On solutions to the Mondrian Puzzle

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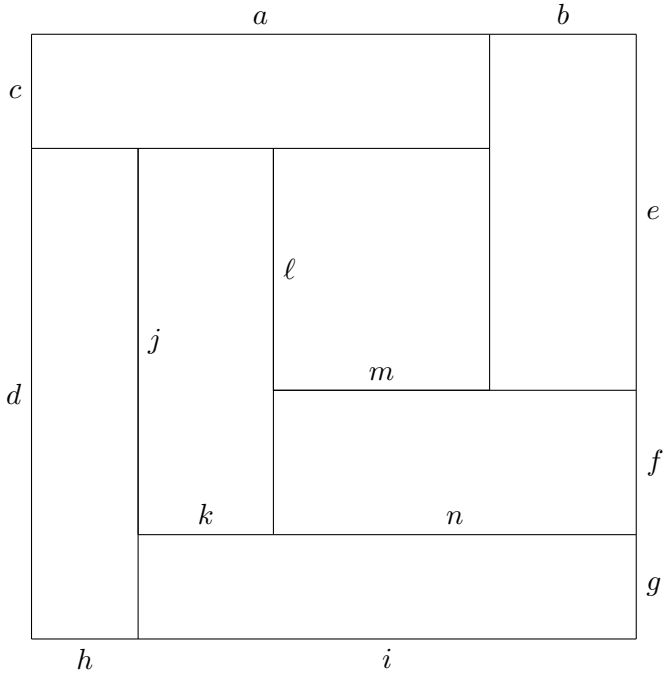
Abstract. We show that for a positive proportion of integers n, it is impossible to partition an $n \times n$ square into distinct rectangles with equal area and integer side lengths. We also obtain an analogue of this result in dimension 3.

1. Introduction

Half a century ago, Blanche Descartes (the pseudonym of a group of four British mathematicians) [7] proved that it is possible to partition a square into at least two rectangles with the following properties. First, the rectangles have the same area. Second, no two rectangles have the same height (and, by extension, the same width). We reprint Descartes' construction below.



Date: Thursday 22nd October, 2026, 13:59.

2020 *Mathematics Subject Classification.* 11N25, 11N37, 11K65, 05B45.

Key words and phrases. Mondrian puzzle, integer partitions, divisor sums, Erdős–Hooley Delta function, sets of multiples, primitive bases, natural density.

Here, $n = 210$, and, with $r = \sqrt{19}$, $a = 14(7 + r)$, $b = 14(8 - r)$, $c = 15(7 - r)$, $d = 15(7 + r)$, $e = 10(8 + r)$, $f = 50$, $g = 10(8 - r)$, $h = 14(7 - r)$, $i = 14(8 + r)$, $j = 25(1 + r)$, $k = 14(-1 + r)$, $\ell = 25(-1 + r)$, $m = 14(1 + r)$.

In light of this observation, we can modify the problem. The Mondrian Puzzle asks if it is ever possible to find a partition with the additional restriction that all of the rectangles have integral side lengths. (The puzzle is named for Piet Mondrian, a Dutch painter and art theorician best known for paintings of multi-coloured rectangular partitions.) From here on, we refer to these decompositions as *Mondrian decompositions*. In addition, we say that an integer n is a *Mondrian integer* if the $n \times n$ square has a Mondrian decomposition. A Mondrian integer is called *primitive* if it is not a multiple of a smaller Mondrian integer.

Finally, we let $M(x)$ be the number of Mondrian integers not exceeding x and let $N(x) := \lfloor x \rfloor - M(x)$ denote the cardinality of the complement of Mondrian integers not exceeding x.

Conjecture 1.1. *Mondrian integers do not exist.*

It is thus expected that $N(x) = \lfloor x \rfloor$. Dalfó, Fiol, and López [4] proved that $N(x) \gg x(\log_2 x)^2/\log x$.[1] Soon after, O'Kuhn and Fellman [15] obtained $N(x) \gg x/\log_2 x$. Additionally, García-Colín et al. [10] proved that there is no Mondrian integer $\leqslant 1000$. This note is devoted to showing that $N(x) \gg x$.

Theorem 1.2. *The set of Mondrian integers has upper density* < 1.

The basic idea of the proof is as follows. Since the Mondrian integers form a set of multiples, we study its primitive basis. A horizontal or vertical cross-section of a primitive Mondrian decomposition represents n as a sum of distinct divisors of n^2, and primitivity guarantees that, for every proper divisor r of n, such a cross-section can be chosen to contain a summand not divisible by r. Taking $r = P^+(n)$, the largest prime factor of n, produces a nonzero subset sum divisible by $P^+(n)$. A quantitative estimate for Erdős-Hooley's Delta function then enables showing that there are few possible subset sums, which allows a suitable upper bound for the size of the primitive basis. Behrend's inequality then provides the density conclusion.

As observed by O'Kuhn and Fellman [15] at the end of their paper, the multiplicative structure of the set of Mondrian integers implies that it has positive lower density as soon as it is non-empty. Therefore, showing that $M(x) = o(x)$ is equivalent to proving that $M(x) = 0$ for all x.

In the final section of the paper, we discuss a generalization of this problem to higher dimensions.

Theorem 1.3. *For a positive proportion of numbers n, it is impossible to partition a 3-dimensional cube with side length n into rectangular prisms with different heights and same volume.*

Here, "height" can refer to any direction. The only restriction is that along one specific axis, all of the prisms have different lengths. It seems

[1] Here and throughout, we let $\log_k$ denote the k-fold iterated logarithm.

plausible that this result would extend to higher dimensions, but the current proof would not apply. In addition, it would be interesting to know if the condition that the prisms have different heights can be weakened to the simpler restriction that no two prisms are identical.

2. An upper bound on the number of rectangles

In this section, we obtain an a.a. upper bound for the number of rectangles in a Mondrian decomposition of an $n \times n$ square. In the following statement and throughout this article we use the notation a.a. to indicate that a relation holds on a set of density 1.

Theorem 2.1. *The total number $R(n)$ of rectangles in a Mondrian decomposition of the square $n \times n$ satisfies $R(n) \leqslant (\log n)^{\log 3 - 1 + o(1)}$* a.a.

We introduce the following definition—see [12, ch. 5, 6].

Definition. Let $\Delta(n,u) := |\{d \geqslant 1 : d|n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|$ $(n \geqslant 1,\, u \in \mathbb{R})$. Then *Erdős-Hooley's Delta function* is defined as

$$\Delta(n) := \max_{u \in \mathbb{R}} \Delta(n,u) \qquad (n \geqslant 1).$$

The proof of Theorem 2.1 below requires an upper bound on $\Delta(n)$, which we now discuss. Maier and Tenenbaum [13], see also [12, th. 56], proved that if $\psi(n) \to \infty$, then $\Delta(n) \leqslant \psi(n) \log_2 n$ a.a. More precise bounds have subsequently been obtained. La Bretèche and Tenenbaum [2, th. 1.3] showed that, for any

$$\gamma > (\log 2)^2/(1 - \log 2 + (\log 2)^2) \approx 0.610,$$

we have $\Delta(n) \leqslant (\log_2 n)^\gamma$ a.a. As for a lower bound, improving on an estimate of Maier and Tenenbaum [14], Ford, Green, and Koukoulopoulos [9] proved that $\Delta(n) \geqslant (\log_2 n)^{c - o(1)}$ a.a. with $c \approx 0.353$.

For our theorem, we shall however only need the validity of the bound

$$\Delta(n) = (\log n)^{o(1)} \quad \text{a.a.} \tag{2.1}$$

As a preliminary step, we state an estimate for the size of the divisor function.

Lemma 2.2. *Let $k \geqslant 1$. Then $\tau(n^k) = (\log n)^{\log(k+1) + o(1)}$* a.a.

Proof. Let $\Omega(n)$ and $\omega(n)$ denote the number of divisors of n counted with and without multiplicity. By the Hardy-Ramanujan theorem—see e.g. [17, th. III.3.4]—the normal order of both functions is $\log_2 n$.

Let $n = p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}$. Then

$$\tau(n^k) = \prod_{1 \leqslant j \leqslant r} (k\nu_j + 1), \quad (k+1)^{\omega(n)} = (k+1)^r, \quad (k+1)^{\Omega(n)} = \prod_{1 \leqslant j \leqslant r} (k+1)^{\nu_j}.$$

However, $k + 1 \leqslant k\nu + 1 \leqslant (k+1)^\nu$ for all $\nu \in \mathbb{Z}_+$. Hence,

$$(k+1)^{\omega(n)} \leqslant \tau(n^k) \leqslant (k+1)^{\Omega(n)},$$

with equality exactly when n is squarefree. This plainly implies the stated conclusion. □

Proof of Theorem 2.1. Let us start with a weaker result, namely that

$$R(n) \leqslant (\log n)^{\log 3+o(1)} \quad \text{a.a.}$$

Let $\mathcal{R}$ be a decomposition of an $n \times n$ square, $R(n) = R$, and A the common area of the rectangles. By definition, $AR = n^2$. Lemma 2.2 implies that $\tau(n^2) = (\log n)^{\log 3+o(1)}$ a.a. By assumption, every rectangle has a different height. That height divides A, which in turn divides n^2. Therefore, the heights of the rectangles are all distinct divisors of n^2, which implies that $R \leqslant \tau(n^2) \leqslant (\log n)^{\log 3+o(1)}$ a.a.

Now consider the dimensions of the rectangles more closely. Suppose one of our rectangles has height h and width w. In order to fit inside the original square, we must have $\max(h, w) \leqslant n$. However,

$$wh = A = n^2/R \geqslant n^2/(\log n)^{\log 3+o(1)},$$

hence $\min(h, w) \geqslant n/(\log n)^{\log 3+o(1)}$.

Thus, we can modify the bound in our first paragraph. In addition to being distinct divisors of n^2, the heights of the rectangles must also lie in the interval $[n/(\log n)^{\log 3+o(1)}, n)$. Since, by definition, any interval of logarithmic length 1 contains at most $\Delta(n^2)$ divisors of n^2, we see that

$$R(n) \leqslant 2(\log_2 n)\Delta(n^2) \quad \text{a.a.} \tag{2.2}$$

Since we have $\Delta(n^2) = (\log n)^{(\log 3)-1+o(1)}$ a.a. by a recent result of La Bretèche and Tenenbaum [3], the proof is complete. □

Remark. Our proof of Theorem 1.2 only requires that $R(n) \leqslant (\log n)^c$ a.a. for some $c < 1$. In view of (2.1), this follows with any $c \in (\log 2, 1)$ from the classic inequality $\Delta(ab) \leqslant \Delta(a)\tau(b)$, valid for all integers a, b.

The following observation will turn out to be crucial for our approach.

Lemma 2.3. *Let n be a primitive Mondrian integer and let $r > 1$ be a proper divisor of n. Put $I(n) := \{d : d|n^2,\ n/\tau(n^2) \leqslant d < n\}$. Then there is a representation*

$$n = \sum_{d \in I(n)} \varepsilon_d d, \tag{2.3}$$

where $\varepsilon_d \in \{0, 1\}$, $\sum_{d\in I(n)} \varepsilon_d \geqslant 2$, and $r \nmid d$ for at least one $d \in I(n)$ such that $\varepsilon_d = 1$.

Proof. Fix a Mondrian decomposition into $R \geqslant 2$ rectangles of common area A. As previously observed, if d is the height or width of any rectangle, we have

$$n/\tau(n^2) \leqslant d < n.$$

The side lengths encountered by such a line are distinct, since rectangles of equal area having the same height or width would be identical. Therefore, any horizontal or vertical line through the square therefore furnishes a representation the form (2.3) except possibly for the condition involving r.

If every height and every width in the decomposition were divisible by r, then dividing all side lengths by r would produce a Mondrian decomposition of an $(n/r) \times (n/r)$ square, contradicting the primitivity of n. Thus, some rectangle has a side not divisible by r. Assume without loss of generality, that this side is the width. Passing a horizontal line through the interior of that rectangle in the perpendicular direction furnishes the required representation. □

3. Lemmas on sets of multiples

The argument of the proof of Theorem 1.2 is modelled on Erdős' treatment of primitive bases in [8]. We first recall the necessary density results for sets of multiples.

Notation. Given a set of positive integers S, we let $\mathcal{M}(S)$ denote the set of integers which are multiples of some element of S, and write $\overline{D}(S)$ (resp. $\underline{D}(S)$) for the upper (resp. lower) asymptotic density of $\mathcal{M}(S)$. Whenever these quantities are equal, their common value is denoted by $D(S)$.

The following fundamental theorem [5, 6] (see also [16, eqn. (1.1)]) is due to Davenport and Erdős.

Theorem 3.1. *For every set S of positive integers, $\mathcal{M}(S)$ has a logarithmic density, which is equal to its lower asymptotic density. Moreover,*

$$\underline{D}(S) = \lim_{N\to\infty} D(S \cap [1, N]).$$

This statement enables us to readily extend to arbitrary sets Behrend's inequality [1], initially stated for sets of multiples of finite sets—see, e.g., exercises 247–250 of [17], with solutions in [18].

Theorem 3.2 (Behrend)**.** *If S and T are sets of positive integers, then*

$$1 - \underline{D}(S \cup T) \geqslant (1 - \underline{D}(S))(1 - \underline{D}(T)).$$

Erdős [8, p. 124] observed that Behrend's inequality implies the following result.

Lemma 3.3 ([8, lemma 1])**.** *Let $S \subset \mathbb{Z}_+ \setminus \{1\}$. Suppose that*

$$\overline{D}(S \cap [N, \infty)) = o(1) \quad (N \to \infty).$$

Then $D(S)$ is well-defined and is < 1.

Corollary 3.4. *Let $S \subset \mathbb{Z}_+ \setminus \{1\}$. If $\sum_{n\in S} 1/n < \infty$, then $D(S)$ is well-defined and is < 1.*

Proof. For every $N \geqslant 1$,

$$\overline{D}(S \cap [N, \infty)) \leqslant \sum_{\substack{n\in S \\ n\geqslant N}} \frac{1}{n}.$$

The right-hand side tends to zero, so Lemma 3.3 applies. □

4. An average estimate for moments of $\Delta(n^2)$

The following result is proved by adapting the method of [11, th. 3].

Theorem 4.1. *Let $y > 0$, $t \geqslant 1$, and put $\gamma(y,t) := y - 1 + \big(3^t y - y - t\big)^+$. For any fixed $\kappa > \sqrt{2t}$, we have*

$$\sum_{n\leqslant x} \Delta(n^2)^t y^{\omega(n)} \ll x(\log x)^{\gamma(y,t)} \mathrm{e}^{\kappa\sqrt{\log_2 x\, \log_3 x}}. \tag{4.1}$$

Remarks. (i) Given $r \in \mathbb{Z}_+^*$, this statement readily extends by the same method to the estimate

$$\sum_{n\leqslant x} \Delta(n^r)^t \varrho(n) \ll x(\log x)^{\gamma_r(y,t)} \mathrm{e}^{\kappa\sqrt{\log_2 x \log_3 x}} \tag{4.2}$$

with $\gamma_r(y,t) := y - 1 + ((r+1)^t y - y - t)^+$, fixed $\kappa > \sqrt{2t}$, and ϱ is a multiplicative function polynomially bounded on prime powers and such that, for suitable $c = c(\varrho) > 0$,

$$\sum_{p\leqslant x} \varrho(p)\log p = yz + O\Big(\frac{z}{(\log z)^c}\Big) \qquad (z \geqslant 2).$$

(ii) For $t = 1$, as shown in [2], the quantity $\log_3 x$ can removed from the exponential factor of (4.1) at the cost of selecting a suitable $\kappa = \kappa(y,t)$. Adapting the method of [2] clearly leads to removing this term in all cases.

Proof. Define

$$M_q(n) := \int_{\mathbb{R}} \Delta(n,u)^q \,\mathrm{d}u \quad (q \geqslant 1,\, n \geqslant 1),$$

and, for integer $q \geqslant t$, $1 < \sigma \leqslant 2$, consider the Dirichlet series

$$L(\sigma) := \sum_{n\geqslant 1}^{*} \frac{M_q(n^2)^{t/q} y^{\omega(n)}}{n^\sigma}.$$

where, here and in the sequel, the asterisk indicates that the sum is restricted to squarefree integers. We have

$$-L'(\sigma) = y\sum_p \frac{\log p}{p^\sigma} \sum_{\substack{m\geqslant 1\\ p\nmid m}}^{*} \frac{M_q(m^2p^2)^{t/q} y^{\omega(m)}}{m^\sigma}. \tag{4.3}$$

From the identity

$$\Delta(m^2p^2,u) = \sum_{0\leqslant j\leqslant 2} \Delta(m^2, u - j\log p),$$

we get $M_q(m^2p^2) = 3M_q(m^2) + W_q(m,p)$ with

$$W_q(m,p) = \sum_{\substack{j_0+j_1+j_2=q\\ \max j_h<q}} \binom{q}{j_0,j_1,j_2} \int_{\mathbb{R}} \prod_{0\leqslant h\leqslant 2} \Delta(m^2, u - h\log p)^{j_h} \,\mathrm{d}u.$$

Arguing as in [11, lemma 3] or [3, prop. 4.5], we get

$$\sum_p \frac{W_q(m,p)\log p}{p} \leqslant C6^{2q}3^{\omega(m)}M_{q-1}(m^2)$$
$$\leqslant C6^{2q}3^{q\omega(m)/(q-1)}M_q(m^2)^{(q-2)/(q-1)},$$

where the latter inequality, trivial if $q=2$, is obtained for $q\geqslant 3$ by applying Minkowski's inequality and then Hölder's with exponents $(q-1)/(q-2)$, $q-1$. Substitution into (4.3) and applying [11, lemma 1]

$$\sum_p \frac{\log p}{p^\sigma} < \frac{1}{\sigma-1} \qquad (\sigma>1),$$

yields, provided $q\geqslant t$,

$$\begin{aligned}-L'(\sigma) &\leqslant \frac{y3^{t/q}L(\sigma)}{\sigma-1} + \frac{y}{(\sigma-1)^{1-t/q}}\sideset{}{^*}\sum_{m\geqslant 1}\frac{y^{\omega(m)}}{m^\sigma}\Big(\sum_p \frac{W(m,p)\log p}{p}\Big)^{t/q}\\ &\leqslant \frac{y3^{t/q}L(\sigma)}{\sigma-1} + \frac{yC^{t/q}6^{2t}}{(\sigma-1)^{1-t/q}}\sideset{}{^*}\sum_{m\geqslant 1}\frac{y^{\omega(m)}3^{t\omega(m)/(q-1)}M_q(m^2)^{t(q-2)/q(q-1)}}{m^\sigma}\\ &\leqslant \frac{y3^{t/q}L(\sigma)}{\sigma-1} + \frac{yC^{t/q}6^{2t}L(\sigma)^{(q-2)/(q-1)}}{(\sigma-1)^{1-t/q}}\Big(\sideset{}{^*}\sum_{m\geqslant 1}\frac{(3^ty)^{\omega(m)}}{m^\sigma}\Big)^{1/(q-1)}.\end{aligned}$$

In view of the bound

$$\sideset{}{^*}\sum_{m\geqslant 1}\frac{(3^ty)^{\omega(m)}}{m^\sigma} = \prod_p\Big(1+\frac{3^ty}{p^\sigma}\Big) < \zeta(\sigma)^{3^ty} < \Big(\frac{2}{\sigma-1}\Big)^{3^ty} \qquad (1<\sigma\leqslant 2),$$

we finally obtain

$$-L'(\sigma) \leqslant \frac{y3^{t/q}L(\sigma)}{\sigma-1} + \frac{yC^{t/q}2^{3^ty/(q-1)}6^{2t}L(\sigma)^{(q-2)/(q-1)}}{(\sigma-1)^{1-t/q+3^ty/(q-1)}}.$$

Assume momentarily that $y\geqslant t/(3^t-1)$. A solution of the associated differential equation is

$$X(\sigma) := \frac{K}{(\sigma-1)^{\alpha(t,q)}}, \qquad \alpha(t,q) := 3^ty - t(1-1/q),$$

and

$$K := \frac{y^{q-1}C^{t(q-1)/q}2^{3^ty}6^{2t(q-1)}}{\{\alpha(t,q)-3^{t/q}y\}^{q-1}}.$$

Since $C\geqslant 1$,

$$L(2) \leqslant \sideset{}{^*}\sum_{n\geqslant 1}\frac{(3^ty)^{\omega(n)}}{n^2} < \prod_p\Big(1+\frac{3^ty}{p^2}\Big) < 2^{3^ty},$$

and $\alpha(t,q)<3^ty$, we have $L(2)<X(2)$. By [12, lemma 70.2], we therefore infer that

$$L(\sigma) < X(\sigma) \qquad (1<\sigma\leqslant 2).$$

Now $\alpha(t,q)-3^{t/q}y \geqslant t(3^t-3^{t/q})/(3^t-1)-t+t/q \geqslant t/q$ since $q \geqslant t$. Therefore $K \leqslant (C_0q)^q$ with $C_0 = C_0(y,t)$. Selecting $q := \left\lfloor \sqrt{2t\log_2 x/\log_3 x} \right\rfloor$ and $\sigma := 1+1/\log x$, we obtain (4.1) provided $y \geqslant t/(3^t-1)$. Here, we have used the inequality [12, th. 72]

$$\Delta(n) \leqslant 2M_q(n)^{1/q} \qquad (n \geqslant 1),$$

and the bound

$$\sum_{n\leqslant x} \Delta(n^2)^t y^{\omega(n)} \ll \frac{x}{\log x} \sum_{n\leqslant x} \frac{\Delta(n^2)^t y^{\omega(n)}}{n},$$

which can be proved by the argument of [12, th. 61].

We still need to deal with the $y < t/(3^t-1)$ case. Then, for suitable $T > t$, we have $y = T/(3^T-1)$. Estimating the left-hand side of (4.1) with T in place of t and applying Hölder's inequality with exponents T/t, $T/(T-t)$ yields the required estimate. □

5. Proof of Theorem 1.2

Let $\mathcal{A}$ be the set of Mondrian integers and let $\mathcal{B}$ be its primitive basis. We want to show that $\overline{D}(\mathcal{B}) < 1$

Fix a sufficiently small $\varepsilon > 0$ and split

$$\mathcal{B} \subset \mathcal{B}_1 \cup \mathcal{B}_2, \qquad \mathcal{B}_1 := \{b > 1 : \Omega(b) > (1+\varepsilon)\log_2 b\}, \tag{5.1}$$

with $\mathcal{B}_2 := \mathcal{B} \setminus \mathcal{B}_1$. An adaptation of the Hardy–Ramanujan theorem in the form of [17, th. III.3.9] yields

$$\lim_{N\to\infty} \overline{D}\big(\mathcal{B}_1 \cap [N,\infty)\big) = 0.$$

Hence Lemma 3.3 implies

$$D(\mathcal{B}_1) < 1. \tag{5.2}$$

We claim that, for some $\kappa > 1$,

$$\big|\mathcal{B}_2 \cap (1,x]\big| \ll \frac{x}{(\log x)^\kappa} \quad (x \geqslant 2). \tag{5.3}$$

This implies the convergence of $\sum_{b\in\mathcal{B}_2} 1/b$, and in turn $D(\mathcal{B}_2) < 1$ by Corollary 3.4. Taking (5.2) into account we deduce $\overline{D}(\mathcal{B}) < 1$ from Theorem 3.2.

Let us now embark on proving (5.3). By the simple estimate proved in [17, th. III.5.1], the number of $b \leqslant x$ for which

$$P^+(b) \leqslant x^{1/(4\log_2 x)}$$

is $\ll x/(\log x)^2$. Standard estimates for friable integers then yield that the same bound holds for those b such that $P^+(b)^2 \mid b$.

Next, we show that, for suitable $c < 1$, we have

$$T(b) := \left|\left\{d \mid b^2 : \frac{b}{(\log b)^2} < d < b\right\}\right| \leqslant (\log x)^c \tag{5.4}$$

for all but at most $\ll x/(\log x)^\kappa$ integers $b \leqslant x$. Since $T(b) \ll \Delta(b^2)\log_2 x$ for $b \leqslant x$, we bound $\Delta(b^2)$. Theorem 4.1 with $t = 2$ and $y = 1/4$ furnishes the bound

$$\sum_{\substack{b\leqslant x\\ b\in\mathcal{B}_2}} \Delta(b^2)^2 \leqslant \sum_{b\leqslant x} \Delta(b^2)^2 y^{\omega(b)-(1+\varepsilon)\log_2 b} \ll x(\log x)^{-3/4+(1+\varepsilon)\log 4+o(1)}.$$

Consequently, for every fixed c with $\frac{1}{8} + \log 2 < c < 1$, we deduce from Markov's inequality that, on taking ε sufficiently small, the bound (5.4) holds for all but $\ll x/(\log x)^\kappa$ integers $b \leqslant x$, provided $1 < \kappa < 2c + \frac{3}{4} - \log 4$.

It remains to count those $b \in \mathcal{B}_2 \cap (1, x]$ satisfying all previous conditions. Write

$$b = mp, \qquad p = P^+(b) > P^+(m), \qquad p > x^{1/(4\log_2 x)}.$$

Here p is a proper divisor of b: a prime number cannot satisfy the representation supplied by Lemma 2.3. Since $\Omega(b) \leqslant (1+\varepsilon)\log_2 x$, we have

$$\tau(b^2) \leqslant 3^{\Omega(b)} < (\log x)^2$$

for small ε and large x. Applying Lemma 2.3 with $r = p$, we obtain

$$b = mp = \sum_{\substack{d\mid m^2p^2\\ mp/(\log x)^2<d<mp}} \varepsilon_d d, \tag{5.5}$$

where at least one selected d is not divisible by p.

Let

$$D := \sum_{\substack{d\mid m^2\\ mp/(\log x)^2<d<mp}} \varepsilon_d d.$$

Then $D \geqslant 1$, and reducing (5.5) modulo p gives $p \mid D$. There are at most $T(b) \leqslant (\log x)^c$ summands in (5.5), so the number of possible values of D does not exceed $\tau(b^2)^{T(b)} \leqslant \mathrm{e}^{2(\log x)^c \log_2 x}$. Since $D \leqslant b \leqslant x$, each D has $\ll \log x$ distinct prime factors. Moreover

$$m \leqslant x^{1-1/(4\log_2 x)}.$$

Therefore the number of remaining $b = mp$ is

$$\ll x^{1-1/(4\log_2 x)}\mathrm{e}^{2(\log x)^c\log_2 x}\log x \ll \frac{x}{(\log x)^2},$$

because $c < 1$. This proves (5.3) and so completes the proof.

6. Higher dimensions

Definition. Let k be an integer exceeding 1. An integer n is said to be *k-Mondrian* if it is possible to partition a k-dimensional hypercube with side length n into distinct k-dimensional rectangular prisms with the same volume and integral side lengths. We say that n is a *strong k-Mondrian integer* if the additional restriction that the heights of the prisms are distinct can be imposed.

The notions of 2-Mondrian and strong 2-Mondrian integer coincide. As stated in Theorem 1.3, we can extend Theorem 1.2 to 3-strong Mondrian integers.

Proof of Theorem 1.3. First we obtain a result similar to Lemma 2.3. Suppose we have a Mondrian decomposition of a cube with side length n. Then the height of every rectangular prism is a distinct divisor of n^3. So, the number of prisms is at most $\tau(n^3)$. Since all volumes are equal and all side lengths do not exceed n, we deduce that every side length is at least $n/\tau(n^3)$ and we conclude as before that, writing $I(n) := [n/\tau(n^3), n)$, we have

$$n = \sum_{\substack{d \in I(n) \\ d \mid n^3}} \varepsilon_d d$$

where $\varepsilon_d \in \{0, 1\}$.

If in addition n is primitive and r is a proper divisor of n, then all side lengths cannot be divisible by r for dividing all coordinates by r would produce a strong 3-Mondrian decomposition of n/r, contradicting primitivity.

We may then argue as in the proof of Theorem 1.2, by considering the primitive basis $\mathcal{B}$ of the set of strong 3-Mondrian integers and, after selecting $\varepsilon > 0$ sufficiently small, splitting $\mathcal{B} \subset \mathcal{B}_1 \cup \mathcal{B}_2$ as in (5.1) and showing that, for suitable $\kappa > 1$,

$$B_2(x) := |\mathcal{B}_2 \cap (\sqrt{x}, x]| \ll \frac{x}{(\log x)^\kappa} \quad (x \geqslant 2). \tag{6.1}$$

Let us make the convention that, here and throughout, the letter ℓ denotes a prime number. Define $z := x^{1/(4\log_2 x)}$, $b_z := \prod_{\substack{\ell^\nu \| b \\ \ell \leqslant z}} \ell^\nu$ for $b \geqslant 1$, and let

$$S := \{\sqrt{x} < b \leqslant x : b \in \mathcal{B}_2,\ P^+(b) > z,\ q^2 | b \Rightarrow q \leqslant z,\ b_z \leqslant b^{1/4}\}.$$

We then have

$$\big|\{\sqrt{x} < b \leqslant x : b \in \mathcal{B}_2 \setminus S\}\big| \ll x/(\log x)^2.$$

This is established as in the planar case, except for the property involving b_z, which readily follows from [17, ex. 299]—see [18] for the solution.

For $b = mp$, with $p := P^+(b)$, and $x \geqslant 2$, put $T_3(b) := \big|\{d | m^3 : d \in I(b)\}\big|$. We shall show that for suitable $c < 1 < \kappa$, we have

$$|\{b \leqslant x : b \in S,\ T_3(b) > (\log x)^c\}| \ll x/(\log x)^\kappa. \tag{6.2}$$

This is sufficient to complete the proof arguing exactly as in the 2-dimensional case.

Let $T_2(b)$ denote the subsum of $T_3(b)$ comprising those d dividing b^2. Then we know that the stated bound holds for any $c_0 > 1/8 + \log 2 \approx 0.81847$. Note that the replacement of $\tau(b^2)$ by $\tau(b^3)$ in the definition of $I(b)$ has no influence once we use the bound $\tau(b^3) \leqslant (\log x)^2$ a.a., and appeal to a relevant bound for the size of $\Delta(b^2)$.

Put $U(b) := T_3(b) - T_2(b)$. By the definition of S, a divisor d counted by $U(b)$ must have $P^+(d) > z$. We may thus write

$$U(b)^2 \leqslant V_1(b) + V_2(b),$$

with

$$V_1(b) := \sum_{\substack{q^\nu \| m^3 \\ 1 \leqslant \nu \leqslant 3}} \sum_{\substack{d_1, d_2 | m^3/q^\nu \\ d_j q^\nu \in I(b) \\ (j=1,2)}} 1, \quad V_2(b) := \sum_{\substack{q_1 \neq q_2 \\ q_1^{\nu_1} q_2^{\nu_2} \| m^3 \\ 1 \leqslant \nu_j \leqslant 3}} \sum_{\substack{d_j | m^3/q_j^{\nu_j} \, (j=1,2) \\ d_j q_j^{\nu_j} \in I(b)}} 1,$$

where q, with or without subscript, denotes a prime number $> z$.

Consider

$$\mathfrak{S}_j(x) := \sum_{b \leqslant x} V_j(b) \qquad (j = 1, 2).$$

Inverting summations, we see that, in $\mathfrak{S}_1(x)$, d_1/d_2 lies in an interval of multiplicative length $\leqslant (\log x)^2$. Therefore, writing $L = \log x$, $\xi = \log_2 x$,

$$\begin{aligned}
\mathfrak{S}_1(x) &\ll \sum_{\substack{h \leqslant x/z \\ \Omega(h) \leqslant (1+\varepsilon)\xi}} \sum_{\substack{d_1, d_2 | h^3 \\ |\log d_2/d_1| \leqslant 2\xi}} \sum_{2 \leqslant \nu_1, \nu_2 \leqslant 3} \sum_{\substack{hp/d_j L^3 < q^{\nu_j - 1} < hp/d_j \\ (j=1,2)}} \sum_{z < p \leqslant x/hq} 1 \\
&\ll \sum_{\substack{h \leqslant x/z \\ \Omega(h) \leqslant (1+\varepsilon)\xi}} \sum_{\substack{d_1, d_2 | h^3 \\ |\log d_2/d_1| \leqslant 2\xi}} \sum_{2 \leqslant \nu_1, \nu_2 \leqslant 3} \sum_{\substack{hp/d_j L^3 < q^{\nu_j - 1} < hp/d_j \\ (j=1,2)}} \frac{x\xi}{hqL} \\
&\ll \sum_{\substack{h \leqslant x/z \\ \Omega(h) \leqslant (1+\varepsilon)\xi}} \frac{x\xi^3}{hL^3} \sum_{\substack{d_1, d_2 | h^3 \\ |\log d_2/d_1| \leqslant 2\xi}} 1.
\end{aligned}$$

Introducing the multiplicative function $\tau(n, \vartheta) := \sum_{d|n} d^{i\vartheta}$ $(\vartheta \in \mathbb{R})$, we see by a standard application of the Fourier inversion formula

$$\Big(1 - \frac{v}{4\xi}\Big)^+ = \frac{2\xi}{\pi} \int_{\mathbb{R}} \Big(\frac{\sin 2\xi\vartheta}{2\xi\vartheta}\Big)^2 \mathrm{e}^{-iv\vartheta} \, \mathrm{d}\vartheta$$

that the inner sum is

$$\ll \xi \int_0^1 |\tau(h^3, \vartheta)|^2 \, \mathrm{d}\vartheta.$$

Writing $\vartheta_\ell = \vartheta \log \ell$, we have for any prime ℓ,

$$|\tau(\ell^3, \vartheta)|^2 = 16\{\cos(\tfrac{1}{2}\vartheta_\ell)\}^6 = \tfrac{1}{2}\big\{10 + 15\cos\vartheta_\ell + 6\cos(2\vartheta_\ell) + \cos(3\vartheta_\ell)\big\}.$$

Applying [12, lemma 30.1] as in the proof of [12, lemma 30.3], we get, for fixed $y \in (0, 1]$,

$$\int_0^1 \sum_{h \leqslant x} \frac{\tau(h^3, \vartheta)^2 y^{\Omega(h) - (1+\varepsilon)\xi}}{h} \, \mathrm{d}\vartheta \ll (\log x)^{5y + (11y-1)^+ - (1+\varepsilon)\log y} (\log_2 x)^{\delta(1,y)},$$

where $\delta(1, y) := \mathbf{1}_{y=1}$. Carrying this back into our previous bound for $\mathfrak{S}_1(x)$, we get, on selecting $y = \frac{1}{11}$,

$$\mathfrak{S}_1(x) \ll x(\log x)^{(1+\varepsilon)\log 11 - 18/11} (\log_2 x)^4.$$

Therefore, if ε has been chosen sufficiently small, we get that, for any $c_1 > \frac{1}{2}\log 11 - \frac{9}{11}$, the number of $b \in]\sqrt{x}, x] \cap S$ such that $V_1(b) > (\log x)^{2c_1}$ does not exceed $\ll x/(\log x)^{\kappa_1}$ provided $\kappa_1 < 2c_1 - \log 11 + \frac{18}{11}$. For $c_1 = \frac{9}{10}$, we may select $\kappa_1 > 1$.

Let us now bound $\mathfrak{S}_2(x)$. We have

$$\begin{aligned}
\mathfrak{S}_2(x) &\ll \sum_{\substack{h\leqslant x/z\\ \Omega(h)\leqslant(1+\varepsilon)\xi}} \sum_{d_1,d_2\mid h^3} \sum_{2\leqslant\nu_1,\nu_2\leqslant 3} \sum_{\substack{hp/(d_jL^2)<q_j^{\nu_j-1}<hp/d_j\\ (j=1,2)}} \sum_{z<p\leqslant x/hq_1q_2} 1\\
&\ll \sum_{\substack{h\leqslant x/z\\ \Omega(h)\leqslant(1+\varepsilon)\xi}} \sum_{d_1,d_2\mid h^3} \sum_{2\leqslant\nu_1,\nu_2\leqslant 3} \sum_{\substack{hp/(d_jL^2)<q_j^{\nu_j-1}<hp/d_j\\ (j=1,2)}} \frac{x\xi}{hq_1q_2L}\\
&\ll \frac{x\xi^3}{L^3} \sum_{\substack{h\leqslant x/z\\ \Omega(h)\leqslant(1+\varepsilon)\xi}} \frac{\tau(h^3)^2}{h}\\
&\ll x(\log x)^{4(1+\varepsilon)\log 2-2}(\log_2 x)^3.
\end{aligned}$$

It follows that, given $c_2 > 2\log 2 - 1$ and suitably small $\varepsilon > 0$, the number of $b \in]\sqrt{x}, x] \cap S$ such that $V_2(b) > (\log x)^{2c_2}$ does not exceed $\ll x/(\log x)^{\kappa_2}$, provided $\kappa_2 < 2 + 2c_2 - 4\log 2$. Selecting $c_2 = \frac{9}{10}$ ensures that $\kappa_2 > 1$ is possible.

We conclude from these computations that (6.2) holds with $c = \frac{9}{10} > c_0$ and suitable $\kappa > 1$. □

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