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Generalized Mertens sums*

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To Krishna Alladi, half-way,
as a token of a life-long friendship.

Let

$$S_k(x) := \sum_{p_1\cdots p_k\leqslant x} \frac{1}{p_1\cdots p_k} \qquad (x \geqslant 2),$$

where p_j denotes a prime number. It is a well known result of Mertens that

$$S_1(x) = \log_2 x + c_1 + O\Big(\frac{1}{\log x}\Big) \qquad (x \geqslant 3),$$

with (see, e.g., [3], p. 18)

$$c_1 := \gamma - \sum_p \Big\{ \log\Big(\frac{1}{1-1/p}\Big) - \frac{1}{p}\Big\} \approx 0.261497. \tag{1}$$

Here and in the sequel, γ is Euler's constant, p stands for a prime number and $\log_2$ denotes the two-fold iterated logarithm. The number c_1 is called Mertens' constant, also known as the Meissel-Mertens, or the Kronecker, or the Hadamard-La Vallée-Poussin constant.

In [1], [2], Popa used elementary techniques to derive similar asymptotic formulae in the cases $k = 2$ and 3, with a main term equal to a polynomial of degree k in $\log_2 x$ and a remainder term $\ll (\log_2 x)^k/\log x$. In this note we investigate the general case. We define classically Γ as the Euler gamma function.

Theorem 1. *Let $k \geqslant 1$. We have*

$$S_k(x) = P_k\big(\log_2 x\big) + O\bigg(\frac{(\log_2 x)^{k-1}}{\log x}\bigg) \qquad (x \geqslant 3),$$

where $P_k(X) := \sum_{0\leqslant j\leqslant k} \lambda_{j,k} X^j$, and

$$\lambda_{j,k} := \sum_{0\leqslant m\leqslant k-j} \binom{k}{m, j, k-m-j} (c_1-\gamma)^{k-m-j} \Big(\frac{1}{\Gamma}\Big)^{(m)}(1) \qquad (0 \leqslant j \leqslant k).$$

Proof. Write $P(s) := \sum_p 1/p^s$, so that we have

$$P(s) = \log\zeta(s) - g(s), \qquad g(s) := \sum_{m\geqslant 2} \frac{1}{m} \sum_p \frac{1}{p^{ms}}$$

in any simply connected zero and pole-free region of the zeta function where the series $g(s)$ converges. (Here $\log\zeta(s)$ is the branch that is real for real $s > 1$.) Moreover, for $s+1$ in the same region, we have

$$P(s+1) = \log(1/s) + h(s),$$

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[*] We include here some corrections with respect to the published version.

with $h(s)=\log\{s\zeta(s+1)\}-g(s+1)$ and where $\log(1/s)$ is understood as the principal branch. The function $h(s)$ is clearly holomorphic in a disk around $s=0$.

Now, for any $c>0$, we have

$$S_k(x)=\frac{1}{2\pi i}\int_{c+i\mathbb{R}}P(s+1)^k x^s\frac{\mathrm{d}s}{s}\qquad(x\in\mathbb{R}^+\smallsetminus\mathbb{N}).\tag{2}$$

By following, *mutatis mutandis*, the argument of the Selberg-Delange method (see [3], ch. II.5 & II.6) we readily obtain

$$S_k(x)=\frac{1}{2\pi i}\int_{\mathcal{H}}\Big\{\log\Big(\frac{1}{s}\Big)+h(0)\Big\}^k x^s\frac{\mathrm{d}s}{s}+O\Big(\frac{(\log_2 x)^{k-1}}{\log x}\Big)\qquad(x\geqslant 2),$$ (1)

where $\mathcal{H}$ is a Hankel contour around $\mathbb{R}^-$, positively oriented.

We also observe that, by (1), we have

$$h(0)=-\sum_p\Big\{\log\Big(\frac{1}{1-1/p}\Big)-\frac{1}{p}\Big\}=c_1-\gamma.$$

It remains to compute

$$I_m(x):=\frac{1}{2\pi i}\int_{\mathcal{H}}\Big\{\log\frac{1}{s}\Big\}^m x^s\frac{\mathrm{d}s}{s}\qquad(m\geqslant 0).$$

To this end, we consider Hankel's formula (see, e.g., [3], th. II.0.17)

$$\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{x^s}{s^{1+z}}\,\mathrm{d}s=\frac{(\log x)^z}{\Gamma(z+1)}\qquad(z\in\mathbb{C})$$

and derive

$$I_m(x)=\sum_{0\leqslant j\leqslant m}\binom{m}{j}(\log_2 x)^j\Big(\frac{1}{\Gamma}\Big)^{(m-j)}(1).$$

Rearranging the terms, we arrive at the announced formula for $P_k(X)$. □

Specialization. Noting that $(1/\Gamma)'(1)=\gamma$, $(1/\Gamma)''(1)=\gamma^2-\frac{1}{6}\pi^2$, $(1/\Gamma)'''(1)=2\zeta(3)-\frac{1}{2}\pi^2\gamma+\gamma^3$, $(1/\Gamma)^{(4)}(1)=\frac{1}{60}\pi^4+8\gamma\zeta(3)+\pi^2\gamma^2+\gamma^4$, as may be deduced from classical formulae for the logarithmic derivative of the Euler function (see, e.g., [3], chap. II.0), we find

$$P_1(X)=X+c_1,\quad P_2(X)=(X+c_1)^2-\tfrac{1}{6}\pi^2,\qquad P_3(X)=(X+c_1)^3-\tfrac{1}{2}\pi^2(X+c_1)+2\zeta(3),$$
$$P_4(X)=(X+c_1)^4-\pi^2(X+c_1)^2+8\zeta(3)(X+c_1)+\tfrac{1}{60}\pi^4.$$

Remark. By retaining, in the integrand of (2), the first $N+1$ terms of the Taylor expansion of $h(s)$ at the origin, the above method readily yields, for arbitrary integer $N\geqslant 0$, an asymptotic formula of the type

$$S_k(x)=\sum_{0\leqslant j\leqslant N}\frac{P_{j,k}(\log_2 x)}{(\log x)^j}+O\Big(\frac{(\log_2 x)^{k-1}}{(\log x)^{N+1}}\Big)$$

where $P_{j,k}$ is an explicit polynomial of degree $\leqslant k-1$.

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References

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(1) Due to an oversight, the exponent of $\log_2 x$ has been set to k instead of $k-1$ in the published version of this work.