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On moments of the Erdős–Hooley Delta-function

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To Roger Heath-Brown,
on the occasion of
his seventy-fifth birthday

Abstract. For integer $n \geqslant 1$ and real u, let $\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|$. The Erdős–Hooley Delta-function is then defined by $\Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u)$. We provide new upper bounds for weighted real moments of this function.

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1. Introduction

For integer $n \geqslant 1$ and real u, put

$$\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \quad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u).$$

Introduced by Erdős [5] (see also [6]) and studied by Hooley [11], the Δ-function and specifically its (possibly weighted) mean-value proved very useful in several branches of number theory — see, e.g., [16], [10], and [2] for further references. If $\tau(n)$ denotes the total number of divisors of n, then $\Delta(n)/\tau(n)$ coincides with the concentration of the numbers $\log d$, $d|n$.

Asymptotic estimates for

$$S(x) := \sum_{n\leqslant x} \Delta(n)$$

have a rather long history since Hooley's pioneer work [11]: see [3] for references. Recent progress regarding this mean-value is due to Koukoulopoulos and Tao [12], Ford, Koukoulopoulos and Tao [8], and the authors [3]. The current best estimates are

$$x(\log_2 x)^{1+\eta+o(1)} \ll S(x) \ll x(\log_2 x)^{5/2},$$

where $\eta \approx 0.353327$, the lower and upper bounds being respectively proved in [8] and [3]. Here and in the sequel we let $\log_k$ denote the k-fold iterated logarithm.

In [4], we applied the latter upper bound to Waring type problems, showing that, for certain integer vectors $(2, \ell_1, \ldots, \ell_t) \in \mathbb{N}^{*(t+1)}$ satisfying $1/2 + \sum_{1\leqslant j\leqslant t} 1/\ell_j = 1$, the number of integers n not exceeding x and representable by the form $m_0^2 + \sum_{1\leqslant j\leqslant t} m_j^{\ell_j}$ is $\gg x/(\log_2 x)^{5/2}$.

Given $z > 0$, we let $\mathcal{M}_z$ denote the class of those non-negative multiplicative functions ϱ that are bounded on the set of prime powers and satisfy, for suitable $c = c(\varrho) > 0$,

$$\sum_{p\leqslant y} \varrho(p)\log p = zy + O\Big(\frac{y}{(\log y)^c}\Big) \qquad (y \geqslant 2). \tag{1·1}$$

We then have

$$\sum_{P^+(n)<x} \frac{\mu(n)^2\varrho(n)}{n} = \prod_{p<x}\Big(1 + \frac{\varrho(p)}{p}\Big) \asymp (\log x)^z \qquad (x \geqslant 2).$$

Here and throughout, μ denotes the Möbius function, the letter p denotes a prime number. A standard instance of an element of $\mathcal{M}_z$ is the function defined by $\varrho(n) := z^{\omega(n)}$.

In this work, we investigate upper bounds for the weighted moments

$$S_{t,\varrho}(x) := \sum_{n\leqslant x} \varrho(n)\Delta(n)^t \qquad (t \geqslant 1,\ \varrho \in \mathcal{M}_z,\ x \geqslant 1). \tag{1·2}$$

Here and in the sequel, t and z are fixed parameters, $\varrho \in \mathcal{M}_z$, and implicit constants may depend upon t and ϱ. Estimates for $S_{t,\varrho}(x)$ are potentially useful for determining dominant values of $\varrho(n)\Delta(n)$ in certain averages. In [17], this was used with $t = 1 + \varepsilon$, for arbitrary small ε.

Writing $\beta = \beta(t, z) := 2^t z - t$, $z_t := t/(2^t - 1)$, we have, for real $t \geqslant 1$,

$$x(\log x)^{\max\{\beta,z\}-1} \ll S_{t,\varrho}(x) \ll x(\log x)^{\max\{\beta,z\}-1}\mathrm{e}^{\{\kappa+o(1)\}\sqrt{(\log_2 x)\log_3 x}}, \tag{1·3}$$

with

$$\kappa := \begin{cases} 2t & \text{if } z \geqslant z_t, \\ 2\sqrt{1-\log(1/z)/\log_2 2} & \text{if } z < z_t. \end{cases}$$

The upper bound is proved in [9][(1)] while the lower bound follows from the inequality (see, e.g., [10; th. 60])

$$\Delta(n) \geqslant \max\Big\{1, \frac{\tau(n)}{1+\log n}\Big\} \qquad (n \geqslant 1). \tag{1·4}$$

Our approach to bounding $S_{t,\varrho}(x)$ is primarily based on an iterative procedure restricting the parameter t to integer values. While the induction step is based on the method developed in [12], the initialization turns out to be non trivial and necessitates new ideas—see Propositions 4.1 and 6.1 *infra*. The extension to real values of the exponent is derived secondarily by exploiting the flexibility offered by the parameter z.

We obtain the following results. Here and in the sequel, we let $\delta = \delta(t, z) := \mathbf{1}_{\beta=z} = \mathbf{1}_{z=z_t}$.

Theorem 1.1. *Let $t \geqslant 1$, $z > 0$, and $\varrho \in \mathcal{M}_z$.*

(i) For $z \geqslant z_t$ and all large x, we have

$$S_{t,\varrho}(x) \ll x(\log x)^{\beta-1}(\log_2 x)^{t+1+\delta}. \tag{1·5}$$

(ii) For $0 < z < z_t$ and all large x, we have

$$S_{t,\varrho}(x) \ll x(\log x)^{z-1}(\log_2 x)^{t+2t/s}, \tag{1·6}$$

where $s > t$ is defined by $z_s = z$.

Remarks. (i) As shown in [3], the upper bound (1·5) can be improved to $x(\log_2 x)^{5/2}$ in the case $z = t = 1$.

(ii) As will be clear from the proofs, all our estimates are locally uniform in z and t in the complement of the line $\beta = z$. In any admissible compact domain in (z, t), they only depend on the constants involved in (1·1).

Inserting (1·5) into the proof of [13; cor. 4] readily yields the following statement, where [3; th. 1.1] has been taken into account for the case $t = 1$.

Corollary 1.2. *Let $F \in \mathbb{Z}[X]$ be irreducible with no fixed prime divisor. Then, for any $t \geqslant 1$ and $\varepsilon > 0$, we have*

$$\sum_{x<n\leqslant x+y} \Delta\big(|F(n)|\big)^t \ll y(\log x)^{\beta(t,1)-1}(\log_2 x)^{t+1+\delta(t,1)/2}$$

provided $x^\varepsilon \leqslant y \leqslant x$.

For exponents at least equal to 2 and sufficiently large z, we obtain the true order of magnitude.

Theorem 1.3. *Let $t \geqslant 2$, $z > z_t^+ := t/(2^t - 2^{t/2}) = z_t(1 + 2^{-t/2})$, $\varrho \in \mathcal{M}_z$. We have*

$$S_{t,\varrho}(x) \asymp x(\log x)^{\beta-1} \qquad (x \geqslant 2). \tag{1·7}$$

We have $z_2^+ = 1$. This motivates the next statement.

Theorem 1.4. *Let $\varrho \in \mathcal{M}_1$. We have*

$$S_{2,\varrho}(x) = \sum_{n\leqslant x} \varrho(n)\Delta(n)^2 \ll x(\log x)(\log_2 x)\sqrt{\log_3 x} \qquad (x \geqslant 16). \tag{1·8}$$

This last estimate improves by a factor essentially $\log_2 x$ the current upper bound for the quadratic mean-value of the Delta-function: see [3; th. 1.2] where an argument from [10] is made explicit.

From (1·8) and [13; th. 1], we derive the following corollary, which seems out of reach by standard techniques.

Corollary 1.5. *Let $\varepsilon \in]0, 1[$. Uniformly for $N \geqslant 1$, $x \geqslant 3$, $x^\varepsilon \leqslant y \leqslant x$, we have*

$$\sum_{N<m,n\leqslant 2N} \bigg(\Big\lfloor \frac{x+y}{[m,n]} \Big\rfloor - \Big\lfloor \frac{x}{[m,n]} \Big\rfloor \bigg) \ll y(\log x)(\log_2 x)\sqrt{\log_3 x}.$$

1. The proof of [9] is written for $\varrho(n) := z^{\omega(n)}$ but it readily extends to $\varrho \in \mathcal{M}_z$.

2. Notation

Put

$$E := \{n \geqslant 1 : \mu(n)^2 = 1\}, \quad n_y := \prod_{p|n,\, p<y} p \quad (n \geqslant 1,\, y \geqslant 1),$$

Given $t > 0$ and $T \geqslant 2$, put

$$\mathfrak{b} = \mathfrak{b}_{t,z} := \frac{1}{2^t z \log 2 - 1}, \tag{2·1}$$

$$f_T(y) := \mathfrak{c} \min\left(\log(T \log 3y), \frac{(\log_2 3y - \mathfrak{b}_{t,z} \log T)^2}{\log T}\right) \quad (y \geqslant 1), \tag{2·2}$$

where $\mathfrak{c}$ is a strictly positive constant to be chosen sufficiently small in the sequel.

Define

$$\begin{aligned} E_T^{t,z} &:= \big\{n \in E : \tau(n_y) \leqslant \mathrm{e}^{-f_T(y)} T \log 3y \ (y \geqslant 1)\big\}, \\ E_T &:= \big\{n \in E : \tau(n_y) \leqslant T \log 3y \ (y \geqslant 1)\big\}. \end{aligned} \tag{2·3}$$

It is useful to bear in mind that $n \in E_T^{t,z}$ implies that $d \in E_T^{t,z}$ for any divisor d of n.

Let $P^+(n)$—resp. $P^-(n)$—denote the largest—resp. the smallest—prime factor of $n > 1$ with the convention that $P^+(1) := 1$, $P^-(1) = \infty$. Given $x \geqslant 2$, we consider the probability $\mathbb{P}_{x,\varrho}$ defined on E by

$$\mathbb{P}_{x,\varrho}(\{n\}) := \frac{\varrho(n)}{n} \prod_{2 \leqslant p < x} \Big(1 + \frac{\varrho(p)}{p}\Big)^{-1} \asymp \frac{\varrho(n)}{n(\log x)^z} \qquad \big(P^+(n) < x\big),$$

and let $\mathbb{E}_{x,\varrho}$ denote the corresponding expectation.

Throughout this article, the symbol $\sum^x$ indicates a summation over squarefree integers whose prime factors are restricted to the interval $[2, x[$. For the sake of future reference, we note that, if $\varrho \in \mathcal{M}_z$, then

$$\sum_{n \geqslant 1}^{x} \frac{\mu(n)^2 \varrho(n)}{n} \asymp (\log x)^z \qquad (x \geqslant 2). \tag{2·4}$$

3. Basic lemmas

Lemma 3.1. *Let $t \geqslant 1$, $z > 0$ and $\varrho \in \mathcal{M}_z$. We have*

$$S_{t,\varrho}(x) \asymp x(\log 3x)^{z-1}\, \mathbb{E}_{x,\varrho}(\Delta^t) \qquad (x \geqslant 1). \tag{3·1}$$

Proof. This is a trivial extension of [10; th. 61]. We omit the details. □

As in [12], the first step to the proof of Theorem 1.1 consists in defining a set of integers with useful multiplicative constraints.

The next lemma is analogous to [12; prop. 4.1]. The essential feature consists in bounding $\mathbb{P}_{x,\varrho}(E \smallsetminus E_T^{t,z})$ by a multiple of $1/T^t$. The fact that the resulting estimate is trivial for small T will have no consequence.

Recall notation $z_t := t/(2^t - 1)$, so that $z = z_t$ if, and only if, $\beta = z$.

Lemma 3.2. *Let $t \geqslant 1$, $z \geqslant z_t$. We have*

$$\mathbb{P}_{x,\varrho}\big(E \smallsetminus E_T\big) \leqslant \mathbb{P}_{x,\varrho}\big(E \smallsetminus E_T^{t,z}\big) \ll \frac{(\log x)^{\beta - z}}{T^t} \qquad (x \geqslant 3,\, T \geqslant 3). \tag{3·2}$$

Proof. We may assume $T^t \geqslant (\log 3x)^{\beta - z}$ since the stated estimate is otherwise trivial. Put

$$\kappa_{y,z} := \frac{1}{\log 2}\Big(1 + \frac{\log T - f_T(y)}{\log_2 3y}\Big),$$

and note right away that, since $f_T(y) \leqslant \mathfrak{c} \log T + \mathfrak{c} \log_2 3y$, we have

$$\kappa_{y,z} \geqslant \frac{t + (1 - \mathfrak{c})(\beta - z) - \mathfrak{c}t}{t \log 2} = \frac{(1 - \mathfrak{c})(2^t - 1)z}{t \log 2} = \frac{(2^t - 1)z}{t \log 2} - O(\mathfrak{c}) > z, \tag{3·3}$$

provided $1 \leqslant y \leqslant x$ and $\mathfrak{c}$ is chosen sufficiently small.

We have

$$E_{T,x}^{t,z} := E_T^{t,z} \cap \{n \geqslant 1 : P^+(n) < x\} = \{n \in E : P^+(n) < x, \sup_{1\leqslant y\leqslant x} (\omega(n_y)/\log_2 3y) \leqslant \kappa_{y,z}\}.$$

If $n \in E \smallsetminus E_{T,x}^{t,z}$, then $\omega(n) \geqslant \kappa_{y,z} \log_2 3y - c$ for some absolute constant c and all y in a suitable interval $[y_0, y_0^2]$ with $y_0 \leqslant x$. Therefore,

$$\begin{aligned}
\mathbf{1}_{E\smallsetminus E_{T,x}^{t,z}}(n) &\ll \int_{y_0}^{y_0^2} \sum_{k\geqslant \kappa_{y,z}\log_2 3y-c} \mathbf{1}_{\omega(n_y)=k} \frac{\mathrm{d}y}{y\log 3y},\\
\mathbb{P}_{x,\varrho}(E \smallsetminus E_T^{t,z}) &\ll \frac{1}{(\log 3x)^z} \int_1^{x^2} \sum_{k\geqslant \kappa_y \log_2 3y-c} \frac{1}{k!}\Big(\sum_{p\leqslant y}\frac{\varrho(p)}{p}\Big)^k \sum_{n\geqslant 1}^{y,x} \frac{\varrho(n)}{n}\frac{\mathrm{d}y}{y\log 3y}\\
&\ll \int_1^{x^2} \frac{\mathrm{d}y}{y(\log 3y)^{1+zQ(\kappa_{y,z}/z)}\sqrt{\log_2 3y}},
\end{aligned}$$

where $Q(v) := v\log v - v + 1$ $(v > 0)$. Here the sum over k has been estimated by a standard bound for partial sums of the exponential series such as [14; lemma 4.7], taking (3·3) into account. Now observe that

$$Q(2^t) = 2^t \log 2^t - 2^t + 1, \quad Q'(2^t) = \log 2^t, \quad Q''(v) \gg 1$$

for some $v = \vartheta 2^t + (1-\vartheta)\kappa_{y,z}/z$ with $0 \leqslant \vartheta \leqslant 1$. Therefore, there is a constant $\mathfrak{c}_0 = \mathfrak{c}_0(z,t) > 0$ such that

$$\begin{aligned}
1 + zQ(\kappa_{y,z}/z) &\geqslant 1 + zQ(2^t) + (\kappa_{y,z} - 2^t z)Q'(2^t) + \mathfrak{c}_0\big(\kappa_{y,z} - 2^t z\big)^2\\
&= 1 + z - 2^t z + \kappa_{y,z} t\log 2 + \mathfrak{c}_0(\kappa_{y,z} - 2^t z)^2\\
&= 1 + z - \beta + \frac{t(\log T - f_T(y))}{\log_2 3y} + \mathfrak{c}_0(\kappa_{y,z} - 2^t z)^2,
\end{aligned}$$

whence

$$(\log 3y)^{1+zQ(\kappa_{y,z}/z)} \geqslant T^t(\log 3y)^{1+z-\beta}\mathrm{e}^{-f_T(y)+\mathfrak{c}_0(\kappa_{y,z}-2^t z)^2\log_2 3y}.$$

However, writing $D := \mathfrak{b}\log T - \log_2 3y$ with notation (2·1), we have

$$(\kappa_{y,z} - 2^t z)\log_2 3y = \frac{D}{\mathfrak{b}t\log 2} - \frac{f_T(y)}{\log 2}.$$

If $D \ll \log T$, and so $\log_2 3y \ll \log T$ hence $f_T(y) \ll \mathfrak{c}D^2/\log T \ll \mathfrak{c}D$, the above quantity is $\gg D$, which implies

$$\mathfrak{c}_0(\kappa_{y,z} - 2^t z)^2 \log_2 3y \geqslant (t+1) f_T(y) \tag{3·4}$$

for a suitable choice of $\mathfrak{c}$. Now, if $\log T \leqslant \mathfrak{c}_1 D$ for a small constant $\mathfrak{c}_1$, we must have $\log T \ll \log_2 3y$, and so $f_T(y) \ll \mathfrak{c}D^2/\log_2 3y$, which again implies (3·4) provided $\mathfrak{c}$ is chosen sufficiently small. Therefore,

$$\mathbb{P}_{x,\varrho}(E \smallsetminus E_T^{t,z}) \ll \frac{1}{T^t}\int_1^{x^2} \frac{(\log y)^{\beta-z-1}\mathrm{e}^{-f_T(y)}}{y\sqrt{\log_2 3y}}\,\mathrm{d}y \ll \frac{(\log 3x)^{\beta-z}}{T^t}.$$

The last upper bound is clear if $\beta < z$. In the case $\beta = z$, it follows by splitting the integral at $\exp T^{\mathfrak{b}/2}$ and $\exp T^{2\mathfrak{b}}$, say. □

Remark. The factor $\mathrm{e}^{-f_T(y)}$ appearing in the definition (2·3) of the set $E_T^{t,z}$ will only be useful in the case $\beta = z$, i.e. $z = z_t$. However, this will turn out to be crucial in the proof of part (ii) of Theorem 1.1.

4. Moments of quadratic moments

In all the sequel, we write $\delta = \delta(t,z) := \mathbf{1}_{\beta=z} = \mathbf{1}_{z=z_t}$.

Proposition 4.1. *Let $z > 0$ and $\varrho \in \mathcal{M}_z$.*

(i) *If $t \in \mathbb{R}^{+*}$, $z \geqslant z_t$, we have*

$$\mathbb{E}_{x,\varrho}\big(M_2^t/\tau^t\big) \gg (\log x)^{\beta-z} \qquad (x \geqslant 3). \tag{4·1}$$

(ii) *If $t \in \mathbb{N}^*$, $z \geqslant z_t$, we have*

$$\mathbb{E}_{x,\varrho}\big(M_2^t/\tau^t\big) \asymp (\log x)^{\beta-z}(\log_2 x)^{\delta} \qquad (x \geqslant 3), \tag{4·2}$$

$$\mathbb{E}_{x,\varrho}\big(\mathbf{1}_{E_T^{t,z}} M_2^{t+1}/\tau^{t+1}\big) \ll T(\log x)^{\beta-z} \qquad (T \geqslant 3,\ x \geqslant 3). \tag{4·3}$$

If moreover $z > z_t$, then

$$\mathbb{E}_{x,\varrho}\big(\mathbf{1}_{E_T} M_2^{t+1}/\tau^{t+1}\big) \ll T(\log x)^{\beta-z} \qquad (T \geqslant 3,\ x \geqslant 3). \tag{4·4}$$

Proof. (i) Put $\tau(n,\vartheta) := \sum_{d\mid n} d^{i\vartheta}$ $(n \geqslant 1, \vartheta \in \mathbb{R})$. By Parseval's formula (see [10; (6.23)]) we have

$$\frac{M_2(n)}{\tau(n)} \asymp \int_0^1 \frac{|\tau(n,\vartheta)|^2}{\tau(n)}\,\mathrm{d}\vartheta. \tag{4·5}$$

Estimate (4·1) follows from the observation that, for $x \geqslant 3$, $\mu(n)^2 = 1$, $P^+(n) < x$, we have

$$\frac{|\tau(n,\vartheta)|^2}{\tau(n)} = \prod_{p\mid n}\{1+\cos(\vartheta\log p)\} \geqslant \tau(n)\prod_{p\mid n}\Big(1-\frac{\log p)^2}{4(\log x)^2}\Big) \qquad \big(0 \leqslant \vartheta \leqslant 1/\log x\big).$$

Indeed, we infer from this and (4·5) that

$$\begin{aligned}\mathbb{E}_{x,\varrho}\bigg(\frac{M_2^t}{\tau^t}\bigg) &\gg \frac{1}{(\log x)^{z+t}}\sum_{n\in E}^{x}\frac{\varrho(n)\tau(n)^t}{n}\prod_{p\mid n}\Big(1-\frac{(\log p)^2}{4(\log x)^2}\Big)^t\\ &\gg \frac{1}{(\log x)^{z+t}}\exp\bigg\{\sum_{p\leqslant x}\frac{\varrho(p)2^t}{p}\Big(1-\frac{(\log p)^2}{4(\log x)^2}\Big)^t\bigg\} \asymp (\log x)^{\beta-z},\end{aligned}$$

where we used (1·1) to estimate the last sum over p.

(ii) First apply (4·5) to get

$$\begin{aligned}\mathbb{E}_{x,\varrho}\Big(\frac{M_2^t}{\tau^t}\Big) &\asymp \frac{1}{(\log x)^z}\int_{\boldsymbol{\vartheta}\in[0,1]^t}\sum_{n\geqslant 1}^{x}\frac{\varrho(n)}{n}\prod_{1\leqslant j\leqslant t}\frac{|\tau(n;\vartheta_j)|^2}{\tau(n)}\mathrm{d}\boldsymbol{\vartheta}\\ &\asymp \frac{1}{(\log x)^z}\int_{\boldsymbol{\vartheta}\in[0,1]^t}\exp\bigg\{\sum_{p\leqslant x}\frac{\varrho(p)}{p}\prod_{1\leqslant j\leqslant t}\frac{|\tau(p;\vartheta_j)|^2}{2}\bigg\}\mathrm{d}\boldsymbol{\vartheta}.\end{aligned} \tag{4·6}$$

At this stage some notation is necessary. We use those of [1]. Given an integer $s \geqslant 1$, we designate by $\{e_j : 1 \leqslant j \leqslant s\}$ the canonical basis of $\mathbb{R}^s$ and by $\mathcal{W}_s$ the set of linear forms $w \in \mathcal{L}^*(\mathbb{R}^s) := \mathcal{L}(\mathbb{R}^s,\mathbb{R})$ such that $w(e_j) \in \{-1,0,1\}$ for all j. Note that $0_{\mathcal{L}^*(\mathbb{R}^s)} \in \mathcal{W}_s$. We define the length of $w \in \mathcal{W}_s$ by

$$|w| := \sum_{1\leqslant j\leqslant s}|w(e_j)|. \tag{4·7}$$

Given a prime number p and $\boldsymbol{\vartheta} \in \mathbb{R}^t$, we have

$$\prod_{1\leqslant j\leqslant t}\frac{|\tau(p;\vartheta_j)|^2}{\tau(p)} = \prod_{1\leqslant j\leqslant t}\big(1+\tfrac12(p^{i\vartheta_j}+p^{-i\vartheta_j})\big) = \sum_{w\in\mathcal{W}_t}\frac{\cos\big(w(\boldsymbol{\vartheta})\log p\big)}{2^{|w|}}.$$

In view of the estimate

$$\sum_{p\leqslant y}\frac{\varrho(p)\cos(\psi\log p)}{p} = z\log\Big(\frac{\log y}{1+\psi\log y}\Big) + O(1) \quad (y \geqslant 2,\ 0 \leqslant \psi \leqslant 1), \tag{4·8}$$

that follows from (1·1) by partial summation, we derive that

$$\begin{aligned}\sum_{p\leqslant x}\frac{\varrho(p)}{p}\prod_{1\leqslant j\leqslant t}\frac{|\tau(p;\vartheta_j)|^2}{2}&=\sum_{w\in\mathcal{W}_t}\frac{1}{2^{|w|}}\sum_{p\leqslant x}\frac{\varrho(p)\cos\big(w(\boldsymbol{\vartheta})\log p\big)}{p}\\&=\sum_{w\in\mathcal{W}_t}\frac{z}{2^{|w|}}\log\Big(1+\frac{\log x}{1+|w(\boldsymbol{\vartheta})|\log x}\Big)+O(1).\end{aligned}$$

Defining

$$I_{t,z}(X):=\int_{[0,X]^t}\prod_{w\in\mathcal{W}_t}\Big(1+\frac{X}{1+|w(\boldsymbol{\vartheta})|}\Big)^{z/2^{|w|}}\mathrm{d}\boldsymbol{\vartheta},\tag{4·9}$$

we get, after homothetical change of variables,

$$\mathbb{E}_{x,\varrho}\Big(\frac{M_2^t}{\tau^t}\Big)\asymp\frac{I_{t,z}(\log x)}{(\log x)^t}\cdot$$

It hence remains to evaluate $I_{t,z}(X)$, bearing in mind that

$$\sum_{w\in\mathcal{W}_t}\frac{1}{2^{|w|}}=\sum_{0\leqslant j\leqslant t}\frac{1}{2^j}\sum_{\substack{w\in\mathcal{W}_t\\|w|=j}}1=\sum_{0\leqslant j\leqslant t}\binom{t}{j}=2^t.\tag{4·10}$$

We claim that

$$I_{t,z}(X)\asymp X^{2^tz}\int_{[0,X]^t}\prod_{w\in\mathcal{W}_t}\Big(\frac{1}{1+|w(\boldsymbol{\vartheta})|}\Big)^{z/2^{|w|}}\mathrm{d}\boldsymbol{\vartheta}\asymp X^{2^tz}(\log X)^\delta,\tag{4·11}$$

which plainly implies (4·2).

Let us first establish the lower bound included in (4·11). Since we plainly have $w(\boldsymbol{\vartheta})\ll\vartheta_t$ if

$$0<\tfrac12\vartheta_t/4^{t-j}\leqslant\vartheta_j\leqslant\vartheta_t/4^{t-j}\quad(1\leqslant j\leqslant t)$$

and since the subsum of (4·10) corresponding to non-zero forms equals 2^t-1, the integral is

$$\gg\int_1^X\frac{\mathrm{d}\vartheta_t}{\vartheta_t^{(2^t-1)z-(t-1)}}\cdot$$

This yields the lower bound included in (4·11).

We now prove the upper bound included in (4·11). We may assume $z_t\leqslant z<z_{t-1}$ since the integrand is a non-increasing function of z. The first step consists in associating to each $\boldsymbol{\vartheta}\in]0,+\infty[^t$ a basis $\mathcal{B}_{\boldsymbol{\vartheta}}$ of $\mathcal{L}(\mathbb{R}^t,\mathbb{R})\cap\mathcal{W}_t$ constructed in the following way : w_1 is the non-zero linear form of $\mathcal{W}_t$ minimising $|w(\boldsymbol{\vartheta})|$, w_2 minimises $|w(\boldsymbol{\vartheta})|$ on $\mathcal{W}_t\smallsetminus\mathrm{Vect}(w_1)$, and for each index $k\in[1,t]$, the form w_k minimises $|w(\boldsymbol{\vartheta})|$ on $\mathcal{W}_t\smallsetminus\mathrm{Vect}(w_1,w_2,\dots,w_{k-1})$.

The set $B:=\{\mathcal{B}\in\mathcal{W}_t^t:\exists\boldsymbol{\vartheta}\in]0,\infty[^t\colon\mathcal{B}=\mathcal{B}_{\boldsymbol{\vartheta}}\}$ is finite. Let us assume temporarily that, for any basis $\mathcal{B}=(w_1,w_2,\dots,w_t)$ in B, we have

$$c_k=c_k(\mathcal{B}):=\sum_{\substack{w\in\mathcal{W}_t\cap\mathrm{Vect}(w_1,w_2,\dots,w_k)\\w\neq0}}\frac{1}{2^{|w|}}\leqslant2^k-1\qquad(1\leqslant k\leqslant t).\tag{4·12}$$

This is shown in Lemma 4.2 below.

Given a basis $\mathcal{B}$ in B, consider the domain

$$\mathcal{D}(\mathcal{B}) := \{\boldsymbol{\vartheta} \in [0, X]^t : \mathcal{B}_{\boldsymbol{\vartheta}} = \mathcal{B}\}. \tag{4·13}$$

We have

$$\prod_{w\in\mathcal{W}_t}\Big(\frac{1}{1+|w(\boldsymbol{\vartheta})|}\Big)^{z/2^{|w|}} \ll \prod_{1\leqslant j\leqslant t}\Big(\frac{1}{1+|w_j(\boldsymbol{\vartheta})|}\Big)^{z(c_j-c_{j-1})} \qquad \big(\boldsymbol{\vartheta}\in\mathcal{D}(\mathcal{B})\big),$$

where $c_j := c_j(\mathcal{B})$ $(1 \leqslant j \leqslant t)$ and, by convention, $c_0 = 0$. Performing the change of variables $\boldsymbol{\varphi} = (w_1(\boldsymbol{\vartheta}), \ldots, w_t(\boldsymbol{\vartheta}))$, and noting that the associated Jacobian is constant, we get

$$\int_{\mathcal{D}(\mathcal{B})}\prod_{w\in\mathcal{W}_t}\Big(\frac{1}{1+|w(\boldsymbol{\vartheta})|}\Big)^{z/2^{|w|}}\mathrm{d}\boldsymbol{\vartheta} \ll \int_{|\varphi_1|\leqslant\ldots\leqslant|\varphi_t|\leqslant tX}\prod_{1\leqslant j\leqslant t}\Big(\frac{1}{1+|\varphi_j|}\Big)^{z(c_j-c_{j-1})}\mathrm{d}\boldsymbol{\varphi}. \tag{4·14}$$

Integrate successively according to variables φ_1, φ_2, etc. As $z(2^j - 1) < j$ for all $j \leqslant t-1$, we obtain at step k, a term

$$\ll \prod_{k+1<j\leqslant t}\Big(\frac{1}{1+|\varphi_j|}\Big)^{z(c_j-c_{j-1})}\frac{1}{1+|\varphi_{k+1}|^{zc_{k+1}-k}}. \tag{4·15}$$

At the last stage, we get that the right-hand side of (4·14) is

$$\ll \int_0^{tX}\frac{\mathrm{d}\varphi_t}{1+\varphi_t^{z(2^t-1)-(t-1)}} \ll (\log X)^{\delta}.$$

This implies (4·11), as required.

It remains to establish (4·3) and (4·4). To this end, we adapt the approach of [3; prop. 2.1]. We focus on (4·3) for $z = z_t$, and leave the proof of (4·4) for $z > z_t$ to the reader, noting that this implies (4·3) for $z > z_t$. We have

$$\mathbb{E}_{x,\varrho}\bigg(\mathbf{1}_{E_T^{t,z}}\frac{M_2^{t+1}}{\tau^{t+1}}\bigg) \ll \frac{1}{(\log x)^z}\int_{\boldsymbol{\vartheta}\in[0,1]^{t+1}}\sum_{n\in E_T^{t,z}}^{x}\frac{\varrho(n)}{n}\prod_{1\leqslant j\leqslant t+1}\frac{|\tau(n;\vartheta_j)|^2}{\tau(n)}\mathrm{d}\boldsymbol{\vartheta}. \tag{4·16}$$

Consider the map described earlier associating to each $\boldsymbol{\vartheta}\ \in]0,1[^{t+1}$ a basis $\mathcal{B}_{\boldsymbol{\vartheta}}$ of $\mathcal{L}(\mathbb{R}^{t+1},\mathbb{R})$ and recall that the set $B := \{\mathcal{B} \in \mathcal{W}_{t+1}^{t+1} : \exists\boldsymbol{\vartheta} \in]0,\infty[^{t+1}\colon \mathcal{B}_{\boldsymbol{\vartheta}} = \mathcal{B}\}$ is finite. Given $\mathcal{B} \in B$, define the domain $\mathcal{D}(\mathcal{B}) := \{\boldsymbol{\vartheta} \in [0,1]^{t+1} : \mathcal{B}_{\boldsymbol{\vartheta}} = B\}$ and let $I(\mathcal{D}(\mathcal{B}))$ designate the restriction to $\mathcal{D}(\mathcal{B})$ of the integral in (4·16). In this subintegral, we may perform the change of variables $\boldsymbol{\varphi} = (w_1(\boldsymbol{\vartheta}), \ldots, w_{t+1}(\boldsymbol{\vartheta}))$, where, as previously, the w_j are the linear forms furnishing the successive minima of the lengthes $|w(\boldsymbol{\vartheta})|$ as w runs through $\mathcal{L}(\mathbb{R}^{t+1},\mathbb{R})$. Parallel to computations performed in [3], and, here and throughout, writing $y_t := \exp(1/|\varphi_t|)$ for notational simplicity, we then consider the set of integers

$$D_{h,\ell}(\varphi_t,\varphi_{t+1};T) := \left\{ n \in E_T^{t,z} : \begin{array}{c} \dfrac{1}{2^{h+1}} < \dfrac{|\varphi_t|\tau(n_{y_t})}{T\mathrm{e}^{-f_T(y_t)}} \leqslant \dfrac{1}{2^h} \\ \dfrac{1}{2^{\ell+1}} < \dfrac{|\varphi_{t+1}|\tau(n_{y_{t+1}})}{T\mathrm{e}^{-f_T(y_{t+1})}} \leqslant \dfrac{1}{2^\ell} \end{array} \right\} \qquad (h,\ell\geqslant 0).$$

Let $\mathfrak{I}_{h,\ell}(\mathcal{D}(\mathcal{B}))$ denote the contribution to $I(\mathcal{D}(\mathcal{B}))$ of the subsum corresponding to those n ranging in $D_{h,\ell}(\varphi_t,\varphi_{t+1};T)$. Each integer n appearing in the summation satisfies $\omega(n;y_{t+1}) = r_\ell(\varphi_{t+1})$ for a suitable integer $r_\ell(\varphi_{t+1})$. Multiplying the general term by

$$\frac{T\mathrm{e}^{-f_T(y_t)/2-f_T(y_{t+1})/2}}{2^{(h+\ell)/2}\sqrt{\tau(n_{y_t})\tau(n_{y_{t+1}})|\varphi_t\varphi_{t+1}|}}$$

does not change the order of magnitude. Therefore

$$\begin{aligned}\mathfrak{I}_{h,\ell}\big(\mathcal{D}(\mathcal{B})\big) &\ll \frac{T}{2^{(h+\ell)/2}}\int_{\boldsymbol{\vartheta}\in\mathcal{D}(\mathcal{B})}\sum_{\substack{n\geqslant 1\\ n\in D_{h,\ell}(\varphi_t,\varphi_{t+1};T)\\ \omega(n;y_{t+1})=r_\ell(\varphi_{t+1})}}^{x}\frac{\varrho(n)\mathrm{e}^{-f_T(y_{t+1})/2}}{\sqrt{\tau(n_{y_t})\tau(n_{y_{t+1}})}\sqrt{|\varphi_t\varphi_{t+1}|}}\prod_{1\leqslant j\leqslant t+1}\frac{|\tau(n;\vartheta_j)|^2}{\tau(n)}\mathrm{d}\boldsymbol{\vartheta}\\ &\ll \frac{T}{2^{(h+\ell)/2}}\int_{\boldsymbol{\vartheta}\in\mathcal{D}(\mathcal{B})}\exp\bigg\{\sum_{p\leqslant x}\frac{\varrho(p)\prod_{1\leqslant j\leqslant t+1}\frac{1}{2}|\tau(p;\vartheta_j)|^2}{p\sqrt{\tau(p_{y_t})\tau(p_{y_{t+1}})}}\bigg\}\frac{\mathrm{e}^{-f_T(y_{t+1})/2}}{\sqrt{|\varphi_t\varphi_{t+1}|\log(1/|\varphi_{t+1}|)}}\mathrm{d}\boldsymbol{\vartheta}.\end{aligned}$$

Denote the last integral by $I(\mathcal{D}(\mathcal{B}))$. By (4·8) and (4·10), we have, uniformly for $\boldsymbol{\vartheta}\in\mathcal{D}(\mathcal{B})$,

$$\begin{aligned}\exp\Big\{\sum_{p\leqslant x}\frac{\varrho(p)}{p\sqrt{\tau(p_{y_t})\tau(p_{y_{t+1}})}}\prod_{1\leqslant j\leqslant t+1}\tfrac12|\tau(p;\vartheta_j)|^2\Big\}\\ \ll\exp\Big\{\sum_{w\in\mathcal{W}_{t+1}}\frac{1}{2^{|w|}}\sum_{p\leqslant x}\frac{\varrho(p)\cos\big(w(\boldsymbol{\vartheta})\log p\big)}{p}-z2^t\log(1/|\varphi_{t+1}|)-za_t\log(|\varphi_t/\varphi_{t+1}|)\Big\}\\ \ll\prod_{1\leqslant j\leqslant t+1}\Big(1+\frac{\log x}{1+|w_j(\boldsymbol{\vartheta})|\log x}\Big)^{z(c_j-c_{j-1})}|\varphi_{t+1}|^{z2^t}\Big(\frac{|\varphi_t|}{|\varphi_{t+1}|}\Big)^{za_t},\end{aligned}$$

where we have put $a_t:=\{2^{t+1}-(c_{t+1}-c_t)\}/\sqrt2=(1+c_t)/\sqrt2$, by (4·10) with $t+1$ in place of t. Exploiting the change of variables $\boldsymbol{\varphi}=\big(w_1(\boldsymbol{\vartheta}),\dots,w_{t+1}(\boldsymbol{\vartheta})\big)$ and recalling definition (4·12) for $c_j=c_j(\mathcal{B})$, keeping in mind that here $\mathcal{B}$ has dimension $t+1$, we obtain

$$\frac{I(\mathcal{D}(\mathcal{B}))}{(\log x)^{z2^t-t}}\ll\int_{\mathcal{H}}\prod_{1\leqslant j\leqslant t+1}\Big(\frac{1}{1+|\varphi_j|}\Big)^{z(c_j-c_{j-1})}\varphi_{t+1}^{z2^t-1}\Big(\frac{\varphi_t}{\varphi_{t+1}}\Big)^{za_t-1/2}\frac{\mathrm{e}^{-f_T(y_{t+1})/2}}{\sqrt{\log(1/\varphi_{t+1})}}\,\mathrm{d}\boldsymbol{\varphi},$$

where $\mathcal{H}:=\{\boldsymbol{\varphi}\in\mathbb{R}^{t+1}:0\leqslant\varphi_1\leqslant\dots\leqslant\varphi_{t+1}\}$. Parallel to (4·15), we integrate successively according to variables φ_1, φ_2, etc. By Lemma 4.2, we have $zc_j\leqslant z(2^j-1)<j$ for all $j\leqslant t-1$ since $z=z_t$. Thus, at step $k\leqslant t-1$, we obtain a term

$$\ll\prod_{k+1<j\leqslant t+1}\Big(\frac{1}{1+\varphi_j}\Big)^{z(c_j-c_{j-1})}\frac{\varphi_{t+1}^{z2^t-1}(\varphi_t/\varphi_{t+1})^{za_t-1/2}\mathrm{e}^{-f_T(y_{t+1})/2}}{1+\varphi_{k+1}^{zc_{k+1}-k}\sqrt{\log(1/\varphi_{t+1})}}.$$

At step $t-1$ the above product is empty. By (4·10), we have $c_{t+1}=2^{t+1}-1$, and also $z2^t-t=z$ as $z=z_t$. We get

$$\begin{aligned}I\big(\mathcal{D}(\mathcal{B})\big)&\ll(\log x)^z\int\int_{0\leqslant\varphi_t\leqslant\varphi_{t+1}}\frac{\varphi_{t+1}^{z2^t-1}(\varphi_t/\varphi_{t+1})^{za_t-1/2}\mathrm{e}^{-f_T(y_{t+1})/2}}{(1+\varphi_t)^{zc_t-t+1}(1+\varphi_{t+1}^{z(2^{t+1}-1-c_t)})\sqrt{\log(1/\varphi_{t+1})}}\,\mathrm{d}\varphi_{t+1}\\&\ll(\log x)^z\int_0^\infty\frac{\mathrm{e}^{-f_T(y_{t+1})/2}}{(1+\varphi_{t+1})\sqrt{\log(1/\varphi_{t+1})}}\,\mathrm{d}\varphi_{t+1}\ll(\log x)^z,\end{aligned}$$

where the inequality $za_t-1/2-(zc_t-t)>0$ has been taken into account in order to justify the integration according to φ_t.

Summing $T\mathcal{I}_{h,\ell}(\mathcal{D}(\mathcal{B}))/\{2^{(h+\ell)/2}(\log x)^z\}$ over h, ℓ, and $\mathcal{B}$ yields (4·3). □

Lemma 4.2. *Let $t\in\mathbb{N}^*$. For any basis $\mathcal{B}$ in B, we have*

$$c_k(\mathcal{B})\leqslant2^k-1\qquad(1\leqslant k\leqslant t).\tag{4·17}$$

Proof. Let us first observe that $\mathcal{W}_t\cap\mathrm{Vect}(w_1)=\{0,\pm w_1\}$, hence $c_1(\mathcal{B})\leqslant1$, with equality if $|w_1|=1$, i.e. if w_1 belongs to the canonical dual basis $\{e_j^*\}_{j=1}^t$ of $\mathbb{R}^t$. Inequality (4·17) hence holds for $k=1$.

When $k=t$, we have $\mathcal{W}_t\cap\mathrm{Vect}(w_1,w_2,\dots,w_t)=\mathcal{W}_t$. Inequality (4·17) is then an immediate consequence of (4·10), bearing in mind that the zero form is not counted by the left-hand side of (4·17).

Next, note that inequality (4·17) is actually an equality if $\mathrm{Vect}(w_1,w_2,\dots,w_k)=\mathrm{Vect}(e_j^*\,;j\in J)$ with $|J|=k$. Indeed, we then have

$$\mathcal{W}_t\cap\mathrm{Vect}(w_1,w_2,\dots,w_k)=\mathcal{W}_t\cap\mathrm{Vect}(e_j^*\,;j\in J)=\Big\{\sum_{j\in J}\alpha_je_j^*;\alpha_j\in\{-1,0,1\}\,(\forall j\in J)\Big\},\tag{4·18}$$

whence

$$\sum_{\substack{w\in\mathcal{W}_t\cap\mathrm{Vect}(e_j^*\,;j\in J)\\ w\neq0}}\frac{1}{2^{|w|}}=\sum_{1\leqslant h\leqslant k}\frac{1}{2^h}\sum_{\substack{w\in\mathcal{W}_t\cap\mathrm{Vect}(e_j^*\,;j\in J)\\ |w|=h}}1=\sum_{1\leqslant h\leqslant k}\binom{k}{h}=2^k-1.$$

We prove inequality (4·17) by double induction on k and t. Integers k and t being fixed such that $1 \leqslant k \leqslant t$, denote by $\mathcal{H}_k(t)$ the hypothesis that any base $\mathcal{B}$ in $\mathcal{W}_t$ satisfies $c_k(\mathcal{B}) \leqslant 2^k - 1$.

By our first remark, $\mathcal{H}_1(t)$ holds for all $t \geqslant 1$. Let then $t \geqslant 2$ and $k \in [2, t]$. Assume that $\mathcal{H}_\ell(s)$ is satisfied for all $\ell \in [1, s]$ if $s < t$ and for all ℓ in $[1, k-1]$ if $s = t$. We then have to verify $\mathcal{H}_k(t)$. To this end, consider a basis $\mathcal{B} = \{w_j\}_{j=1}^t$ in $\mathcal{W}_t$ and put $V_k := \mathcal{W}_t \cap \mathrm{Vect}(w_1, \ldots, w_k)$. Let $u_1 \in V_k$ be a non-zero vector of minimal length.[2] Define

$$V_k' := \{v \in V_k \,:\, \forall j \leqslant t \;(u_1(e_j) \neq 0) \Rightarrow (v(e_j) = 0)\}.$$

The set V_k' is the intersection of W_t with a linear space. It always contains the zero form.

If $\operatorname{rank} V_k' = k-1$, let $(v_2, \ldots, v_k)$ be a subset of independent vectors in V_k' chosen in such a way that any form v in V_k is uniquely representable as a sum $v = \alpha_1 u_1 + w$ with $\alpha_1 \in \{-1, 0, 1\}$, $w \in \mathrm{Vect}(v_2, \ldots, v_k)$, and $|v| = |\alpha_1||u_1| + |w|$. It follows that

$$\begin{aligned} c_k(\mathcal{B}) &\leqslant \frac{1}{2^{|u_1|-1}} + \sum_{\substack{\alpha_1 \in \{-1,0,1\} \\ w \in \mathcal{W}_t \cap \mathrm{Vect}(v_2,\ldots,v_k) \\ w \neq 0}} \frac{1}{2^{|\alpha_1||u_1| + |w|}} \\ &\leqslant \frac{1}{2^{|u_1|-1}} + \Big(1 + \frac{1}{2^{|u_1|-1}}\Big) \sum_{\substack{w \in \mathcal{W}_t \cap \mathrm{Vect}(v_2,\ldots,v_k) \\ w \neq 0}} \frac{1}{2^{|w|}} \\ &\leqslant \frac{1}{2^{|u_1|-1}} + \Big(1 + \frac{1}{2^{|u_1|-1}}\Big)(2^{k-1} - 1) = 2^{k-1}\Big(1 + \frac{1}{2^{|u_1|-1}}\Big) - 1 \leqslant 2^k - 1, \end{aligned}$$

where the induction hypothesis has been used for bounding the last sum upon w, and where the trivial lower bound $|u_1| \geqslant 1$ has been taken into account. Note that the inequality is strict if $|u_1| \geqslant 2$.

If $\operatorname{rank} V_k' < k-1$, let h be an index such that $w_1(e_h) \neq 0$. Each linear form w_j is uniquely representable as $w_j = \alpha_j e_h^* + v_j$ with $\alpha_j \in \{-1, 0, 1\}$ and $v_j(e_h) = 0$ for all $1 \leqslant j \leqslant t$. The forms v_j may be considered as elements of $\mathcal{L}^*(\mathbb{R}^{t-1})$.

Note that $\operatorname{rank}(v_1, \ldots, v_k) \in \{k-1, k\}$ since $\operatorname{rank}(v_1, \ldots, v_k, e_h^*) \geqslant k$. If $\operatorname{rank}(v_1, \ldots, v_k) = k-1$, then $v_1 \in \mathrm{Vect}(v_2, \ldots, v_k)$ hence $e_h^* \in \mathrm{Vect}(w_1, \ldots, w_k)$. We may then apply the conclusion of the case $\operatorname{rank} V_k' = k-1$ by selecting $u_1 = e_h^*$.

Since $\mathcal{H}_t(t)$ always holds, we may assume $k < t$. If $\operatorname{rank}(v_1, \ldots, v_k) = k$, this family satisfies $\mathcal{H}_k(t-1)$, whence

$$\sum_{\substack{v \in \mathcal{W}_t \cap \mathrm{Vect}(v_1,\ldots,v_k) \\ v \neq 0}} \frac{1}{2^{|v|}} \leqslant 2^k - 1.$$

The elements w of V_k may hence be written as $w = \alpha e_h^* + v$ with $v \in \mathrm{Vect}(v_2, \ldots, v_k)$, $\alpha \in \{-1, 0, 1\}$, and $|v| = |\alpha| + |v|$. We write $w = \sum_{1 \leqslant j \leqslant k} x_j w_j = \alpha e_h^* + v$ with

$$v = \sum_{1 \leqslant j \leqslant k} x_j v_j, \qquad \alpha = \sum_{1 \leqslant j \leqslant k} x_j \alpha_j \in \{-1, 0, 1\}.$$

By the hypothesis $\operatorname{rank}(v_1, \ldots, v_k) = k$, the x_j are completely determined by v, hence also the w and α. In particular, $v = 0$ if, and only if, $w = 0$. We therefore have

$$\sum_{\substack{w \in \mathcal{W}_t \cap \mathrm{Vect}(w_1,\ldots,w_k) \\ w \neq 0}} \frac{1}{2^{|w|}} = \sum_{\substack{v \in \mathcal{W}_t \cap \mathrm{Vect}(v_1,\ldots,v_k) \\ v \neq 0 \\ \alpha \in \{-1,0,1\}}} \frac{1}{2^{|v| + |\alpha|}} \leqslant \sum_{\substack{v \in \mathcal{W}_t \cap \mathrm{Vect}(v_1,\ldots,v_k) \\ v \neq 0}} \frac{1}{2^{|v|}} \leqslant 2^k - 1.$$

This furnishes the required inequality at rank k and so completes the proof of $\mathcal{H}_k(t)$. Note that in this case the inequality is strict because at least one w is such that $\alpha \neq 0$. □

2. This minimality is not *stricto sensu* necessary but it helps fixing ideas.

5. A functional inequality

Denote by $\mathcal{P}$ the set of all prime numbers. Consider a non-negative arithmetic function f supported on the set of squarefree integers. Assume furthermore that $f(1) = 1$ and that, for a suitable function $w : \mathbb{N}^* \times \mathcal{P} \to \mathbb{R}^+$ vanishing if the first variable is not squarefree, we have

$$f(mp) \leqslant f(m) + w(m, p) \qquad \big(m \geqslant 1,\, p \nmid m\big). \tag{5·1}$$

The following lemma furnishes a useful bound for

$$\mathcal{F}(x) := \sum_{n \geqslant 1}^{x} \frac{\varrho(n) f(n)}{n}\cdot$$

It has been established in [12] for $\varrho = \mathbf{1}$, and hence $z = 1$—see also [3].

Lemma 5.1. *Let* $z > 0$ *and* $\varrho \in \mathcal{M}_z$. *For* f, w *satisfying* (5·1), *we have*

$$\frac{\mathcal{F}(x)}{(\log x)^z} \ll 1 + \int_1^{x^2} \sum_{m \geqslant 1}^{y} \frac{\varrho(m)}{m} \sum_{\sqrt{y} \leqslant p < y} \frac{\varrho(p) w(m,p)}{p} \frac{\mathrm{d}y}{y (\log 3y)^{1+z}}\cdot \tag{5·2}$$

We shall show that (5·2) holds for $f = \mathbf{1}_S M_q^r / \tau^r$ with $1 \leqslant r \leqslant t$ and $q \geqslant 3$, where S is some set of integers such that $(mp \in S) \Rightarrow (m \in S)$.

Proof. Writing $n = mp$ with $P^+(m) < p$, we get

$$\mathcal{F}(x) \leqslant 1 + \sum_{p < x} \sum_{m \geqslant 1}^{p} \frac{\varrho(p)\varrho(m) f(pm)}{pm}\cdot$$

Applying (5·1) when $\mu(pm)^2 = 1$ yields

$$\mathcal{F}(x) \leqslant 1 + \sum_{p<x} \sum_{m \geqslant 1}^{p} \frac{\varrho(p)\varrho(m)(f(m) + w(m,p))}{pm} = \sum_{p<x} \frac{\varrho(p)\mathcal{F}(p)}{p} + V(x),$$

with

$$V(x) := 1 + \sum_{p<x} \frac{\varrho(p)}{p} \sum_{m}^{p} \frac{\varrho(m) w(m,p)}{m}\cdot$$

Iterating this as in [12] and taking (2·4) into account, we get

$$\mathcal{F}(x) \leqslant V(x) + \sum_{2 \leqslant P(n) < x} \frac{\mu^2(n)\varrho(n) V(P^-(n))}{n} \ll V(x) + \sum_{p<x} \frac{\varrho(p) V(p)}{p} \Big(\frac{\log x}{\log p}\Big)^z,$$

whence

$$\frac{\mathcal{F}(x)}{(\log x)^z} \ll \frac{V(x)}{(\log x)^z} + \sum_{p<x} \frac{\varrho(p) V(p)}{p (\log p)^z}\cdot \tag{5·3}$$

Indeed, at the first step we obtain

$$\sum_{p<x} \frac{\varrho(p)\mathcal{F}(p)}{p} \leqslant \sum_{p_1 < x} \frac{\varrho(p_1)}{p_1} \Big\{ \sum_{p_2 < p_1} \frac{\varrho(p_2)\mathcal{F}(p_2)}{p_2} + V(p_1) \Big\},$$

then

$$\begin{aligned} \sum_{p<x} \frac{\varrho(p)\mathcal{F}(p)}{p} &\leqslant \sum_{p_1<x} \frac{1}{p_1} \Big(\varrho(p_1) V(p_1) + \sum_{p_2 < p_1} \frac{\varrho(p_1 p_2)}{p_2} \Big\{ \sum_{p_3 < p_2} \frac{\varrho(p_3)\mathcal{F}(p_3)}{p_3} + V(p_2) \Big\} \Big) \\ &\leqslant \sum_{p_1 < x} \frac{\varrho(p_1) V(p_1)}{p_1} + \sum_{p_2 < p_1 < x} \frac{\varrho(p_1 p_2) V(p_2)}{p_1 p_2} + \sum_{p_3 < p_2 < p_1 < x} \frac{\varrho(p_1 p_2 p_3)\mathcal{F}(p_3)}{p_1 p_2 p_3}, \end{aligned}$$

and so on.

Since $n_y = n_p$ when $P^+(n) < p < y \leqslant p^2$, we may write, still following [12],

$$\begin{aligned} V(x) &\ll 1 + \sum_{p<x} \sum_{m}^{p} \frac{\varrho(p)\varrho(m) w(m,p)}{pm} \int_p^{p^2} \frac{\mathrm{d}y}{y \log 3y} \\ &= 1 + \int_1^{x^2} \sum_{m}^{y} \frac{\varrho(m)}{m} \sum_{\sqrt{y} \leqslant p < y} \frac{\varrho(p) w(m,p)}{p} \frac{\mathrm{d}y}{y \log 3y}\cdot \end{aligned}$$

Inverting summation and integration in (5·3) and appealing to the uniform bound

$$\sum_{\sqrt{y}\leqslant p<x}\frac{\varrho(p)}{p(\log p)^z}\ll\frac{1}{(\log y)^z},$$

stemming from (1·1), we obtain (5·2) as stated. □

6. Bounding integral moments for bounded q

Define

$$M_q(n):=\int_{\mathbb{R}}\Delta(n,u)^q\,\mathrm{d}u\qquad(n\geqslant 1,\ q\geqslant 1).\tag{6·1}$$

In section 8 below, we generalize the method displayed in [12], which is based on bounding $\mathbb{E}_{x,\mathbf{1}}(\mathbf{1}_S M_q/\tau)$ for sets S of decreasing size and applying Markov's inequality to estimate $\mathbb{P}_{x,\mathbf{1}}(M_q/\tau>T)$. The approach may easily be adapted in order to include a weight ϱ. However, the initialisation step requires a bound for $\mathbb{P}_{x,\varrho}(M_q/\tau>T)$ decreasing as $1/T^t$. This is provided by estimating $\mathbb{E}_{x,\varrho}(\mathbf{1}_S M_q^t/\tau^t)$. The generality introduced in Lemma 5.1 then enables an iteration of the process.

This section is devoted to establishing this initialization step in the case of integral t. In the next section, the result is extended to real t.

We now describe the sets S needed in further estimates for conditional expectations.

Let $\gamma\in\{0,1\}$ and $\{\vartheta_{j,T}^\gamma\}_{j=0}^\infty\in(\mathbb{R}^+)^{\mathbb{N}}$ denote a sequence such that $\vartheta_{0,T}^\gamma=\vartheta_{1,T}^\gamma=1$. Define further

$$H_{T,\gamma}^q:=\big\{n\in E_T: M_j(n)\leqslant\tau(n)\vartheta_{j,T}\ (1\leqslant j\leqslant q)\big\},\tag{6·2}$$

$$H_{T,\gamma}^{t,z,q}:=\big\{n\in E_T^{t,z}: M_j(n)\leqslant\tau(n)\vartheta_{j,T}\ (1\leqslant j\leqslant q)\big\},\tag{6·3}$$

$$H_T^{t,z,q}:=H_{T,\delta}^{t,z,q},\tag{6·4}$$

so that $H_{T,\gamma}^0=H_{T,\gamma}^1=E_T$.

Note that, by (6·12) *infra*, $M_j(n)/\tau(n)$ is multiplicatively increasing, so if $n\in H_T^{t,z,q}$ then any divisor of n also lies in $H_T^{t,z,q}$.

The quantities $\vartheta_{j,T}$ will be formally defined later—see (9·1). For now, we only specify that

$$\vartheta_{0,T}=\vartheta_{1,T}=1,\qquad\vartheta_{2,T}\asymp T(\log T)^\gamma,\qquad\vartheta_{j,T}\ll_j T^{j-1}(\log T)^{\gamma(j-1)}\quad(j\geqslant 3).\tag{6·5}$$

When conditioning over $H_{T,\gamma}^q$, we shall need estimates with $\gamma=1$ when $\delta=1$.

Recall notation $\sum^x$ and put

$$\mathcal{T}_{t,z}(x;q):=\mathbb{P}_{x,\varrho}\big(H_T^{t,z,q-1}\big)\mathbb{E}_{x,\varrho}\Big(\frac{M_q}{\tau}\Big|H_T^{t,z,q-1}\Big)=\prod_{p<x}\Big(\frac{1}{1+\varrho(p)/p}\Big)\sum_{n\in H_T^{t,z,q-1}}^{x}\frac{\varrho(n)M_q(n)}{n\tau(n)}.\tag{6·6}$$

Proposition 8.1 below furnishes a recursive bound for $\mathcal{T}_{t,z}(x;q)$. The initialisation step described above requires a more general estimate that will only be used for bounded values of q.

We thus consider the quantities

$$\mathfrak{S}_{r,z}^\gamma(x;q):=\mathbb{P}_{x,\varrho}\big(H_{T,\gamma}^{q-1}\big)\mathbb{E}_{x,\varrho}\Big(\frac{M_q^r}{\tau^r}\Big|H_{T,\gamma}^{q-1}\Big)=\prod_{p<x}\Big(\frac{1}{1+\varrho(p)/p}\Big)\sum_{n\in H_{T,\gamma}^{q-1}}^{x}\frac{\varrho(n)M_q(n)^r}{n\tau(n)^r}.\tag{6·7}$$

Note that the conditioning is over $H_{T,\gamma}^{q-1}$: we shall exploit the fact that this does not depend on t nor z.

Proposition 6.1. *Let $t\in\mathbb{N}^*$, $r\in\mathbb{N}^*\cap[1,t]$, $z\in\mathbb{R}^+$, $\varrho\in\mathcal{M}_z$, $q\geqslant 3$, $\gamma\in\{0,1\}$, and assume*

$$\begin{cases} z\geqslant z_t & \text{if } \gamma=1,\\ z>z_t & \text{if } \gamma=0.\end{cases}\tag{6·8}$$

The sets $H_{T,\gamma}^j$ being defined by (6·3) with $\vartheta_{j,T}$ satisfying (6·5), we have, uniformly for $T\geqslant 2$, $x\geqslant 2$,

$$\mathfrak{S}_{r,z}^\gamma(x;q)\ll T^{r(q-2)}(\log T)^{\gamma r(q-1)}(\log x)^{\beta-z}.\tag{6·9}$$

Proof. Put

$$N_{j,q}(n,v):=\int_{\mathbb{R}}\Delta(n,u)^j\Delta(n,u-\log v)^{q-j}\,\mathrm{d}u\quad(n\geqslant 1,\ v\geqslant 1,\ 1\leqslant j\leqslant q)\tag{6·10}$$

$$W_q(n,v):=\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}N_{j,q}(n,v)\leqslant 2^{q-1}M_q(n)\quad(n\geqslant 1,\ v\geqslant 1).\tag{6·11}$$

From [15; (17)], we have

$$2M_q(n)\leqslant M_q(np)\leqslant 2M_q(n)+2W_q(n,p)\leqslant 2^qM_q(n)\quad(p\nmid n),\tag{6·12}$$

and so

$$\frac{M_q(n)}{\tau(n)}\leqslant\frac{M_q(np)}{\tau(np)}\leqslant\frac{M_q(n)}{\tau(n)}+\frac{W_q(n,p)}{\tau(n)}\leqslant 2^{q-1}\frac{M_q(n)}{\tau(n)}\quad(p\nmid n).\tag{6·13}$$

For $q\geqslant 2$ and any $v\geqslant 1$, we also have

$$W_q(n,v)\leqslant 2^{q-1}\int_{\mathbb{R}}\Delta(n,u)\Delta(n,u-\log v)\big\{\Delta(n,u-\log v)^{q-2}+\Delta(n,u)^{q-2}\big\}\,\mathrm{d}u.\tag{6·14}$$

Define $f:=\mathbf{1}_{H_{T,\gamma}^{q-1}}M_q^r/\tau^r$. By (6·13) and the inequality $(a+b)^r\leqslant a^r+O(ba^{r-1})$ valid for $b\ll a$, we have, for a suitable constant $C_{q,r}$,

$$f(np)\leqslant f(n)+C_{q,r}\frac{W_q(n,p)}{\tau(n)}\mathbf{1}_{H_{T,\gamma}^{q-1}}(n)\frac{M_q(n)^{r-1}}{\tau(n)^{r-1}}\cdot$$

Lemma 5.1 is applicable to $\varrho\in\mathcal{M}_z$ and w defined by

$$w(m,p)=W_{q,r}(m,p):=C_{q,r}\frac{W_q(m,p)}{\tau(m)}\mathbf{1}_{H_{T,\gamma}^{q-1}}(m)\frac{M_q(m)^{r-1}}{\tau(m)^{r-1}}\cdot$$

Moreover, appealing to (6·10) and (6·14), we get, for all $m\geqslant 1$,

$$\int_{\mathbb{R}}W_{q,r}(m,\mathrm{e}^v)\,\mathrm{d}v\ll\frac{M_{q-1}(m)M_q(m)^{r-1}}{\tau(m)^{r-1}}\cdot\tag{6·15}$$

It follows that

$$\mathfrak{S}_{r,z}^{\gamma}(x;q)\ll 1+\int_1^{x^2}\sum_{m\in H_{T,\gamma}^{q-1}}^{y}\frac{\varrho(m)}{m}\sum_{\sqrt{y}\leqslant p<y}\frac{W_{q,r}(m,p)}{p}\frac{\mathrm{d}y}{y(\log 3y)^{1+z}},$$

whence, by partial integration—see (8·4) *infra*—,

$$\mathfrak{S}_{r,z}^{\gamma}(x;q)\ll 1+\int_1^{x^2}\frac{Z_{r,z}^{\gamma}(y;q)}{y(\log 3y)^{2+z}}\,\mathrm{d}y,\tag{6·16}$$

where we have set

$$Z_{r,z}^{\gamma}(y;q):=\sum_{m\in H_{T,\gamma}^{q-1}}^{y}\frac{\varrho(m)}{m}\int_{\mathbb{R}}W_{q,r}(m,\mathrm{e}^v)\,\mathrm{d}v\ll\sum_{m\in H_{T,\gamma}^{q-1}}^{y}\frac{\varrho(m)M_{q-1}(m)M_q(m)^{r-1}}{m\tau(m)^{r-1}}\cdot\tag{6·17}$$

We now set out to proving (6·9) by induction on $r\geqslant 1$.

First fix $\gamma=0$ and assume $z>z_t$.

The initialisation step, corresponding to $r=1$, is established by induction on $q\geqslant 3$. By (4·2), (6·4), and (6·5), we have

$$Z_{1,z}^0(y;3)\ll\sum_{m\in H_T}^{y}\frac{\varrho(m)M_2(m)}{m}\ll T\log y\sum_{m\in H_T}^{y}\frac{\varrho(m)M_2(m)}{\tau(m)m}\ll T(\log y)^{\beta+1}.$$

Carrying back into (6·16) yields (6·9) since $\beta>z$. This establishes (6·9) for $r=1$, $t\geqslant 1$, $q=3$, $z>z_t$.

Next, under the assumption that (6·9) holds for $q-1 \geqslant 3$, $r=1$, $t \geqslant 1$, we apply the hypothesis $\tau(m) \leqslant T \log y$ $(m \in E_T)$ to derive from (6·17) that

$$Z^0_{1,z}(y;q) \ll T(\log 3y)^{z+1}\mathfrak{S}^0_{1,z}(x;q-1) \ll T^{q-2}(\log 3y)^{\beta+1}.$$

This furnishes the required estimate at rank q and thereby establishes (6·9) for $r=1$, $t \geqslant 1$, $q \geqslant 3$, $z > z_t$.

Consider now $r \geqslant 2$ and assume that (6·9) holds for $r-1$, $q \geqslant 3$, $t \geqslant r$, $z > z_t$. We need an upper bound for $Z^0_{t,z}(y;q,r)$. By (6·4) and (6·5), we have $M_{q-1}(n) \leqslant \vartheta_{q-1,T}\tau(n)$, which yields

$$Z^0_{r,z}(y;q) \ll T^{q-2} \sum_{m \in H^{q-1}_{T,0}}^{y} \frac{\varrho(m)\tau(m)M_q(m)^{r-1}}{m\tau(m)^{r-1}} \ll T^{q-2}(\log 3y)^{2z}\mathfrak{S}^0_{r-1,2z}(x;q).$$

Since $2z_t > z_{t-1}$ for $t > 1$ and $\beta(2z, t-1) = \beta + 1$, the induction hypothesis is applicable to bound the last quantity. We obtain

$$Z^0_{r,z}(y;q) \ll T^{r(q-2)}(\log y)^{\beta+1}. \tag{6·18}$$

Carrying back into (6·16) provides the required bound (6·9).

Let us now fix $\gamma = 1$ and assume $z \geqslant z_t$. To establish initialisation step, corresponding to $r=1$, we again proceed by induction on $q \geqslant 3$. If $q=3$, we consider three cases: (i) $z > z_t$, (ii) $z = z_t$ and $t \geqslant 2$, (iii) $z = z_1 = 1$ and $t = 1$.

In case (i), i.e. $z > z_t$, we deduce from (4·2), (6·2) and (6·5) that

$$Z^1_{1,z}(y;3) \ll \sum_{m \in H_T}^{y} \frac{\varrho(m)M_2(m)}{m} \ll T\log y \sum_{m \in H_T}^{y} \frac{\varrho(m)M_2(m)}{\tau(m)m} \ll T(\log y)^{\beta+1}.$$

Carrying back into (6·16), we get (6·9).

In case (ii), i.e. $z = z_t$, $t \geqslant 2$, we have

$$Z^1_{1,z_t}(y;3) \ll T \sum_{m \geqslant 1}^{y} \frac{\varrho(m)\tau(m)}{m} \ll T(\log 3y)^{2z_t}.$$

Since $z_t < 1$ for $t \geqslant 2$, the integral in (6·16) is bounded and (6·9) holds.

In the last case (iii), i.e. $t = z = z_1 = r = 1$, we have

$$Z^1_{1,1}(y;3) \ll T \sum_{m \geqslant 1}^{y} \frac{\varrho(m)\min\big(\tau(m), \sqrt{T\tau(m)\log 3y}\big)}{m} \ll \frac{T^{3/2}(\log 3y)^2}{\sqrt{T} + (\log 3y)^{3/2-\sqrt{2}}}.$$

This implies that the integral in (6·16) is $\ll \log T$ by splitting it at $(\log 3y)^{3/2-\sqrt{2}} = \sqrt{T}$.

This completes the proof of (6·9) in the case $q=3$, $r=1$, $t \geqslant 1$, $z \geqslant z_t$.

Now, assuming that (6·9) holds for $q-1 \geqslant 3$, $r=1$, $t \geqslant 1$, $z \geqslant z_t$, the hypothesis $\tau(m) \leqslant T\log y$ $(m \in E_T)$ enables us to deduce from (6·17) that

$$Z^1_{1,z}(y;q) \ll T(\log 3y)^{z+1}\mathfrak{S}^1_{t,z}(x;q-1,1) \ll T^{q-2}(\log T)^{q-2}(\log 3y)^{\beta+1}.$$

If $z > z_t$, this furnishes the required estimate at rank q. If $z = z_t$, we apply (6·5) in the form

$$Z^1_{1,z}(y;q) \ll \vartheta_{q-1,T} \sum_{m \geqslant 1}^{y} \frac{\varrho(m)\min\big(\tau(m), \sqrt{T\tau(m)\log 3y}\big)}{m} \ll \frac{T^{q-3/2}(\log T)^{q-2}(\log 3y)^{2z}}{\sqrt{T} + (\log 3y)^{(2-\sqrt{2})z-1/2}}.$$

Carrying back into (6·16) furnishes (6·9) at rank q. This completes the induction on q and thus initialises the induction by providing (6·9) for $r=1$, $q \geqslant 3$, $t \geqslant 1$, $z \geqslant z_t$.

Next, consider $r \geqslant 2$ and assume that (6·9) holds for $r-1$, $q \geqslant 3$, $t \geqslant r$, $z \geqslant z_t$. We need an upper bound for $Z^1_{r,z}(y;q)$. By (6·4) and (6·5), we have $M_{q-1}(n) \leqslant \vartheta_{q-1,T}\tau(n)$, which yields

$$\begin{aligned} Z^1_{r,z}(y;q) &\ll T^{q-2}(\log T)^{q-2} \sum_{m \in H^{q-1}_{T,1}}^{y} \frac{\varrho(m)\tau(m)M_q(m)^{r-1}}{m\tau(m)^{r-1}} \\ &\ll T^{q-2}(\log T)^{q-2}(\log 3y)^{2z}\mathfrak{S}^1_{t-1,2z}(x;q,r-1). \end{aligned}$$

Since $2z_t > z_{t-1}$ for $t > 1$ and $\beta(2z, t-1) = \beta + 1$, the induction hypothesis is applicable to bound the last quantity. We obtain

$$Z^1_{r,z}(y;q) \ll T^{r(q-2)}(\log T)^{(q-1)r-1}(\log y)^{\beta+1}. \tag{6·19}$$

Carrying back into (6·16) provides the required bound (6·9) in the case $z > z_t$. If $z = z_t$, (6·19) is insufficient since it would involve an extra factor $\log_2 x$. We circumvent the difficulty by writing

$$\begin{aligned} Z_{r,z}(y;q) &\ll \vartheta_{q-1,T} \sum_{m\in H^{q-1}_{T,1}}^{y} \frac{\varrho_1(m)2^{b_t\omega(m)}M_q(m)^{r-1}}{m\tau(m)^{r-1}} \\ &\ll \vartheta_{q-1,T}(T\log 3y)^{b_t} \sum_{m\in H^{q-1}_{T,1}}^{y} \frac{\varrho_1(m)M_q(m)^{r-1}}{m\tau(m)^{r-1}}, \end{aligned}$$

with $\varrho_1(m) := \varrho(m)(z_{t-1}/z_t)^{\omega(m)}$, so that $\varrho_1 \in \mathcal{M}_{z_{t-1}}$, and $b_t := \{\log(2z_t/z_{t-1})\}/\log 2$. Applying the induction hypothesis with $(z_{t-1}, t-1)$ in place of (z,t) and noting that $\beta(t-1, z_{t-1}) = z_{t-1}$, we get

$$Z^1_{r,z}(y;q) \ll T^{r(q-2)+b_t}(\log T)^{(q-1)r-1}(\log 3y)^{z_{t-1}+b_t}. \tag{6·20}$$

Observe that $z_{t-1} + b_t < z_t + 1$. Split then the integral in (6·16) at $\log 3y = T^\lambda$ with $\lambda := b_t/(1 + z_t - z_{t-1} - b_t)$, applying (6·20) to the upper range and (6·19) to the lower range. This yields

$$\mathfrak{S}^1_{r,z}(x;q) \ll T^{r(q-2)}(\log T)^{(q-1)r}(\log x)^{\beta-z}.$$

This completes the proof of Proposition 6.1 . □

7. Bounding real moments for bounded q

Recall definition (6·7). When the parameter t is integral, Proposition 6.1 enables initialising the induction conducted in the proof of Proposition 8.1 below by providing the upper bound (6·9) for $\mathfrak{S}_{t,z}(x;q)$. The next statement extends this to real values of t provided

$$z \geqslant \mathfrak{z}_t := 2^{\lceil t\rceil - t}\lceil t\rceil/(2^{\lceil t\rceil} - 1).$$

Note that $\mathfrak{z}_t = z_t$ if $t \in \mathbb{N}^*$.

Lemma 7.1. *Let $z > 0$, $t > 0$, $\varrho \in \mathcal{M}_z$.*

(i) *Let $q \geqslant 3$, $\gamma \in \{0,1\}$. Assume $z > \mathfrak{z}_t$ if $\gamma = 1$, and $z \geqslant \mathfrak{z}_t$ if $\gamma = 0$. Then we have*

$$\mathfrak{S}^\gamma_{t,z}(x;q) \ll T^{t(q-2)}(\log T)^{\gamma t(q-1)}(\log x)^{\beta-z} \qquad (x \geqslant 2,\ T \geqslant 3). \tag{7·1}$$

(ii) *Under condition $z > \mathfrak{z}_t$, we have furthermore*

$$\mathfrak{S}^\gamma_{t,z}(x;2) \ll (\log x)^{\beta-z} \qquad (x \geqslant 2,\ T \geqslant 3). \tag{7·2}$$

(iii) *If $z \geqslant \mathfrak{z}_t$, the bound* (4·3) *holds. If $z > \mathfrak{z}_t$ the bound* (4·4) *is satisfied, and, more accurately, with $c_t := \lceil t\rceil - t$,*

$$\mathbb{E}_{x,\varrho}\big(\mathbf{1}_{E^{t,z}_T} M_2^{t+1}/\tau^{t+1}\big) \ll T(\log x)^{\beta-z}\mathrm{e}^{-c_t f_T(x)/(\lceil t\rceil+1)} \qquad (x \geqslant 3). \tag{7·3}$$

Remark. When t is not integral, the assumption $z \geqslant \mathfrak{z}_t$ implies $z > z_t$.

Proof. (i) As noticed above, when $t \in \mathbb{N}^*$, estimate (7·1) follows from (6·9).

Assume then $t \in \mathbb{R}^* \smallsetminus \mathbb{N}$, and put $k := \lceil t\rceil > t$. Let u, v be positive real numbers such that $uv = z$. By Hölder's inequality and (6·9) with $r = t$, we have, provided $u^{k/t} \geqslant \mathfrak{z}_k = k/(2^k - 1)$,

$$\mathfrak{S}^\gamma_{t,z}(x;q) \leqslant \mathbb{E}_{x,\varrho}\bigg(\frac{M_q^k u^{k\omega/t}}{\tau^k z^\omega}\mathbf{1}_{H^{q-1}_T}\bigg)^{t/k}\mathbb{E}_{x,\varrho}\bigg(\frac{v^{k\omega/(k-t)}}{z^\omega}\bigg)^{1-t/k} \ll T^{t(q-2)}(\log x)^{B-z}(\log T)^{\gamma t(q-1)}$$

with

$$B := \Big(2^k u^{k/t} - k\Big)\frac{t}{k} + v^{k/(t-k)}\Big(1 - \frac{t}{k}\Big).$$

Select $u := 2^{-t(1-t/k)}z^{t/k}$ so that $B = z2^t - t = \beta$. Observe that $u^{k/t} = 2^{t-k}z$. Hence condition $u^{k/t} \geqslant \mathfrak{z}_k$ amounts to $z \geqslant 2^{k-t}k/(2^k - 1) = \mathfrak{z}_t$, and $u^{k/t} > \mathfrak{z}_k$ amounts to $z > \mathfrak{z}_t$. This furnishes (7·1) under the stated hypotheses.

(ii) Similarly, (7·2) follows from (4·2) on selecting the same value for u.

(iii) Consider first the validity of (4·3) under the more general hypothesis $z \geqslant \mathfrak{z}_t$. Introducing as previously parameters u, v such that $z = uv$. Writing $u_1 := u^{(k+1)/(t+1)}$, $v_1 := v^{(k+1)/(k-t)}$ for legibility, we have, appealing to the bound $2^{\omega(n)} \leqslant T \log 3x$ ($n \in E_T$, $P^+(n) < x$) and using (4·3) for exponent $k+1$,

$$\begin{aligned}\mathbb{E}_{x,\varrho}\Big(\mathbf{1}_{E_T^{t,z}} \frac{M_2^{t+1}}{\tau^{t+1}}\Big) &\leqslant \mathbb{E}_{x,\varrho}\Big(\mathbf{1}_{E_T^{t,z}} \frac{M_2^{k+1}u_1^\omega}{\tau^{k+1}z^\omega}\Big)^{(t+1)/(k+1)} \mathbb{E}_{x,\varrho}\Big(\frac{v_1^\omega}{z^\omega}\Big)^{(k-t)/(k+1)} \\ &\leqslant \mathbb{E}_{x,\varrho}\Big(\mathbf{1}_{E_T^{t,z}} \frac{M_2^{k+1}u_1^\omega}{\tau^{k+1}z^\omega}\Big)^{(t+1)/(k+1)} (T\log x)^{(k-t)/(k+1)}\mathbb{E}_{x,\varrho}\Big(\frac{v_1^\omega}{(2z)^\omega}\Big)^{(k-t)/(k+1)}\end{aligned}$$

Selecting $u := 2^{-(k-t)(t+1)/(k+1)} z^{(t+1)/(k+1)}$ and checking that $2^k u_1 = 2^t z$ so that $E_T^{t,z} = E_T^{k+1,u_1}$, we may conclude as previously. The proof that (4·4) is valid under the extended hypothesis $z > \mathfrak{z}_t$ is similar. Estimate (7·3) is derived by exploiting the condition $2^{\omega(n)} \leqslant \mathrm{e}^{-f_T(x)} T \log 3x$, valid for $n \in E_T^{t,z}$, $P^+(n) < x$. □

Lemma 7.2. *Let $z > 0$, $t \geqslant 1$, $\varrho \in \mathcal{M}_z$.*

(i) *Let $q \geqslant 3$, $\gamma \in \{0,1\}$. Assume $z > z_t$ if $\gamma = 1$, and $z \geqslant z_t$ if $\gamma = 0$. Assume furthermore that* (6·8) *holds. Then* (7·1) *persists.*

(ii) *If $z > z_t$ the bounds* (4·3) *and* (4·4) *are satisfied. Moreover, if $z \geqslant z_t$, we have*

$$\mathbb{E}_{x,\varrho}\big(\mathbf{1}_{E_T^{t,z}} M_2^{t+1}/\tau^{t+1}\big) \ll T(\log T)^{\delta/2}(\log x)^{\beta - z}. \tag{7·4}$$

Proof. (i) We may plainly assume that t is not an integer and hence that $t > 1$. Arguing as for (6·16), we get

$$\mathfrak{S}_{t,z}^\gamma(x;q) \ll 1 + \int_1^{x^2} \frac{Z_{t,z}^\gamma(y;T)}{y(\log 3y)^{2+z}}\,\mathrm{d}y. \tag{7·5}$$

Appealing to the inequality $M_{q-1}(m) \leqslant \vartheta_{q-1,T}\tau(m)$ taking (6·5) into account, we infer that

$$Z_{t,z}^\gamma(y;q) \ll T^{q-2}(\log T)^{\gamma(q-2)}(\log 3y)^{2z}\mathfrak{S}_{t-1,2z}^\gamma(y). \tag{7·6}$$

Now, observe that $2z_t > \mathfrak{z}_{t-1}$ since $t > \lceil t-1\rceil$.[3] We may hence deduce from (7·1) that

$$\mathfrak{S}_{t-1,2z}^\gamma(y;q) \ll T^{(q-2)(t-1)}(\log T)^{(t-1)(q-1)\gamma}(\log 3y)^{\beta+1-2z}, \tag{7·7}$$

since $\beta(t-1,2z) = \beta(t,z)+1 = \beta+1$. Indeed, since $2z > z_{t-1}$, the bound for $\mathfrak{S}_{t-1,2z}$ is available whenever $\gamma \in \{0,1\}$. Carrying back into (7·6), we obtain

$$Z_{t,z}^\gamma(y;q) \ll T^{t(q-2)}(\log T)^{(t-1)(q-1)\gamma}(\log y)^{\beta+1},$$

which yields the required estimate (7·1) by (7·5) provided $\delta = 0$, and so $\beta + 1 - (z+2) < -1$. If $\delta = 1$, i.e. $z = z_t$ (hence $\gamma = 1$), this is insufficient to get the expected bound. We get round the difficulty by introducing a parameter $\mathfrak{z}_{t-1}^* \in \big]\mathfrak{z}_{t-1}, 2\mathfrak{z}_{t-1}\big]$ and writing

$$\begin{aligned}Z_{t,z}^1(y;q) &\ll \vartheta_{q-1,T}\sum_{m\in H_{T,1}^{q-1}}^{y} \frac{\varrho_2(m)2^{\lambda_t\omega(m)}M_q(m)^{t-1}}{m\tau(m)^{t-1}} \\ &\ll \vartheta_{q-1,T}\big\{T(\log 3y)\big\}^{\lambda_t}\sum_{m\in H_{T,1}^{q-1}}^{y} \frac{\varrho_2(m)M_q(m)^{t-1}}{m\tau(m)^{t-1}}\end{aligned}$$

with $\varrho_2(m) := \varrho(m)(\mathfrak{z}_{t-1}^*/z_t)^{\omega(m)}$ and $\lambda_t := \{\log(2z_t/\mathfrak{z}_{t-1}^*)\}/\log 2$. Taking (6·5) into account, the bound (7·1) yields

$$Z_{t,z}^1(y;q) \ll T^{t(q-2)+\lambda_t}(\log T)^{t(q-1)\gamma-\gamma}(\log 3y)^{\mu_t} \tag{7·8}$$

with

$$\mu_t := \beta(t-1,\mathfrak{z}_{t-1}^*) + \lambda_t < z_t + 1,$$

provided $\mathfrak{z}_{t-1}^*$ is chosen sufficiently close to $\mathfrak{z}_{t-1}$. To see the latter inequality, observe that $\mu_t - z_t - 1 = \varphi(a,b) := (a-b) - \{\log(a/b)\}/\log 2$ with $a = f(\lceil t-1\rceil)$, $b = f(t)$, and $f(s) := s/(1-1/2^s) > 1/\log 2$. Since $\varphi(a,b)$ is a non-increasing function of a and $a > b \geqslant 1/\log 2$, the inequality follows.

3. This follows from the fact that $s \mapsto s2^s/(2^s-1)$ is increasing.

Now split the integral in (7·5) at $\log 3y = T^\nu$ with $\nu := \lambda_t/(1+z_t-\mu_t)$, applying (7·8) to the upper range and (7·7) to the lower range. This yields (7·1) as required.

(ii) Let us focus on proving (4·3) for real $t\geqslant 1$, $z\geqslant z_t$, and leave the proof of (4·4) for $z>z_t$ to the reader. We may plainly assume $t\notin\mathbb{N}^*$. Arguing as in the proof of (6·16), we get

$$\mathbb{E}_{x,\varrho}\bigg(\frac{\mathbf{1}_{E_T^{t,z}}M_2^{t+1}}{\tau^{t+1}}\bigg)\ll 1+\int_1^{x^2}\frac{Z_{t,z}(y;T)}{y(\log 3y)^{2+z}}\,\mathrm{d}y,$$

with now

$$Z_{t,z}(y;T):=\sum_{m\in E_T^{t,z}}^{y}\frac{\varrho(m)\tau(m)M_2(m)^t}{m\tau(m)^t}\cdot$$

Since $2z_t>\mathfrak{z}_{t-1}$, we may apply (7·3) with $(2z,t-1)$ in place of (z,t) to get

$$Z_{t,z}(y;T)\ll T(\log y)^{1+\beta}\mathrm{e}^{-(\lceil t\rceil-t)f_T(y)/(\lceil t\rceil+1)}.$$

This implies

$$\mathbb{E}_{x,\varrho}\big(\mathbf{1}_{E_T^{t,z}}M_2^{t+1}/\tau^{t+1}\big)\ll T(\log T)^{\delta/2}(\log x)^{\beta-z}.$$

□

8. Bounding moments inductively

We are now in a position to obtain a bound for $\mathfrak{T}_{t,z}(x;q)$, as defined in (6·6), that is uniform in x, T and q. Note that $\mathfrak{T}_{t,z}(x;q)$ is a subsum of $\mathfrak{S}_{t,z}^{\gamma}(x;q)$ if $\gamma=\delta$. The initialisation step being granted by Proposition 6.1 and Lemma 7.2, we can adapt the proof of [3; prop. 2.2], itself resting upon [12; prop. 6.2]. The following result is actually valid whether or not the parameter t is an integer.

Proposition 8.1. *Let $t>0$, $z\geqslant z_t$, $\varrho\in\mathcal{M}_z$, and let C_0 be a sufficiently large absolute constant. Assume $\vartheta_{1,T}=1$, and*

$$\vartheta_{j,T}\geqslant\frac{j!}{j^2}C_0^{j-1}T^{j-1}(\log T)^{\delta(j-1)}\quad(j\geqslant 2),\tag{8·1}$$

$$\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\vartheta_{j,T}\vartheta_{q-j,T}\leqslant\frac{\vartheta_{q,T}}{C_0T(\log T)^{\delta/2}}\quad(q\geqslant 3)\cdot\tag{8·2}$$

Then

$$\mathfrak{T}_{t,z}(x;q)\leqslant\frac{C_0\vartheta_{q,T}(\log x)^{\beta-z}}{q^2T^t}\qquad\big(q\geqslant\max(t,3),\,x\geqslant 2,\,T\geqslant 3\big).\tag{8·3}$$

Proof. We argue by induction on $q\geqslant\max(t,3)$. Albeit, in Lemmas 7.1 and 7.2, γ may take two values independently, we fix here $\gamma=\delta$. By Lemma 7.2 with $\gamma=\delta$ and (8·1), the required bound holds for $t\leqslant q\leqslant 2t$, which initialises the induction.

We now assume $q\geqslant\max(2t,3)$.

By (6·12), the function $\mathbf{1}_{H_T^{t,z,q-1}}M_q/\tau$ satisfies (5·1) with $w(n,p)=\mathbf{1}_{H_T^{t,z,q-1}}(n)W_q(n,p)/\tau(n)$. By partial integration and appeal to the Brun-Titchmarsh inequality as in [12; (6.9)], we get

$$\sum_{\sqrt{y}\leqslant p<y}\frac{\varrho(p)W_q(m,p)}{p}\ll\Big(1+2^j\frac{\log 3y}{y^{1/4}}\Big)\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\frac{M_{q-j}(m)M_j(m)}{\log y}\cdot\tag{8·4}$$

Applying Lemma 5.1, we therefore obtain

$$\mathfrak{T}_{t,z}(x;q)\ll 1+\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\int_1^{x^2}\sum_{m\in H_T^{t,z,q-1}}^{y}\Big(1+2^j\frac{\log 3y}{y^{1/4}}\Big)\frac{\varrho(m)M_{q-j}(m)M_j(m)}{m\tau(m)y(\log 3y)^{2+z}}\,\mathrm{d}y.\tag{8·5}$$

The contribution of the term involving $y^{-1/4}$ may be handled as in [12], invoking the inequality

$$\int_1^\infty(\log 3y)^q\frac{\mathrm{d}y}{y^{5/4}}\ll 4^qq!$$

and using the trivial bound

$$M_{q-j}(m)M_j(m) \leqslant M_{q-1}(m)M_1(m) \leqslant \tau(m)^q \leqslant T^{q-2}(\log y)^{q-2}\tau(m)^2,$$

valid whenever $m \in E_T$ and $P^+(m) < y$. This provides an overall term at most $C^q q! T^{q-2}$ where C is a suitable absolute constant. By (8·1) we may choose C_0 such that

$$C^q q! T^{q-2} \ll C(C/C_0)^{q-1} q^2 \vartheta_{q-1,T} \ll \vartheta_{q-1,T}/2^q.$$

Therefore, since $\beta - z \geqslant 0$,

$$T_{t,z}(x;q) \ll \frac{\vartheta_{q-1,T}(\log x)^{\beta-z}}{2^q} + \sum_{1\leqslant j\leqslant q/2} \binom{q}{j} \int_1^{x^2} \frac{Z_{q,j}(y,T)}{y(\log 3y)^{2+z}}\,\mathrm{d}y,$$

with

$$Z_{q,j}(y,T) := \sum_{m\in H_T^{t,z,q-1}}^{y} \frac{\varrho(m)M_{q-j}(m)M_j(m)}{m\tau(m)}.$$

We aim at establishing (8·3) by induction on $q \geqslant \max(2t,3)$. By (6·4), we have, whenever m is counted in $Z_{q,j}$,

$$M_j(m) \leqslant \vartheta_{j,T} T(\log 3y)\,\mathrm{e}^{-f_T(y)} \qquad \Big(m \in H_T^{t,z,q-1},\ P^+(m) < y,\ 1 \leqslant j \leqslant q-1,\ y \geqslant 1\Big).$$

Therefore

$$Z_{q,j}(y,T) \ll \vartheta_{j,T} T(\log 3y)^{1+z}\,\mathrm{e}^{-f_T(y)}\mathfrak{T}_{t,z}(y;q-j).$$

For $j \leqslant q/2$, we have $q - j \geqslant q/2 \geqslant t$. We may hence bound $\mathfrak{T}_{t,z}(y;q-j)$ by the induction hypothesis. This yields

$$Z_{q,j}(y,T) \ll \frac{\vartheta_{j,T}\vartheta_{q-j,T}}{q^2T^t} T\,\mathrm{e}^{-f_T(y)}(\log 3y)^{1+\beta},$$

whence

$$\begin{aligned}\mathfrak{T}_{t,z}(x;q) &\ll \sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\frac{\vartheta_{q-j,T}\vartheta_{j,T}}{q^2T^{t-1}}\int_1^{x^2}\frac{\mathrm{e}^{-f_T(y)}(\log 3y)^{\beta-z-1}}{y}\,\mathrm{d}y + \frac{\vartheta_{q-1,T}(\log x)^{\beta-z}}{2^q}\\ &\ll \sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\frac{\vartheta_{q-j,T}\vartheta_{j,T}(\log T)^{\delta/2}}{q^2T^{t-1}}(\log x)^{\beta-z}.\end{aligned} \tag{8·6}$$

Under assumption (8·2), this implies (8·3). □

9. Proof of Theorem 1.1

Observe that the sequence defined by

$$\vartheta_{q,T} = \frac{q!}{q^2}\big(\tfrac{2}{3}\pi^2C_0\big)^{q-1}T^{q-1}(\log T)^{\delta(q-1)} \quad (q \geqslant 1), \tag{9·1}$$

satisfies (8·1) and (8·2) with $\gamma = \delta$. From now on, we consider the sets $H_T^{z,t,q}$ defined in (6·4).

By (3·2) and (7·4), we derive from using Markov's inequality that, for $z \geqslant z_t$

$$\mathbb{P}_{x,\varrho}\big(E \smallsetminus H_T^{t,z}\big) \ll \frac{(\log x)^{\beta-z}}{T^t} \qquad (x \geqslant 2,\ T \geqslant 3). \tag{9·2}$$

Let us put $\lambda := C_1T(\log T)^\delta$ and set out to establishing that, for fixed $t \geqslant 1$, $z \geqslant z_t$, we have

$$\mathbb{P}_{x,\varrho}\Big(\Delta(n) > \lambda\log_2 x\Big) \ll \frac{(\log x)^{\beta-z}}{T^t} \qquad (x \geqslant 3,\ T \geqslant 1). \tag{9·3}$$

By (9·2), it is sufficient to show that

$$\mathbb{P}_{x,\varrho}\Big(n \in H_T^{t,z},\ \Delta > \lambda\log_2 x\Big) \ll \frac{(\log x)^{\beta-z}}{T^t}.$$

However, from (8·3), we have, for $q \geqslant 3$,

$$\mathbb{P}_{x,\varrho}\big(H_T^{t,z,q-1} \smallsetminus H_T^{t,z,q}\big) \leqslant \mathbb{P}_{x,\varrho}\big(H_T^{t,z,q-1}\big)\mathbb{E}_{x,\varrho}\Big(\frac{M_q}{\vartheta_{q,T}\tau}\Big|H_T^{t,z,q-1}\Big) \ll \frac{(\log x)^{\beta-z}}{q^2T^t}\cdot$$

When $3 \leqslant q \leqslant t$, we deduce from Lemma 7.2 that

$$\mathbb{P}_{x,\varrho}\Big(H_T^{t,z,q-1} \smallsetminus H_T^{t,z,q}\Big) \leqslant \mathbb{P}_{x,\varrho}\Big(H_T^{t,z,q-1}\Big)\mathbb{E}_{x,\varrho}\Bigg(\frac{M_q^t}{\vartheta_{q,T}^t\tau^t}\Bigg|H_T^{t,z,q-1}\Bigg) \ll \frac{\mathfrak{S}_{q,t}(x)}{\vartheta_{q,T}^t} \ll \frac{(\log x)^{\beta-z}}{q^2T^t},$$

and so

$$\mathbb{P}_{x,\varrho}\Big(E \smallsetminus \cap_q H_T^{t,z,q}\Big) \ll \frac{(\log x)^{\beta-z}}{T^t}\cdot \tag{9·4}$$

Now observe that, if $n \in \cap_q H_T^{t,z,q}$ and $P^+(n) \leqslant x$, we have

$$\Delta(n) \leqslant 2M_q(n)^{1/q} \ll \tau(n)^{1/q}\vartheta_{q,T}^{1/q} \ll qT(\log T)^\delta(\log x)^{1/q},$$

where the first inequality is [10; (5.56)]. Selecting $q = \lfloor \log_2 x \rfloor$ we deduce that, for a suitably large constant C_1, we have $\Delta(n) \leqslant \lambda \log_2 x$. Thus (9·3) follows from (9·4).

It remains to estimate $\mathbb{E}_{x,\varrho}(\Delta^t)$. We may assume trivially that $2(\log_2 x) \leqslant \Delta(n) \leqslant (\log x)^{z2^t+t}$. Indeed, since $\Delta \leqslant \tau$, the contribution of $\Delta(n) > (\log x)^{z2^t+t}$ in $\mathbb{E}_{x,\varrho}(\Delta^t)$ is bounded by

$$\leqslant \frac{\mathbb{E}_{x,\varrho}(\tau^{t+1})}{(\log x)^{z2^t+t}} \ll \frac{(\log x)^{z2^{t+1}-z}}{(\log x)^{z2^t+t}} \ll (\log x)^{\beta-z}.$$

The contribution of those integers n such that $2^j < \Delta(n)/(\log_2 x) \leqslant 2^{j+1}$ is plainly

$$\ll \frac{2^{tj}(\log_2 x)^{t+\delta}(\log x)^{\beta-z}}{2^{tj}} = (\log_2 x)^{t+\delta}(\log x)^{\beta-z}.$$

Summing over $j \ll \log_2 x$, we finally get

$$\mathbb{E}_{x,\varrho}(\Delta^t) \ll (\log_2 x)^{1+t+\delta}(\log x)^{\beta-z}.$$

Since, by Lemma 3.1, we have

$$S_{t,\varrho}(x) = \sum_{n \leqslant x} \varrho(n)\Delta(n)^t \ll x(\log x)^{z-1}\mathbb{E}_{x,\varrho}(\Delta^t),$$

we obtain the upper bound of (1·5) as required.

To establish (1·6), we apply Hölder's inequality with parameters s/t and $s/(s-t)$ and invoke (1·5) with $t = s$, $\delta = 1$, yielding

$$\mathbb{E}_{x,\varrho}(\Delta^t) \leqslant \big\{\mathbb{E}_{x,\varrho}(\Delta^s)\big\}^{t/s} \ll (\log_2 x)^{t+2t/s}.$$

10. Proof of Theorem 1.3

Write $t = 2s$ with $s \geqslant 1$. By [10; Exercise 47] (see a proof in [3; §4]) we have

$$\Delta(n)^2 \leqslant 4M_2(a)M_2(b) \qquad (ab = n), \tag{10·1}$$

and so

$$\Delta(n)^t \leqslant \sum_{ab=n} \frac{M_2(a)^s M_2(b)^s}{2^{\omega(ab)}}\cdot$$

Appealing to (6·9) with $\varrho_1(n) := \varrho(n)2^{(s-1)\omega(n)}$, we deduce that, provided $2^{s-1}z > z_s$, we have

$$\mathbb{E}_{x,\varrho}(\Delta^t) \ll \frac{1}{(\log x)^z}\Bigg\{(\log x)^{2^{s-1}z}\,\mathbb{E}_{x,\varrho_1}\Bigg(\frac{M_2^s}{\tau^s}\Bigg)\Bigg\}^2 \ll (\log x)^{(2^t-1)z-t}.$$

In view of (3·1) and (1·4), this furnishes the true order of magnitude.

11. Proof of Theorem 1.4

Put $\chi_k(n) := \mathbf{1}_{\omega(n)=k}$ $(n \geqslant 1,\ k \geqslant 0)$, so that

$$\mathbb{E}_{x,\varrho}(\Delta^2) = \sum_{k\geqslant 0} \mathbb{E}_{x,\varrho}(\Delta^2\chi_k). \tag{11·1}$$

The proof of (1·8) relies on the following bound.

Lemma 11.1. *Let $\varrho \in \mathcal{M}_1$. Uniformly for $x \geqslant 3$ and integer $h \geqslant 1$, we have*

$$\mathbb{E}_{x,\varrho}\big(M_2\chi_h/\tau\big) \ll 1. \tag{11·2}$$

Proof. By (4·5), we plainly have

$$\mathbb{E}_{x,\varrho}\Big(\frac{M_2\chi_h}{\tau}\Big) \ll \frac{1}{h!\log x}\int_0^1 \bigg(\sum_{p\leqslant x}\frac{\varrho(p)\{1+\cos(\vartheta\log p)\}}{p}\bigg)^h \mathrm{d}\vartheta.$$

In view of (1·1), the contribution to the above integral of the range $0 \leqslant \vartheta \leqslant 1/\log x$ is, for a suitable, absolute constant C,

$$\ll \frac{(2\log_2 x + C)^h}{h!(\log x)^2} \ll 1.$$

The complementary contribution is

$$\ll \frac{1}{h!\log x}\int_{1/\log x}^1 \big(\log_2 x + \log 1/\vartheta + C\big)^h \,\mathrm{d}\vartheta = \frac{\mathrm{e}^C}{h!}\int_{\log_2 x + C}^{2\log_2 x + C} u^h \mathrm{e}^{-u}\,\mathrm{d}u \ll 1. \qquad \square$$

We are now in a position to complete the proof of Theorem 1.4.

Let $\varepsilon > 0$. For any $z \geqslant 1$, the contribution of $k \geqslant 4(1+\varepsilon)\log_2 x$ to (11·1) is

$$\leqslant \mathbb{E}_{x,\varrho}\Big(z^{\omega(n)-4(1+\varepsilon)\log_2 x}\Delta^2\Big) \asymp \frac{S_{2,\varrho_2}(x)}{(\log x)^{4(1+\varepsilon)\log z}} \ll (\log x)^{4z-3-4(1+\varepsilon)\log z}(\log_2 x)^3,$$

with now $\varrho_2(n) = \varrho(n)z^{\omega(n)}$. Selecting $z = 1+\varepsilon$, we get, for a suitable constant $c > 0$,

$$\sum_{k\geqslant 4(1+\varepsilon)\log_2 x} \mathbb{E}_{x,\varrho}\big(\Delta^2\chi_k\big) \ll (\log x)^{1-c\varepsilon^2}(\log_2 x)^3 \ll \log x,$$

provided $\varepsilon = \varepsilon_x := \sqrt{3\log_3 x/c\log_2 x}$.

Similarly, bounding $\mathbb{E}_{x,\varrho}(\Delta^2)$ with $z = 1 - \varepsilon_x$, we see that the contribution to (11·1) of $k \leqslant 4(1-\varepsilon)\log_2 x$ is $\ll x\log x$.

Only $O\big(\sqrt{(\log_2 x)\log_3 x}\big)$ possible values of k remain. Each squarefree integer n such that $\omega(n) = k$ may be represented as $n = ab$ with $\omega(a) = \lfloor k/2\rfloor$ and $\omega(b) = \lceil k/2\rceil$. However, since there are $\gg 2^{\omega(n)}/\sqrt{\log_2 x}$ representations of n as a product ab of the above form, we infer that

$$\mathbb{E}_{x,\varrho}(\Delta^2\chi_k) \ll (\log x)\sqrt{\log_2 x}\,\mathbb{E}_{x,\varrho}\bigg(\frac{M_2\chi_{\lfloor k/2\rfloor}}{\tau}\bigg)\mathbb{E}_{x,\varrho}\bigg(\frac{M_2\chi_{\lceil k/2\rceil}}{\tau}\bigg) \ll (\log x)\sqrt{\log_2 x},$$

where the last estimate follows from (11·2). Summing over k, we get (1·8) as required.

References

[1] R. de la Bretèche & G. Tenenbaum, Oscillations localisées sur les diviseurs, *J. London Math. Soc. (2)* **85** (2012), 669–693.

[2] R. de la Bretèche & G. Tenenbaum, Two upper bounds for the Erdős–Hooley Delta-function, *Sci. China Math.* **66**, no. 12 (2023), 2683–2692.

[3] R. de la Bretèche & G. Tenenbaum, Note on the mean-value of the Erdős-Hooley Delta-function, *Acta Arith.*, **219.4** (2025), 379–394.

[4] R. de la Bretèche & G. Tenenbaum, Mean values of arithmetical functions and application to sum of powers, *Math. Proc. Cambridge Philos. Soc.*, to appear.

[5] P. Erdős, Problem 218, Solution by the proposer, *Can. Math. Bull.* **16** (1973), 621–622.

[6] P. Erdős & J.-L. Nicolas, Répartition des nombres superabondants, *Bull. Soc. math. France* **103** (1975), 65–90.

[7] K. Ford, B. Green, D. Koukoulopoulos, Equal sums in random sets and the concentration of divisors, *Inventiones Math.* **232**, (2023),1027–1160.
[8] K. Ford, D. Koukoulopoulos, T. Tao, A lower bound on the mean value of the Erdős–Hooley Delta function, *Proc. London Math. Soc.* (3) **129,** no. 1 (2024), e12618.
[9] R.R. Hall & G. Tenenbaum, The average orders of Hooley's Δ_r-functions, II, *Compos. Math.* **60** (1986), 163–186.
[10] R.R. Hall & G. Tenenbaum, *Divisors*, Cambridge tracts in mathematics, no 90, Cambridge University Press (1988).
[11] C. Hooley, On a new technique and its applications to the theory of numbers, *Proc. London Math. Soc.* (3) **38** (1979), 115–151.
[12] D. Koukoulopoulos & T. Tao, An upper bound on the mean value of the Erdős–Hooley Delta function, *Proc. London Math. Soc. (3)* **127**, no.6 (2023), 1865–1885.
[13] M. Nair & G. Tenenbaum, Short sums of certain arithmetic functions, *Acta Math.* **180**, (1998), 119–144.
[14] K. Norton, On the number of restricted prime factors of an integer, I, *Illinois J. Math.* **20**, no. 4 (1976), 681–705.
[15] G. Tenenbaum, Sur la concentration moyenne des diviseurs, *Comment. Math. Helvetici* **60** (1985), 411-428.
[16] G. Tenenbaum, Fonctions Δ de Hooley et applications, Séminaire de théorie des nombres, Paris 1984-85, Prog. Math. 63 (1986), 225–239.
[17] G. Tenenbaum, Sur une question d'Erdős et Schinzel, II, *Invent. Math.* **99** (1990), 215–224.
[18] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, 3rd ed., Graduate Studies in Mathematics 163, Amer. Math. Soc. (2015). See also *Introduction à la théorie analytique et probabiliste des nombres*, 5th ed., Dunod, Sciences Sup (Paris), 2022.

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