) = 1$, inequality (4·2) has been proved by Stewart [25]. If $\gamma = v_p(c(Q)) \geqslant 1$, we have

$$\varrho_Q^+(p^\nu) = p^\gamma \varrho_{Q^*}^+(p^{\nu-\gamma}) \leqslant g p^{\nu(1-1/g)} p^{v_p(c(Q))/g}.$$

with $Q^* := Q/p^\gamma$. Thus (4·2) holds for $t = 1$.

We now proceed by induction on t. Let x_1 be such that Q depends on x_1 and write

$$Q(x_1, x_2, \ldots, x_t) = \sum_{0 \leqslant j \leqslant g} Q_j(x_2, \ldots, x_t) x_1^j$$

with $Q_j \in \mathbb{Z}[X_2, \ldots, X_t]$ and $g_j := \deg Q_j \leqslant g - j$. By the case $t = 1$, we have

$$\varrho_Q^+(p^\nu) \leqslant g p^{\nu(1-1/g)} \sum_{0 \leqslant \gamma \leqslant \nu} p^{\gamma/g+(t-1)(\nu-\gamma)} N(\gamma, p)$$

with $N(\gamma, p) := |\{\boldsymbol{x} \in (\mathbb{Z}/p^\gamma\mathbb{Z})^{t-1} : Q_j(\boldsymbol{x}) \equiv 0 \,(\mathrm{mod}\, p^\gamma) \ (0 \leqslant j \leqslant g)\}|$.

Now the induction hypothesis yields

$$\begin{aligned} N(\gamma, p) &\leqslant \min_{0 \leqslant j \leqslant g} |\{\boldsymbol{x} \in (\mathbb{Z}/p^\gamma\mathbb{Z})^{t-1} : Q_j(\boldsymbol{x}) \equiv 0 \,(\mathrm{mod}\, p^\gamma)\}| \\ &\leqslant \min_{0 \leqslant j \leqslant g} g_j^{t-1}(\nu+1)^{t-2} p^{\gamma(t-1)+\min\{v_p(c(Q_j))-\gamma, 0\}/g_j} \\ &\leqslant g^{t-1}(\nu+1)^{t-2} p^{\gamma(t-1-1/g)+v_p(c(Q))/g}, \end{aligned}$$

since $\min_{0 \leqslant j \leqslant g} v_p(c(Q_j)) = v_p(c(Q))$. This completes the proof of (4·2). □

Let us now state the analogue in our context of [13; lemma 6]. This is a straightforward consequence of the fundamental lemma of the combinatorial sieve. It coincides with [3; lemma 3.3] when $t = 2$ and with [13; lemma 6] when $t = 1$. As in [13; lemma 6], we let α be an arbitrary constant in $]0, 1]$ and define

$$\alpha_1 := \tfrac{3}{25}\alpha, \quad \alpha_2 := \varepsilon_1/6g. \tag{4·3}$$

Lemma 4.3. *Let $t \geqslant 1$, $0 < \alpha < 1$, and let $Q, R_h \in \mathbb{Z}[X_1, \ldots, X_t]$ be polynomials satisfying the hypotheses of Theorem 3.1. Under the conditions*

$$\boldsymbol{a} \in \mathbb{N}^{*r}, \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^{*t}, \ w := \min_j x_j, \ \wp\boldsymbol{a} \leqslant w^{\alpha_1}, \ z \leqslant w^{\alpha_2}, \ x_j^\alpha \leqslant y_j \leqslant x_j,$$

we have

$$\sum_{\substack{x_j < n_j \leqslant x_j + y_j \ (1 \leqslant j \leqslant t) \\ a_h \| R_h(\boldsymbol{n}) \ (1 \leqslant h \leqslant r) \\ (p | Q(\boldsymbol{n}),\, g < p \leqslant z) \Rightarrow p | \wp\boldsymbol{a} \\ (R_h(\boldsymbol{n})/a_h, \wp\boldsymbol{a}) = 1 \ (1 \leqslant h \leqslant r)}} 1 \ll \wp\boldsymbol{y} \frac{\varrho_{\boldsymbol{R}}^\#(\boldsymbol{a})}{\mathcal{K}(\boldsymbol{a})^t} \prod_{\substack{g < p \leqslant z \\ p \nmid \wp\boldsymbol{a}}} \Big(1 - \frac{\varrho_Q^+(p)}{p^r}\Big). \tag{4·4}$$

Note that, since the product in (3·3) only covers primes $p > g$, it avoids any fixed prime divisor of the R_h. It is also worthwhile to keep in mind that

$$\frac{\varrho_{\boldsymbol{R}}^\#(\boldsymbol{a})}{\mathcal{K}(\boldsymbol{a})^t} = \mathrm{dens}\Big\{\boldsymbol{n} \in \mathbb{Z}^t : a_h \| R_h(\boldsymbol{n}) \ (1 \leqslant h \leqslant r),\ (p \mid \wp\boldsymbol{a},\ p \nmid a_h) \Rightarrow (p \nmid R_h(\boldsymbol{n}))\Big\}. \tag{4·5}$$

Proof. In the above statement, the parameters α_1 and α_2 defined in (4·3) coincide respectively with ε_1 and ε_3 from [13; lemma 6]. Lemma 4.3 follows by applying the combinatorial sieve to the sequence

$$\mathcal{A} := \left\{ Q(n_1, \ldots, n_t) : \begin{matrix} x_j < n_j \leqslant x_j + y_j \ (1 \leqslant j \leqslant t), \ a_h \| R_h(\boldsymbol{n}) \ (1 \leqslant h \leqslant r), \\ (p \mid \wp\boldsymbol{a}/a_h, \, p \nmid a_h) \Rightarrow (p \nmid R_h(n)) \end{matrix} \right\}.$$

Thus, we can select $A_1 = g+1$ in the hypotheses of the sieve—see [10; lemma 2.2], condition (Ω_1)—since the upper bound (4·1) implies that

$$\frac{\varrho_Q^+(p)}{p^t} \leqslant \frac{g}{p} \leqslant 1 - \frac{1}{g+1} \qquad (p \geqslant g+1, \, p \nmid \wp\boldsymbol{a}).$$

Further details are omitted. □

Substituting Lemma 4.3 to [13; lemma 6] in the proof of [13; th. 5] readily furnishes estimate (3·3). Note that Lemma 4.2 is crucial in order to get effective bounds in terms of $\|Q\|$.

5. Proof of Theorem 1.1

5·1. Statements

It follows from (1·6) that

$$V_2(x; \boldsymbol{c}, \boldsymbol{\ell}) = \operatorname{card}\Big\{ (\boldsymbol{m}, \boldsymbol{n}) \in (\mathbb{N}^*)^{t+1} \times (\mathbb{N}^*)^{t+1} : \sum_{0 \leqslant j \leqslant t} c_j m_j^{\ell_j} = \sum_{0 \leqslant j \leqslant t} c_j n_j^{\ell_j} \leqslant x \Big\},$$

and so $V_2(x) = V_2(x; \boldsymbol{c}, \boldsymbol{\ell}) = V_2^{\neq}(x) + V_2^{=}(x)$, where the subsums $V_2^{\neq}(x)$ and $V_2^{=}(x)$ respectively correspond to the extra conditions $m_0 \neq n_0$ and $m_0 = n_0$. We obtain the following estimates.

Proposition 5.1. *Let $t \geqslant 1$, $\delta > 0$, $\boldsymbol{c}, \boldsymbol{\ell} \in (\mathbb{N}^*)^{t+1}$, satisfy $\ell_0 \geqslant 2$ and $\delta = \sum_{1 \leqslant j \leqslant t} 1/\ell_j$. Then*

$$V_2^{\neq}(x) \ll x^{2\delta} \mathfrak{S}(x) \ll x^{2\delta} (\log_2 x)^{5/2} \qquad (x \geqslant 3). \tag{5·1}$$

The second bound follows from (1·1). The exponent 5/2 hence measures, in the current state of knowledge, the deviation of this general estimate from optimality. In [17; § 8], Koukoulopoulos and Tao indicate the exponent $= 2^{4L} + 15/4$ for $L = \max \ell_j$, $\ell_0 = 2$, $\delta = \frac{1}{2}$. The condition $m_0 \neq n_0$ has been omitted in [17]. Since it necessary for a general result, the argument given in [17] only yields bound [17; (1.5)] in the historical case $\boldsymbol{\ell} = (2, 4, 4)$ considered by Hooley. The exponent stated in [17] is then $65539 + 3/4$.

Our approach to Theorem 1.1 is based on an inductive argument described in the following statement.

Proposition 5.2. *Let $t \geqslant 2$, $\boldsymbol{c} = \{c_j\}_{j=0}^t \in (\mathbb{N}^*)^{t+1}, \boldsymbol{\ell} = \{\ell_j\}_{j=0}^t \in (\mathbb{N}^*)^{t+1}$ with $\ell_0 \geqslant 2$. Put $\check{\boldsymbol{c}} := (c_1, \ldots, c_t)$, $\check{\boldsymbol{\ell}} = (\ell_1, \ldots, \ell_t)$. We have*

$$V_2^{=}(x; \boldsymbol{c}, \boldsymbol{\ell}) \leqslant \Big(\frac{x}{c_0}\Big)^{1/\ell_0} V_2(x; \check{\boldsymbol{c}}, \check{\boldsymbol{\ell}}) \qquad (x \geqslant 3). \tag{5·2}$$

Moreover, if $t \in \{1, 2\}$ and $\ell_{t-1} = \ell_t \geqslant 3$, or if $t = 3$ and $\ell_{t-2} = \ell_{t-1} = \ell_t \geqslant 26$, then

$$V_2(x; \check{\boldsymbol{c}}, \check{\boldsymbol{\ell}}) \ll x^{t/\ell_t}. \tag{5·3}$$

Proof. The bound (5·2) is clear, and so is (5·3) for $t = 1$. The remaining bounds (5·3) are due to Salberger: for $t = 2$, see [23; cor. 0.7]; for $t = 3$, $\ell_t \geqslant 33$, see [5]; for $t = 3$, $\ell_t \geqslant 26$, see [22; cor. 4.6]. □

As mentioned in the introduction, we have been informed by Salberger that he intends to write up the proof that (5·2) holds for $t = 3$, $\ell_t \geqslant 16$, and hopefully $\ell_t \geqslant 12$.

5·2. A lemma from the literature

Define $P^+(n)$ as the largest prime factor of an integer n, with the convention that $P^+(1) := 1$. The following statement is established in [13; lemmas 2 & 5].

Lemma 5.3. *Let $r, g \geqslant 1$, $A \geqslant 1$, $B \geqslant 1$, $\varepsilon > 0$ and let $\{\sigma_j\}_{j=1}^r$, $\{\vartheta_j\}_{j=1}^r$ be two families of multiplicative arithmetical functions. Assume that, uniformly for prime powers p^ν, we have $\sigma_j(p^\nu) \ll 1$, $\vartheta_j(p^\nu) = 1 + O(1/p)$. Let $\widehat{F} \in \mathcal{M}_r(A^g, B, g\varepsilon)$, and define*

$$H(\boldsymbol{s}) := \widehat{F}(\boldsymbol{s}) \prod_{1 \leqslant j \leqslant r} \sigma_j(s_j) \frac{\varrho_{\boldsymbol{R}}^{\#}(\boldsymbol{s})}{\mathcal{K}(\boldsymbol{s})^t} \qquad (\boldsymbol{s} \in \mathbb{N}^{*r}).$$

Then, for $x \geqslant 3$, we have

$$\sum_{\substack{\boldsymbol{s} \in (\mathbb{N}^{*r} \\ \wp \boldsymbol{s} \leqslant x}} H(\boldsymbol{s}) \prod_{1 \leqslant j \leqslant r} \vartheta_j(s_j) \ll \sum_{\substack{\boldsymbol{s} \in (\mathbb{N}^{*r} \\ \wp \boldsymbol{s} \leqslant x}} H(\boldsymbol{s}), \tag{5·4}$$

$$\sum_{\substack{\boldsymbol{s} \in (\mathbb{N}^{*r} \\ P^+(\wp \boldsymbol{s}) \leqslant x}} H(\boldsymbol{s}) \ll \sum_{\substack{\boldsymbol{s} \in (\mathbb{N}^{*r} \\ \wp \boldsymbol{s} \leqslant x}} H(\boldsymbol{s}). \tag{5·5}$$

5·3. Proof of Proposition 5.1.

Consider

$$Q(X_1, \dots, X_t, Y_1, \dots, Y_t) := \sum_{1 \leqslant j \leqslant t} c_j X_j^{\ell_j} - \sum_{1 \leqslant j \leqslant t} c_j Y_j^{\ell_j}. \tag{5·6}$$

The following statement gathers the necessary estimates regarding function ϱ_Q defined in (2·4).

Lemma 5.4. *Let $t \geqslant 1$, $\boldsymbol{c}$, $\boldsymbol{\ell} \in \mathbb{N}^{*t}$ be such that*

$$L := \max_{1 \leqslant j \leqslant t} \ell_j, \quad \gcd(c_1, \dots, c_t) = 1.$$

For all prime numbers p and all integers $\nu \geqslant 1$ and Q defined by (5·6), we have

$$\varrho_Q(p) = p^{2t-1} + O(p^t), \qquad \varrho_Q(p^\nu) \leqslant L p^{2t\nu - 1}, \qquad \varrho_Q(p^\nu) \ll \nu p^{2(t-\delta)\nu}$$

with $\delta := \sum_{j=1}^t 1/\ell_j$.

Proof. The second estimate follows from Lemma 4.1. The first formula is included in the first estimate of [21; lemma 3.4], itself resting on an estimate of Korobov [16; ch. 1, th. 5]. The third bound is a consequence of the second estimate in [21; lemma 3.4], resting on Korobov's [16; ch. 1, th. 6], observing that the proof given in [21] remains valid as it stands when $\delta < 1/2$. □

We are now ready to complete the proof of Proposition 5.1. Put $L := \max \ell_j$. Proposition 5.1 will follow from Theorem 3.1, applied with $k = 1$, $Q_1 = Q$, $F = \Delta$. Up to dividing through by $\gcd(c_1, \dots, c_t)$, the c_j may be assumed coprime. The polynomial Q is then irreducible in $\mathbb{Z}[X_1, \dots, X_t, Y_1, \dots, Y_t]$, so that $r = 1$. Indeed, we may write $Q = c_1 X_1^{\ell_1} + Q_1$, where $Q_1 \in \mathbb{Z}[X_2, \dots X_t, Y_1, \dots, Y_t]$ has no square factor in $\mathbb{Q}[X_2, \dots, X_t, Y_1, \dots, Y_t]$. This enables applying Eisenstein's criterion associated to a non constant irreducible factor of Q_1 in $\mathbb{Z}[X_2, \dots, X_t, Y_1, \dots, Y_t]$.

Taking the above into account, Lemma 5.4 furnishes

$$\prod_{L < p \leqslant x} \Big(1 - \frac{\varrho_Q^+(p)}{p^{2t}}\Big) \asymp \log x \qquad (x \geqslant 2L).$$

Moreover, the quantity $E_Q(v)$ defined in (3·4) satisfies

$$E_Q(v) \asymp \sum_{n \leqslant v} \frac{\Delta(n)}{n} \qquad (v \geqslant 1). \tag{5·7}$$

This follows from Lemmas 5.3 and 5.4. Indeed, since $k = r = 1$, we have

$$E_Q(v) = \sum_{n \leqslant v} \frac{\Delta(n) \varrho_Q^{\#}(n)}{\mathcal{K}(n)^{2t}} \cdot \tag{5·8}$$

By Lemma 5.4, there exist two multiplicative functions ϑ_1 and ϑ_2 such that

$$\text{(i)}\quad \vartheta_j(p^\nu)=1+O(1/p)\quad (j\in\{1,2\},\ \nu\geqslant 1),$$
$$\text{(ii)}\quad \varrho^{\#}(n)/\mathcal{K}(n)^{2t}=\varrho^{+}(n)\vartheta_1(n)/n^{2t}=\vartheta_2(n)/n\quad (n\geqslant 1).$$

A first application of (5·4) provides the upper bound in (5·7). To establish the corresponding lower bound, we note that, by condition (i), we have $\vartheta_2(p)\neq 0$ for all but only a bounded number of primes p. Let $\vartheta_3(p^\nu)=\vartheta_2(p^\nu)+\mathbf{1}_{\{\vartheta_2(p^\nu)=0\}}(p^\nu)$. Applying (5·5) to replace condition $n\leqslant v$ by $p|n\Rightarrow p\leqslant v$, we hence see that substituting ϑ_3 to ϑ_2 in (5·8) does not alter the order of magnitude of the sum. A second application of (5·4) with $H(n):=\Delta(n)\vartheta_3(n)$ and $\vartheta=1/\vartheta_3$ then furnishes the lower bound in (5·7).

We may now apply Theorem 3.1 in order to derive Proposition 5.1. Put

$$(\boldsymbol{m},\boldsymbol{n})=(m_1,\dots,m_t,n_1,\dots,n_t).$$

By symmetry, we may assume $n_0>m_0$. Let us then split the variation range of m_0+n_0 into dyadic intervals $]N,2N]$ with $N=2^\nu\leqslant 2x^{1/\ell_0}$. Up to neglecting a contribution $\ll x^{2\delta}$, we may assume $N>x^{1/\ell_0-1/2\ell_0\ell_k}$. Indeed, if $N\leqslant x^{1/\ell_0-1/2\ell_0\ell_k}$, then for all fixed $(m_1,\dots,m_k)$, $(n_1,\dots,n_{k-1})$, the power $n_k^{\ell_k}$ lies in a fixed interval of length $\ll N^{\ell_0}\ll x^{1-1/2\ell_k}$. The number of admissible n_k is hence $\ll x^{1/\ell_k-1/2\ell_k^2}$. However, for each fixed n_k, the number of admissible pairs (m_0,n_0) is $\ll x^\varepsilon$ for any $\varepsilon>0$. This plainly implies that the contribution of $N\leqslant x^{1/\ell_0-1/2\ell_0\ell_k}$ is $\ll x^{2\delta-1/2\ell_k^2+\varepsilon}\ll x^{2\delta}$ provided that $\varepsilon\leqslant 1/2\ell_k^2$.

If $(m_0,\boldsymbol{m},n_0,\boldsymbol{n})$ is counted in $V_2^{\neq}(x)$, we have $Q(\boldsymbol{m},\boldsymbol{n})=c_0(n_0^{\ell_0}-m_0^{\ell_0})$, hence the divisor $n_0^{\ell_0-1}+m_0n_0^{\ell_0-1}+\ldots+m_0^{\ell_0-1}$ of $Q(\boldsymbol{m},\boldsymbol{n})$ is of size N^{ℓ_0-1}. As a consequence, for each $2t$-tuple $(\boldsymbol{m},\boldsymbol{n})$, the number of admissible pairs (m_0,n_0) does not exceed $\Delta(Q(\boldsymbol{m},\boldsymbol{n}))$. Now let us split the variation ranges of the $m_j^{\ell_j}$ and $n_h^{\ell_h}$ into intervals of size N^{ℓ_0}, say $m_j^{\ell_j}\in]r_jN^{\ell_0},(r_j+1)N^{\ell_0}]$ and $n_h^{\ell_h}\in]s_hN^{\ell_0},(s_h+1)N^{\ell_0}]$ with $r_j,s_h\ll x/N^{\ell_0}$. For each r_j, the integer m_j is hence restricted to an interval of length $\ll N^{\ell_0/\ell_j}/(r_j+1)^{(1-1/\ell_j)}$.

For each pair (r_j,s_j), Theorem 3.1 furnishes a contribution

$$\ll\prod_{1\leqslant j\leqslant t}\frac{N^{2\ell_0/\ell_j}}{(r_j+1)^{1-1/\ell_j}(1+s_j)^{1-1/\ell_j}}\frac{E_Q(x)}{\log x}\cdot\tag{5·9}$$

Here, the hypothesis $N>x^{1/\ell_0-1/2\ell_0\ell_k}$ has been used in a crucial way since it implies that m_j lies in a range of size $\gg x^{1/\ell_j-1/2\ell_k}\gg x^{1/2\ell_k}$. This enables us to check hypotheses (3·2) of Theorem 3.1, which require that conditions (3·2) be satisfied by

$$x_j=(r_jN^{\ell_0})^{1/\ell_j},\quad x_{j+k}=(s_jN^{\ell_0})^{1/\ell_j},\quad y_j=\{(r_j+1)N^{\ell_0}\}^{1/\ell_j}-(r_jN^{\ell_0})^{1/\ell_j},$$
$$y_{j+k}=((s_j+1)N^{\ell_0})^{1/\ell_j}-(s_jN^{\ell_0})^{1/\ell_j},$$

for $1\leqslant j\leqslant k$. Thus, the choice $\alpha=1/2\ell_k$, $\beta=1/\ell_k-1/2\ell_k^2$, is admissible in (3·2).

Now observe that relations $\big|m_j^{\ell_j}-n_j^{\ell_j}-(r_j-s_j)N^{\ell_0}\big|\leqslant N^{\ell_0}$ and

$$c_0|n_0^{\ell_0}-m_0^{\ell_0}|=\Big|\sum_{1\leqslant j\leqslant t}c_j(m_j^{\ell_j}-n_j^{\ell_j})\Big|\leqslant 2^{\ell_0}c_0N^{\ell_0}$$

imply $\Big|\sum_{1\leqslant j\leqslant t}c_j(r_j-s_j)\Big|\leqslant 2^{\ell_0}c_0+\sum_{1\leqslant j\leqslant t}c_j$. Therefore, when all r_j and all s_h are fixed except s_1, the number of admissible values for s_1 is bounded. Summing (5·9) over r_j and s_h, taking the condition $\sum_{1\leqslant j\leqslant t}1/\ell_j=\delta$ into account then yields a contribution

$$\ll N^{\ell_0/\ell_1}x^{2\delta-1/\ell_1}\frac{E_Q(x)}{\log x}\cdot$$

It remains to sum over ν such that $N^{\ell_0}=2^{\ell_0\nu}\leqslant 2^{\ell_0}x$ to get, in view of (5·7),

$$V_2^{\neq}(x;\boldsymbol{c},\boldsymbol{\ell})\ll x^{2\delta}\frac{E_Q(x)}{\log x}\ll x^{2\delta}\mathfrak{S}(x).$$

Note that the manipulation resting on the condition $m_0+n_0\in]N,2N]$ saved a factor $\log_2 x$.

This completes the proof of Proposition 5.1.

5·4. *Proof of Theorem 1.1.*

Since $V_1(x) \asymp x$ (see [21; lemma 3.1] for further precisions) and taking (1·5) into account the required estimate (1·3) immediately results from a suitable bound for $V_2(x, \boldsymbol{c}, \boldsymbol{\ell})$. In view of (5·1), we write $\boldsymbol{c} = (c_1, \ldots, c_t)$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_t)$.

For $1 \leqslant k \leqslant t$, put $\delta_k = \sum_{t-k \leqslant j \leqslant t} 1/\ell_j$, $\boldsymbol{c}_k = (c_{t-k}, \ldots, c_t)$, $\boldsymbol{\ell}_k = (\ell_{t-k}, \ldots, \ell_t)$. The parameter s being defined in the statement of Theorem 1.1 and $\boldsymbol{\ell}$ satisfying conditions (i), (ii), (iii) of this statement, we shall show by induction on $k \geqslant \max(1, s-1)$ that

$$V_2(x; \boldsymbol{c}_k, \boldsymbol{\ell}_k) \ll x^{\delta_k} \mathfrak{S}(x). \tag{5·10}$$

Note that hypothesis (iii) may be rewritten as $\delta_k \leqslant 1/\ell_{t-k-1}$ for $s-1 \leqslant k \leqslant t-1$.

For $k = s = 1$, $t \geqslant 2$, the stated bound (5·10) follows from [21; lemma 3.11]. The case $k = 1$, $s = 2$, $t \geqslant 2$, follows from (5·3). For $k = 2$, $t \geqslant 3$, $s = 3$, we only need to bound the number of solutions $(\boldsymbol{x}, \boldsymbol{y})$ such that $x_j, y_j \leqslant x^{1/\ell}$ and

$$x_1^\ell + x_2^\ell + x_3^\ell = y_1^\ell + y_2^\ell + y_3^\ell$$

with $\ell = \ell_t$. The required estimate then follows from Proposition 5.2. The induction is thus initialized. In these cases, the factor $\mathfrak{S}(x)$ is actually superfluous.

Let $k \geqslant s-1$ and assume (5·10) holds for k. We want to prove that it persists for $k+1$. By Propositions 5.2 and 5.1, we have

$$V_2(x; \boldsymbol{c}_{k+1}, \boldsymbol{\ell}_{k+1}) \ll x^{2\delta_k} \mathfrak{S}(x) + x^{1/\ell_{t-k-1}} V_2(x; \boldsymbol{c}_k, \boldsymbol{\ell}_k) \ll (x^{2\delta_k} + x^{1/\ell_{t-k-1}+\delta_k}) \mathfrak{S}(x).$$

This provides the desired bound since $\delta_k \leqslant 1/\ell_{t-k-1}$, and $1/\ell_{t-k-1} + \delta_k = \delta_{k+1}$.

The induction step is thus established, and so we get the result for $k = t+1$. This completes the proof of Theorem 1.1 since $\delta_{t+1} = 1$.

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