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Note on the mean value of the Erdős–Hooley Delta-function

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To the memory of Richard R. Hall,
whose talent and elegance
will remain as a constant
source of inspiration

Abstract. For integer $n \geqslant 1$ and real u, let $\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|$. The Erdős–Hooley Delta-function is then defined by $\Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u)$. We improve a recent upper bound for the mean value of this function by showing that, for large x, we have

$$\sum_{n\leqslant x} \Delta(n) \ll x(\log_2 x)^{5/2}.$$

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1. Introduction and statement of results

For integer $n \geqslant 1$ and real u, put

$$\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \quad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u).$$

Introduced by Erdős [3] (see also [4]) and studied by Hooley [13], the Δ-function and specifically its (possibly weighted) mean-value proved very useful in several branches of number theory — see, e.g., [15], [12], and [1] for further references. If $\tau(n)$ denotes the total number of divisors of n, then $\Delta(n)/\tau(n)$ coincides with the concentration of the numbers $\log d$, $d|n$.

Asymptotic estimates for

$$S(x) := \sum_{n\leqslant x} \Delta(n)$$

have a rather long history since Hooley's pioneer work [13]: see [9], [14], [10], [12], and the recent papers [1], [11], and [6] for a description of the main steps. While Hooley's upper bound was $S(x) \ll x(\log x)^{4/\pi-1}$ and following works aimed at reducing the value of the exponent, the first estimate of the type

$$S(x) \ll x\mathrm{e}^{c\sqrt{\log_2 x \log_3 x}} \qquad (x \geqslant 16) \tag{1·1}$$

appears in Tenenbaum's paper [14]. Here and in the sequel, we let $\log_k$ denote the k-fold iterated logarithm. In [1], we could remove the $\log_3 x$ from (1·1). Very recently Koukoulopoulos and Tao made a further breakthrough by showing

$$S(x) \ll x(\log_2 x)^{11/4} \qquad (x \geqslant 3).$$

In this work, we slightly improve this upper bound.

Considering lower bounds, it is shown in [8] that $S(x) \gg x\log_2 x$ and this remained unimproved for over four decades. A very recent work of Ford, Koukoulopoulos and Tao [6] provides the lower bound

$$S(x) \gg x(\log_2 x)^{1+\eta+o(1)} \tag{1·2}$$

where $\eta \approx 0.353327$ is the exponent appearing in the new lower bound for the normal order of $\Delta(n)$ by Ford, Green and Koukoulopoulos [5].

Theorem 1.1. *For all large real x, we have*

$$S(x) \ll x(\log_2 x)^{5/2}. \tag{1·3}$$

Since the work [14], all upper estimates for $S(x)$ rest on properties of moments

$$M_q(n) := \int_{\mathbb{R}} \Delta(n,u)^q \,\mathrm{d}u \qquad (n \geqslant 1,\ q \geqslant 1).$$

As we clearly have $\Delta(np,u) = \Delta(n,u) + \Delta(n,u-\log p)$ for any integer $n \geqslant 1$ and any prime number p such that $p \nmid n$, we may write the inductive inequality

$$2M_q(n) \leqslant M_q(np) \leqslant 2M_q(n) + 2W_q(n,p) \quad (p \nmid n) \tag{1·4}$$

with

$$\begin{aligned} N_{j,q}(n,p) &:= \int_{\mathbb{R}} \Delta(n,u)^j \Delta(n,u-\log p)^{q-j} \,\mathrm{d}u,\\ W_q(n,p) &:= \sum_{1\leqslant j\leqslant q/2} \binom{q}{j} N_{j,q}(n,p). \end{aligned} \tag{1·5}$$

An averaging argument over the largest prime factor of integers enables one to essentially exploit the above in the form

$$M_q(np) \leqslant 2M_q(n) + C \sum_{1\leqslant j\leqslant q/2} \binom{q}{j} M_j(n) M_{q-j}(n) \quad (p \nmid n) \tag{1·6}$$

for some absolute constant C. This is the basis of various induction processes, leading to corresponding upper bounds. However, it turns out that an early appeal to Hölder's inequality in the form

$$M_j(n) \leqslant M_q(n)^{(j-1)/(q-1)} \tau(n)^{(q-j)/(q-1)} \quad (n \geqslant 1,\ q \geqslant 2,\ 1 \leqslant j \leqslant q-1)$$

can only lead to upper bounds of the shape (1·1), with or without the triple logarithm. The decisive progress achieved in [11] consists in finding a framework inside which the full strength of an inequality like (1·6) can be exploited.

To prove (1·3), we follow the approach of [11] and insert a refinement based on an appeal to Fourier transforms: see (2·7). As noticed in § 2.2, this also yields an improvement on an estimate established in [12]—see (2·2) *infra.*

Since the theory of the Δ-function already provided a number of surprises, it is rather risky to make a guess regarding the best possible exponent of $\log_2 x$ in (1·3). Regarding upper bounds, our approach might lead to the exponent 2, but we haven't been able to substantiate this so far.

It is worthwhile mentioning that the upper bound method displayed here (as of course that of [11]) adapts smoothly to weighted averages of the type

$$\sum_{n\leqslant x} g(n)\Delta(n)$$

where g is a non-negative multiplicative function satisfying mild growth assumptions such as [1; (1.1),(1.2)] and having mean-value 1 over the primes, viz.

$$\sum_{p\leqslant x} g(p)\log p = x + O\big(x\mathrm{e}^{-(\log x)^c}\big)$$

for some fixed $c > 0$. This is precisely what is needed to handle averages of the Δ-function at polynomial arguments. Thus we can state that if $F \in \mathbb{Z}[X]$ is an irreducible polynomial with integer coefficients, then

$$\sum_{n \leqslant x} \Delta\big(|F(n)|\big) \ll_F x(\log_2 x)^{5/2} \qquad (x \geqslant 3). \tag{1·7}$$

This is likely to be useful in applications, in particular to Waring's type problems: see [2].

To conclude this introduction we note that the question of the quadratic mean-value of the Δ-function has been solved, up to a power of $\log_2 x$, in [12; Exercice 72]. We state the corresponding estimates in the following statement and, for the reader's convenience, provide the short proof in section 4.

Theorem 1.2 ([12]). *We have*

$$x \log x \ll \sum_{n \leqslant x} \Delta(n)^2 \ll x(\log x)(\log_2 x)^2 \qquad (x \geqslant 3). \tag{1·8}$$

A challenging open question is to determine the true exponent of $\log_2 x$ in this problem.

2. Proof of Theorem 1.1

2·1. The set E_T

As in [11], the first step consists in defining a set of integers with useful multiplicative constraints.

Given a parameter $T \geqslant 3$, put

$$\begin{aligned}
&\mathfrak{b} := \frac{1}{\log 4 - 1}, \quad f_T(y) := \delta \min\Big(\log(T \log 3y), \frac{(\log_2 3y - \mathfrak{b}\log T)^2}{\log T}\Big) \quad (\delta > 0,\, y \geqslant 1),\\
&E := \{n \geqslant 1 : \mu(n)^2 = 1\}, \quad n_y := \prod_{p|n,\, p<y} p \quad (n \geqslant 1,\, y \geqslant 1),\\
&P^+(n) := \max_{p|n} p \quad (n > 1),\ P^+(1) := 1,\\
&E_T := \big\{n \in E : \tau(n_y) \leqslant T(\log 3y)\mathrm{e}^{-f_T(y)}\ (y \geqslant 1)\big\}.
\end{aligned}$$

Here and throughout, μ denotes the Möbius function, the letter p denotes a prime number and δ stands for an absolute, sufficiently small constant. It is useful to bear in mind that $n \in E_T$ implies that $d \in E_T$ for all divisors d of n.

Given $x > y \geqslant 2$, let $\mathbb{P}_{y,x}$ denote the probability on E defined by

$$\mathbb{P}_{y,x}(\{n\}) := \frac{1}{n} \prod_{y \leqslant p < x} \Big(1 + \frac{1}{p}\Big)^{-1} \asymp \frac{\log y}{n \log x} \qquad \big(P^-(n) \geqslant y,\, P^+(n) < x\big),$$

and let $\mathbb{E}_{y,x}$ denote the corresponding expectation. Throughout we let $\sum^{y,x}$ indicate a summation over squarefree integers whose prime factors are restricted to the interval $[y, x[$. When $y = 2$, we simply write $\mathbb{P}_x$, $\mathbb{E}_x$, and $\sum^x$.

It is proved in [11; prop. 5.1] that, provided δ is suitably chosen, we have

$$\mathbb{P}_x(E \smallsetminus E_T) \ll \frac{1}{T} \qquad (x \geqslant 2,\, T \geqslant 3). \tag{2·1}$$

2·2. *The set* E_T^*

It is shown in [11] that one can assume $M_2(n)/\tau(n) \leqslant T \log T$ $(n \in E_T)$ without perturbing (2·1) and we note that this readily follows from [12; th. 46], stated in terms of the function

$$T(n,0) := \sum_{\substack{d,d'|n \\ |\log(d'/d)|\leqslant 1}} 1,$$

which satisfies $M_2(n) \leqslant T(n,0) \leqslant 4M_2(n)$ for all integers $n \geqslant 1$. From [12; thms. 46 & 47] we actually have

$$\frac{1}{T\sqrt{\log T}} \ll \mathbb{P}_x\big(M_2/\tau > T\big) \ll \frac{\log T}{T} \qquad (T \geqslant 3). \tag{2·2}$$

The next statement improves both bounds to $1/T$. The upper estimate will be employed as a property of the set

$$E_T^* := \big\{n \in E_T : M_2(n)/\tau(n) \leqslant T\big\}. \tag{2·3}$$

It is useful to bear in mind that, since $M_2(n)/\tau(n)$ is multiplicatively increasing,(1) we have $M_2(n_y)/\tau(n_y) \leqslant T$ for all $y \geqslant 1$ and all $n \in E_T^*$.

Proposition 2.1. *We have*

$$\mathbb{P}_x(E \smallsetminus E_T^*) \ll \frac{1}{T} \qquad (x \geqslant 2,\, T \geqslant 3). \tag{2·4}$$

Moreover

$$\mathbb{P}_x\big(M_2/\tau > T\big) \asymp \frac{1}{T}\cdot \tag{2·5}$$

Since only (2·4) is necessary for the upper bound of (1·3), we restrict below to the proof of this estimate and postpone the proof of (2·5) to section 3.

Proof of (2·4). By (2·1), it suffices to prove that $\mathbb{P}_x(E_T \smallsetminus E_T^*) \ll 1/T$. Put $\tau(n,\vartheta) := \sum_{d|n} d^{i\vartheta}$ $(n \geqslant 1, \vartheta \in \mathbb{R})$. In view of [12; (3.2)], Parseval's formula provides

$$M_2(n) = \frac{1}{2\pi}\int_{\mathbb{R}} \Big(\frac{\sin\vartheta/2}{\vartheta/2}\Big)^2 |\tau(n,\vartheta)|^2 \,\mathrm{d}\vartheta \qquad (n \geqslant 1), \tag{2·6}$$

from which we derive, by applying the Montgomery–Wirsing lemma on mean values of Dirichlet polynomials (see, e.g., [16; lemma III.4.10]) to intervals $k - 1/2 \leqslant \vartheta \leqslant k + 1/2$ $(k \in \mathbb{Z})$ and taking symmetry into account, that

$$\frac{M_2(n)}{\tau(n)} \ll \int_0^1 \frac{|\tau(n,\vartheta)|^2}{\tau(n)}\,\mathrm{d}\vartheta \qquad (n \geqslant 1). \tag{2·7}$$

1. This follows from (1·4). See also [12; th. 40] for the variant related to $T(n,0)$.

The trivial bound $|\tau(n,\vartheta)|\leqslant\tau(n)$ immediately yields that the contribution of the range $0\leqslant\vartheta\leqslant 1/\log x$ has bounded expectation with respect to $\mathbb{P}_x$. Fix a large constant K, and first assume that $T^K\leqslant\log x$. Let then $m_2(n,T)$ denote contribution of the range $1/\log x\leqslant\vartheta\leqslant 1/T^K$ to the above integral, and let $M_2(n,T)$ denote the complementary contribution. Thus, in view of (2·1), we have, for a suitable absolute constant $c>0$,

$$\mathbb{P}_x\big(M_2/\tau>T\big)\ll\frac{1}{T}+\alpha+\beta,$$

with

$$\alpha:=\mathbb{P}_x\Big(n\in E_T,\,m_2(n,T)>cT\Big),\qquad\beta:=\mathbb{P}_x\Big(n\in E_T,\,M_2(n,T)>cT\Big).$$

Under conditions $n\in E_T$ and $K\geqslant K(\delta)$, we have $\tau(n_{\exp(1/\vartheta)})\leqslant(T/\vartheta)^{1-\delta}$ whenever $\vartheta\leqslant 1/T^K$. Therefore

$$\begin{aligned}\mathbb{E}_x\big(m_2(n,T)\big)&\leqslant\int_0^{1/T^K}\mathbb{E}_x\bigg(\frac{T^{1-\delta}|\tau(n,\vartheta)|^2}{\tau(n)\tau(n_{\exp(1/\vartheta)})\vartheta^{1-\delta}}\bigg)\,\mathrm{d}\vartheta\\&\ll\frac{T^{1-\delta}}{\log x}\int_0^{1/T^K}\sum_{n\geqslant1}^{x}\frac{|\tau(n,\vartheta)|^2}{n\tau(n)\tau(n_{\exp(1/\vartheta)})\vartheta^{1-\delta}}\,\mathrm{d}\vartheta\ll T^{1-\delta}\int_0^{1/T^K}\frac{\mathrm{d}\vartheta}{\vartheta^{1-\delta}},\end{aligned}$$

where the inner sum, written as a product over primes, has been estimated by [12; lemma 30.1] or [16; lemma III.4.13]. For $K>1/\delta$, the above bound is $\ll 1$, hence $\alpha\ll 1/T$.

Consider next

$$V_2(x):=\mathbb{E}_x\big(M_2(n,T)^2\big)\ll\int_{1/T^K}^{1}\int_{1/T^K}^{1}F_x(\vartheta,\varphi)\,\mathrm{d}\vartheta\,\mathrm{d}\varphi,\tag{2·8}$$

with

$$F_x(\vartheta,\varphi):=\frac{1}{\log x}\sum_{n\in E_T}^{x}\frac{|\tau(n,\vartheta)|^2|\tau(n,\varphi)|^2}{n\tau(n)^2}.$$

We use distinct upper bounds for this quantity, according to the sizes of $\tau(n_{\exp(1/\vartheta})$ and $\tau(n_{\exp(1/\varphi)})$. By symmetry, we may assume $0<\vartheta\leqslant\varphi\leqslant1$. Recall that for $n\in E_T$ we have

$$\tau(n_{\exp(1/\psi)})\leqslant\frac{T\mathrm{e}^{-f_T(\exp(1/\psi))}}{\psi}\qquad\big(\psi\in\{\varphi,\vartheta\}\big).$$

For $j\geqslant 0$, we define

$$D_j(\varphi;T):=\bigg\{n\in E_T:\frac{1}{2^{j+1}}<\frac{\varphi\tau(n_{\exp(1/\varphi)})}{T\mathrm{e}^{-f_T(\exp(1/\varphi))}}\leqslant\frac{1}{2^j}\bigg\},$$

let $F_x^{(j)}(\vartheta,\varphi)$ denote the subsum of $F_x(\vartheta,\varphi)$ restricted to summands n from $D_j(\varphi;T)$ and let $V_2^{(j)}(x)$ denote the integral of $F_x^{(j)}(\vartheta,\varphi)$ over $[1/T^K,1]^2$. Writing $\psi_p:=\psi\log p$ $(\psi>0)$, we have

$$U_1:=\sum_{p\leqslant\exp(1/\varphi)}\frac{1}{p}\leqslant\log\Big(\frac{1}{\varphi}\Big)+O(1),\tag{2·9}$$

$$U_2:=\sum_{\exp(1/\varphi)<p\leqslant\exp(1/\vartheta)}\frac{1+\cos\varphi_p}{p}\leqslant\log\Big(\frac{\varphi}{\vartheta}\Big)+O(1),\tag{2·10}$$

$$\begin{aligned}U_3&:=\sum_{\exp(1/\vartheta)<p\leqslant x}\frac{(1+\cos\vartheta_p)(1+\cos\varphi_p)}{p}\\&\leqslant\log(1+\vartheta\log x)+\tfrac12\log\Big(\frac{\varphi\log x}{1+(\varphi-\vartheta)\log x}\Big)+O(1),\end{aligned}\tag{2·11}$$

where we used the formulae

$$\begin{aligned}&(1+\cos\vartheta_p)(1+\cos\varphi_p)\\&\qquad=1+\cos\vartheta_p+\cos\varphi_p+\tfrac12\cos(\varphi_p-\vartheta_p)+\tfrac12\cos(\varphi_p+\vartheta_p),\\&\sum_{p\leqslant y}\frac{\cos\psi_p}{p}=\log\Big(\frac{\log y}{1+\psi\log y}\Big)+O(1)\quad(y\geqslant2,\,0\leqslant\psi\leqslant1).\end{aligned}\tag{2·12}$$

Let us now estimate $V_2^{(j)}(x)$. We have

$$F_x^{(j)}(\vartheta,\varphi)\ll\frac{TG_x^{(j)}(\vartheta,\varphi)\mathrm{e}^{-f_T(\exp(1/\vartheta))/2-f_T(\exp(1/\varphi))/2}}{\sqrt{\vartheta\varphi}2^{j/2}\log x},$$

with

$$G_x^{(j)}(\vartheta,\varphi):=\sum_{n\in D_j(\varphi;T)}^{x}\frac{|\tau(n,\vartheta)|^2|\tau(n,\varphi)|^2}{n\tau(n)^2\sqrt{\tau(n_{\exp(1/\vartheta)})\tau(n_{\exp(1/\varphi)})}}.$$

Observe that $n\in D_j(\varphi;T)$ implies

$$\omega(n_{\exp(1/\varphi)})=r_j(\varphi):=\left\lfloor\frac{\log(T/2^j\varphi)-f_T(\exp(1/\varphi))}{\log 2}\right\rfloor.$$

Split $n=uvw$, where $u:=n_{\exp(1/\varphi)}$ and where the prime factors of v, w, belong respectively to $\big]\exp(1/\varphi),\exp(1/\vartheta)\big]$, and $\big]\exp(1/\vartheta),x]$. We have

$$\frac{|\tau(n,\vartheta)|^2|\tau(n,\varphi)|^2}{n\tau(n)^2\sqrt{\tau(n_{\exp(1/\vartheta)})\tau(n_{\exp(1/\varphi)})}}\leqslant\frac{\tau(u)|\tau(v,\varphi)|^2|\tau(w,\vartheta)|^2|\tau(w,\varphi)|^2}{uvw\sqrt{\tau(v)}\tau(w)^2}.$$

Now, by (2·9),

$$\begin{aligned}\sum_{\substack{u,v,w\\ \omega(u)=r_j(\varphi)}}\frac{\tau(u)|\tau(v,\varphi)|^2|\tau(w,\vartheta)|^2|\tau(w,\varphi)|^2}{uvw\sqrt{\tau(v)}\tau(w)^2}&\ll\frac{(2U_1)^{r_j(\varphi)}\mathrm{e}^{\sqrt{2}U_2+U_3}}{r_j(\varphi)!}\\&\ll\frac{\mathrm{e}^{2U_1+\sqrt{2}U_2+U_3}}{\sqrt{1+U_1}}\ll\frac{\mathrm{e}^{\sqrt{2}U_2+U_3}}{\varphi^2\sqrt{\log(3/\varphi)}}.\end{aligned}$$

Applying (2·10) and (2·11), we infer that

$$\begin{aligned}\frac{G_x^{(j)}(\vartheta,\varphi)}{\sqrt{\vartheta\varphi}\log x}&\ll\frac{(\varphi/\vartheta)^{\sqrt{2}}(1+\vartheta\log x)}{\vartheta^{1/2}\varphi^{5/2}\sqrt{\log(3/\varphi)}\log x}\sqrt{\frac{\varphi\log x}{1+(\varphi-\vartheta)\log x}}\\&\ll\frac{\vartheta+1/\log x}{\varphi^{2-\sqrt{2}}\vartheta^{\sqrt{2}+1/2}\sqrt{\varphi-\vartheta+1/\log x}\sqrt{\log(3/\varphi)}}.\end{aligned}$$

Ignoring the factor e^{-f_T} and splitting at $\varphi=2\vartheta$, we hence obtain that the integral of $\varphi\mapsto F_x^{(j)}(\vartheta,\varphi)$ over $\vartheta\leqslant\varphi\leqslant 1$ is

$$\ll\frac{T(\vartheta+1/\log x)^{3/2}\mathrm{e}^{-f_T(\exp(1/\vartheta))/2}}{2^{j/2}\vartheta^{5/2}\sqrt{\log(3/\vartheta)}}.$$

Taking into account that $\vartheta\geqslant 1/\log x$ and integrating over the range $1/T^K\leqslant\vartheta\leqslant 1$ yields, exploiting here the factor $\mathrm{e}^{-f_T(\exp(1/\vartheta))}$, that

$$V_2^{(j)}(x)\ll T/2^{j/2}.$$

Summing over $j\geqslant 0$, we obtain

$$V_2(x)\ll T.$$

Hence $\beta\ll V_2(x)/T^2\ll 1/T$, as required, provided $T^K\leqslant\log x$. However, if $T^K>\log x$, we have $\mathbb{P}_x(M_2/\tau>T)\ll 1/T+\beta$ with now $\beta:=\mathbb{P}_x\big(n\in E_T,\ M_2(n,(\log x)^{1/K})>cT\big)$, and so the desired conclusion persists in view of the above calculations. □

2·3. *Bounding moments inductively*

This is a simple reappraisal of the argument of [11] which we reproduce with some simplifications and some improvement arising from the definition of the set E_T^*.

Let $\{\vartheta_{j,T}\}_{j=0}^{\infty} \in (\mathbb{R}^+)^{\mathbb{N}}$ be a sequence such that $\vartheta_{0,T} = \vartheta_{1,T} = 1$ and define the subset $E_{q,T}$ comprising those integers $n \in E_T^*$ such that

$$(E_{q,T}) \qquad M_j(n) \leqslant \tau(n)\vartheta_{j,T} \qquad (1 \leqslant j \leqslant q),$$

with $E_{0,T} = E_{1,T} = E_T^*$. Note that, by (1·4), if $n \in E_{q,T}$, then any divisor of n also lies in $E_{q,T}$. By Proposition 2.1, provided $\vartheta_{2,T} \geqslant T$, we still have

$$\mathbb{P}_x\big(E \smallsetminus E_{2,T}\big) \ll 1/T \qquad (x \geqslant 2,\ T \geqslant 3).$$

Recall notation $\sum^x$ and put

$$S_q(x) := \mathbb{P}_x\big(E_{q-1,T}\big)\mathbb{E}_x\Big(\frac{M_q}{\tau}\Big|E_{q-1,T}\Big) = \prod_{p<x}\Big(\frac{1}{1+1/p}\Big)\sum_{n\in E_{q-1,T}}^{x} \frac{M_q(n)}{n\tau(n)}\cdot$$

Proposition 2.2. *Let C_0 be a large absolute constant. Assume $\vartheta_{1,T} \geqslant 1$, $\vartheta_{2,T} \geqslant C_0T$, $\vartheta_{3,T} \geqslant 2C_0^2T^2\log T$, and*

$$(2·13) \qquad \vartheta_{j,T} \geqslant j!(C_0T)^{j-1} \quad (j \geqslant 2), \qquad \sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\vartheta_{j,T}\vartheta_{q-j,T} \leqslant \frac{\vartheta_{q,T}}{C_0T\sqrt{\log T}} \quad (q \geqslant 3)\cdot$$

Then

$$(2·14) \qquad S_q(x) \leqslant \frac{C_0\vartheta_{q,T}}{q^2T} \qquad (q \geqslant 3,\ x \geqslant 2,\ T \geqslant 3).$$

Proof. Put $L_x := \prod_{p<x}(1+1/p)$. Writing $n = mp$ with $P(m) < p$, we get

$$L_xS_q(x) \leqslant 1 + \sum_{p<x}\sum_{m\in E_{q-1,T}}^{p} \frac{M_q(pm)}{pm\tau(pm)}\cdot$$

Using $\tau(pm) = 2\tau(m)$ and expanding $M_q(pm)$ by (1·4) yields

$$L_xS_q(x) \leqslant 1 + \sum_{p<x}\sum_{m\in E_{q-1,T}}^{p} \frac{M_q(m)+W_q(m,p)}{pm\tau(m)} = \sum_{p<x}\frac{L_pS_q(p)}{p} + V_q(x),$$

with

$$V_q(x) := 1 + \sum_{p<x}\sum_{m\in E_{q-1,T}}^{p} \frac{W_q(m,p)}{pm\tau(m)}\cdot$$

Iterating this as in [11], we get

$$L_xS_q(x) \leqslant V_q(x) + \sum_{\substack{2\leqslant P^+(n)<x\\ n\in E}} \frac{V_q(P^-(n))}{n} \ll V_q(x) + \sum_{p<x}\frac{V_q(p)}{p}\frac{\log x}{\log p},$$

hence

$$(2·15) \qquad S_q(x) \ll \frac{V_q(x)}{\log x} + \sum_{p<x}\frac{V_q(p)}{p\log p}\cdot$$

Indeed, at the first step we obtain

$$\sum_{p<x}\frac{L_pS_q(p)}{p}\leqslant\sum_{p_1<x}\frac{1}{p_1}\Big\{\sum_{p_2<p_1}\frac{L_{p_2}S_q(p_2)}{p_2}+V_q(p_1)\Big\},$$

then

$$\begin{aligned}\sum_{p<x}\frac{L_pS_q(p)}{p}&\leqslant\sum_{p_1<x}\frac{1}{p_1}\Big(V_q(p_1)+\sum_{p_2<p_1}\frac{1}{p_2}\Big\{\sum_{p_3<p_2}\frac{L_{p_3}S_q(p_3)}{p_3}+V_q(p_2)\Big\}\Big)\\&\leqslant\sum_{p_1<x}\frac{V_q(p_1)}{p_1}+\sum_{p_2<p_1<x}\frac{V_q(p_2)}{p_1p_2}+\sum_{p_3<p_2<p_1<x}\frac{L_{p_3}S_q(p_3)}{p_1p_2p_3},\end{aligned}$$

and so on.

Since $n_y=n_p$ when $P^+(n)<p<y\leqslant p^2$, we may write, still following [11],

$$\begin{aligned}V_q(x)&\ll 1+\sum_{p<x}\sum_{m\in E_{q-1,T}}^{p}\frac{W_q(m,p)}{pm\tau(m)}\int_p^{p^2}\frac{\mathrm{d}y}{y\log 3y}\\&=1+\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\int_1^{x^2}\sum_{m\in E_{q-1,T}}^{y}\frac{1}{m\tau(m)}\sum_{\sqrt{y}\leqslant p<y}\frac{N_{j,q}(m,p)}{p}\frac{\mathrm{d}y}{y\log 3y},\end{aligned}$$

where $N_{j,q}$ is as defined in (1·5). By the argument leading to [11; (6.9)], we get

$$V_q(x)\ll 1+\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\int_1^{x^2}\sum_{m\in E_{q-1,T}}^{y}\Big(1+2^j\frac{\log y}{y^{1/4}}\Big)\frac{M_{q-j}(m)M_j(m)}{m\tau(m)}\frac{\mathrm{d}y}{y(\log 3y)^2}.$$

Inverting summation and integration in (2·15) and appealing to the uniform bound

$$\sum_{\sqrt{y}\leqslant p<x}\frac{1}{p\log p}\ll\frac{1}{\log y},$$

we obtain

$$S_q(x)\ll 1+\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\int_1^{x^2}\sum_{m\in E_{q-1,T}}^{y}\Big(1+2^j\frac{\log y}{y^{1/4}}\Big)\frac{M_{q-j}(m)M_j(m)}{m\tau(m)}\frac{\mathrm{d}y}{y(\log 3y)^3}.$$

The contribution of the term involving $y^{-1/4}$ may be handled as in [11] using the trivial bound(2)

$$M_{q-j}(m)M_j(m)\leqslant\tau(m)^q\leqslant(T\log y)^{q-2}\tau(m)^2,$$

valid whenever $m\in E_T$ and $P^+(m)<y$. This provides an overall term at most $C^qq!T^{q-2}$ where C is a suitable absolute constant. By the first assumption (2·13) we may choose C_0 such that

$$C^qq!T^{q-2}\leqslant C^2(C/C_0)^{q-2}q\vartheta_{q-1,T}\ll\vartheta_{q-1,T}/2^q.$$

Therefore

$$S_q(x)\ll\frac{\vartheta_{q-1,T}}{2^q}+\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\int_1^{x^2}\frac{Z_{q,j}(y,T)}{y(\log 3y)^3}\,\mathrm{d}y,\tag{2·16}$$

with

$$Z_{q,j}(y,T):=\sum_{m\in E_{q-1,T}}^{y}\frac{M_{q-j}(m)M_j(m)}{m\tau(m)}.$$

2. Note that $M_j(m)\leqslant\Delta(m)^{j-1}M_1(m)\leqslant\tau(m)^j$ for $j\geqslant 1$, $m\geqslant 1$.

We aim at establishing (2·14) by induction on $q\geqslant 3$. By $(E_{q,T})$ and (2·3) we have, whenever m is counted in $Z_{q,j}(y,T)$,

$$M_j(m)\ll \vartheta_{j,T}T(\log 3y)\mathrm{e}^{-f_T(y)}\qquad (m\in E_{q-1,T},\ P^+(m)<y,\ 1\leqslant j\leqslant q-1,\ y\geqslant 1).$$

Therefore

$$Z_{q,j}(y,T)\ll \vartheta_{j,T}T(\log 3y)^2S_{q-j}(y)\mathrm{e}^{-f_T(y)}. \tag{2·17}$$

When $q=3$, we infer from $(E_{q,T})$ with $q=2$ and the definition of E_T^* that

$$\begin{aligned}Z_{3,1}(y,T)&=\sum_{m\in E_{2,T}}^{y}\frac{M_2(m)}{m}\leqslant\sum_{m\geqslant 1}^{y}\frac{\min\big(T\tau(m),T^{3/2}\sqrt{\tau(m)\log y}\,\big)}{m}\\&\ll\frac{T^{3/2}(\log y)^2}{\sqrt{T}+(\log y)^{(3-\sqrt{8})/2}}\cdot\end{aligned}\tag{2·18}$$

Hence (2·16) implies

$$S_3(x)\ll T\log T\ll\frac{\vartheta_{3,T}}{T}\cdot$$

This initiates the induction provided C_0 is sufficiently large.

If $q\geqslant 4$, then $q-j>2$ (except for $q=4$ and $j=2$), so we can use the induction hypothesis (2·14). For $q=4$ and $j=2$, we simply note that $Z_{4,2}(y,T)\leqslant Z_{4,1}(y,T)$ since $M_2(m)^2\leqslant M_3(m)\tau(m)$ for all $m\geqslant 1$. Appealing to (2·14) for bounding $S_{q-j}(y)$ for $1\leqslant j\leqslant q/2$, we derive from (2·16) and (2·17)

$$\begin{aligned}S_q(x)&\ll\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\frac{\vartheta_{q-j,T}\vartheta_{j,T}}{q^2T}\int_1^{x^2}\frac{T\mathrm{e}^{-f_T(y)}}{y(\log 3y)}\,\mathrm{d}y+\frac{\vartheta_{q-1,T}}{2^q}\\&\ll\sum_{1\leqslant j\leqslant q/2}\binom{q}{j}\frac{\vartheta_{q-j,T}\vartheta_{j,T}}{q^2}\sqrt{\log T}.\end{aligned}\tag{2·19}$$

Under assumption (2·13), this implies (2·14). □

2·4. Completion of the argument

Observe that the sequence defined by

$$\vartheta_{q,T}:=\frac{q!}{q^2}\Big(\frac{2\pi^2C_0}{3}\Big)^{q-1}T^{q-1}(\log T)^{(q-1)/2}\quad(q\geqslant 1)$$

satisfies (2·13).

Now, we put $\lambda:=C_1T(\log T)^{1/2}$ and aim at establishing

$$\mathbb{P}_x\big(\Delta(n)>\lambda\log_2 x\big)\ll\frac{1}{T}\cdot\tag{2·20}$$

By (2·4), it is sufficient to show that

$$\mathbb{P}_x\Big(n\in E_T^*,\ \Delta(n)>\lambda\log_2 x\Big)\ll\frac{1}{T}\cdot$$

However, from (2·14), for $q\geqslant 3$,

$$\mathbb{P}_x\big(E_{q-1,T}\smallsetminus E_{q,T}\big)\leqslant\mathbb{P}_x\big(E_{q-1,T}\big)\mathbb{E}_x\Big(\frac{M_q(n)}{\vartheta_{q,T}\tau(n)}\Big|E_{q-1,T}\Big)=\frac{S_q(x)}{\vartheta_{q-1,T}}\ll\frac{1}{q^2T},$$

whence

$$\mathbb{P}_x(E_T\smallsetminus\cap_{q\geqslant 3}E_{q,T})\ll\frac{1}{T}\cdot$$

Now observe that if $n \in \cap_{q\geqslant 3} E_{q,T}$ and $P^+(n) \leqslant x$, we have

$$\Delta(n) \leqslant 2M_q(n)^{1/q} \ll \tau(n)^{1/q}\vartheta_{q,T}^{1/q} \ll (T\log x)^{1/q} qT^{1-1/q}(\log T)^{1/2},$$

where the first inequality is [12; (5.56)]. Selecting $q = \lfloor \log_2 x\rfloor$ we have $\Delta(n) \leqslant \lambda \log_2 x$ for a suitable large constant C_1 and hence (2·20).

It remains to estimate $\mathbb{E}_x(\Delta)$. We may assume trivially that $10C_1 \log_2 x \leqslant \Delta(n) \leqslant (\log x)^3$. Indeed,

$$\mathbb{E}_x(\Delta, \Delta > (\log x)^3) \leqslant \frac{\mathbb{E}_x(\tau^2)}{(\log x)^3} \ll 1.$$

The contribution of those integers n such that $2^j < \Delta(n)/\log_2 x \leqslant 2^{j+1}$ is plainly

$$\ll \frac{2^j j^{1/2}\log_2 x}{2^j} = j^{1/2}\log_2 x.$$

Summing over $j \ll \log_2 x$, we finally get

$$\mathbb{E}_x(\Delta) \ll (\log_2 x)^{5/2}.$$

Since, by [12; th. 61], we have $S(x) \ll x\mathbb{E}_x(\Delta)$, we obtain the upper bound of (1·3) as required.

3. Proof of estimate (2·5)

For integer k, let χ_k denote the indicator of the set $\big\{n \in E : \omega(n) = k\big\}$. For notational convenience, we shall also write $\xi := \sqrt{\log_2 x}$ in the sequel. Recall the definition of the constant $\mathfrak{b}$ at the beginning of § 2.1

Lemma 3.1. *Let $0 < \varepsilon < \frac{1}{2}$ be fixed. For integer h, $(\log x)^\varepsilon \leqslant 2^h \leqslant (\log x)^{1-\varepsilon}$, $y := \exp 2^h$, $\varepsilon \log_2 x \leqslant k \leqslant (\log_2 x)/\varepsilon$, we have*

$$\mathbb{E}_y\big(\chi_k\big) \asymp \frac{(h\log 2)^k}{2^h k!}\cdot \tag{3·1}$$

In particular, for $t \in h/\mathfrak{b} + [-\xi, \xi]$, we have

$$\mathbb{E}_y(\chi_{h+t}) \asymp \frac{1}{2^t\xi}\cdot \tag{3·2}$$

Proof. This readily follows from, e.g., [7; lemma III.13], which provides the estimate

$$\sum_{p_1<\cdots<p_k\leqslant y} \frac{1}{p_1\cdots p_k} \asymp \frac{(h\log 2)^k}{k!},$$

and Stirling's formula. □

We may now embark on proving (2·5). In view of (2·1) and (2·4), only the lower bound is to be established. Given $T \geqslant 3$ and a large integer m, put

$$\xi_T := \sqrt{\log T}, \quad \mathcal{H}_T := \Big(\frac{\mathfrak{b}\xi_T^2}{\log 2} + [0, \xi_T]\Big) \cap m\mathbb{N}, \quad \mathcal{Y}_T := \{y_h := \exp 2^h : h \in \mathcal{H}_T\}.$$

Let $K \geqslant 1$ to be chosen later and put $\mathcal{E}_K := \{n \in E : \max_{y\in\mathcal{Y}_T} n_y/y^K \leqslant 1\}$. We have

$$\frac{M_2(n)}{\tau(n)} \geqslant \frac{M_2(n_y)}{\tau(n_y)} > \frac{\tau(n_y)}{2K\log y} \quad (n \in \mathcal{E}_K,\, y \in \mathcal{Y}_T).$$

Let us then put

$$t := \frac{\log 2T}{\log 2}, \quad \mathfrak{q}_T(n) := \max_{y\in\mathcal{Y}_T} \chi_{h+t}(n_y),$$

so that

$$\frac{M_2(n)}{\tau(n)} > \frac{T\mathfrak{q}_T(n)}{2K} \quad (n \in \mathcal{E}_K).$$

Consider $\mathfrak{d}_T := \mathbb{E}_x\big(\mathfrak{q}_T\big)$. In view of Lemma 3.1, we first have

$$\mathbb{E}_x\big(\chi_{h+t}(n_y)\big) \asymp \frac{1}{T\xi_T},$$

hence

$$\mathfrak{d}_T \leqslant \sum_{h\in\mathcal{H}_T} \mathbb{E}_x\big(\chi_{h+t}\big) \ll \frac{1}{T}\cdot$$

Now, the inclusion-exclusion principle implies the lower bound

$$\mathfrak{d}_T \geqslant \sum_{h\in\mathcal{H}_T} \mathbb{E}_x\big(\chi_{h+t}\big) - \sum_{\substack{h,j\in\mathcal{H}_T\\ h<j}} \mathbb{E}_x\big(\chi_{h+t}\chi_{j+t}\big).$$

When $j = h+\ell$, $\ell \geqslant m$, we have

$$\mathbb{E}_x\big(\chi_{h+t}\chi_{j+t}\big) \asymp \frac{(\ell\log 2)^\ell}{T\xi_T 2^\ell\,\ell!} \ll \frac{\mathrm{e}^{-\beta\ell}}{T\xi_T},$$

with $\beta := \log(2/\mathrm{e}\log 2) \approx 0.05966$. It follows that, for suitable absolute constants c_1, $c_2 > 0$,

$$\mathfrak{d}_T \geqslant \frac{c_1}{mT} - \frac{c_2}{T}\sum_{\substack{\ell\geqslant m\\ \ell\equiv 0\,(\mathrm{mod}\,m)}} \mathrm{e}^{-\beta\ell} \geqslant \frac{c_1/m - c_2/(\mathrm{e}^{\beta m}-1)}{T}\cdot$$

Selecting m sufficiently large, we get that $\mathfrak{d}_T \gg 1/T$.

To conclude the proof it only remains to observe that

$$\begin{aligned}\mathbb{E}_x\Big(\mathbf{1}_{E\smallsetminus\mathcal{E}_K}\mathfrak{q}_T\Big) &\ll \frac{1}{\mathrm{e}^K}\sum_{h\in\mathcal{H}_T}\mathbb{E}_x\Big(\chi_{h+t}(n_y)n_y^{1/\log y}\Big)\\ &\ll \frac{1}{\mathrm{e}^K}\sum_{h\in\mathcal{H}_T}\frac{1}{2^h(h+t)!}\Big(\sum_{p\leqslant y}\frac{1}{p^{1-1/\log y}}\Big)^{h+t} \ll \frac{1}{\mathrm{e}^K T},\end{aligned}$$

and get, with a suitable choice of the constant K,

$$\mathbb{P}_x\Big(M_2(n)/\tau(n) > T/2K\Big) \gg 1/T.$$

Up to substituting $2KT$ to T, this implies the lower bound in (2·5).

4. Proof of Theorem 1.2

The lower bound readily follows from the pigeon-hole principle in the form

$$\Delta(n) \gg \tau(n)/\log 2n \quad (n \geqslant 1).$$

In order to establish the upper bound, we first note that by a standard argument (see, e.g., [14; (22)], [12; th. 61]), it is sufficient to show that

$$\mathbb{E}_x(\Delta^2) \ll (\log_2 x)^2\log x. \tag{4·1}$$

Now, we observe that, for all integers $r \geqslant 1$, $s \geqslant 1$ and suitable $u \in \mathbb{R}$, we have

$$\begin{aligned}\Delta(rs) &\leqslant \sum_{\substack{d|r,\,t|s\\ -1/2\leqslant\log dt-u\leqslant 1/2}} 1 \leqslant 2\sum_{d|r}\sum_{t|s}(1-|\log dt-u|)^+\\ &= \frac{1}{\pi}\int_{\mathbb{R}}\Big(\frac{\sin\vartheta/2}{\vartheta/2}\Big)^2\tau(r,\vartheta)\tau(s,\vartheta)\mathrm{e}^{-iu\vartheta}\,\mathrm{d}\vartheta.\end{aligned}$$

By (2·6) and the Cauchy-Schwarz inequality, we get as in [12; Exercise 47]

$$\Delta(rs)^2 \leqslant 4M_2(r)M_2(s) \qquad (r \geqslant 1,\ s \geqslant 1).$$

Therefore

$$\mu(n)^2\Delta(n)^2 \leqslant 4\sum_{rs=n}\frac{\mu(r)^2M_2(r)}{\tau(r)}\frac{\mu(s)^2M_2(s)}{\tau(s)} \qquad (n\geqslant 1).$$

It is then sufficient to apply (2·7) and [12; th. 42], which readily implies that $M_2(n)\mu(n)^2/\tau(n)$ has average order $\ll \log_2 n$, to get (4·1), as required.

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References

[1] R. de la Bretèche & G. Tenenbaum, Two upper bounds for the Erdős–Hooley Delta-function, *Sci. China Math.* **66**, no. 12 (2023), 2683–2692.

[2] R. de la Bretèche & G. Tenenbaum, Mean values of arithmetic functions and application to sums of powers, *Math. Proc. Camb. Phil. Soc.*, to appear.

[3] P. Erdős, Problem 218, Solution by the proposer, *Can. Math. Bull.* **16** (1973), 621-622.

[4] P. Erdős & J.-L. Nicolas, Répartition des nombres superabondants, *Bull. Soc. math. France* **103** (1975), 65-90.

[5] K. Ford, B. Green, D. Koukoulopoulos, Equal sums in random sets and the concentration of divisors, *Inventiones Math.* **232** (2023), 1027–1160.

[6] K. Ford, D. Koukoulopoulos, T. Tao, A lower bound on the mean value of the Erdős–Hooley Delta function, *Proc. London Math. Soc.* (3) **129,** no. 1 (2024), e12618.

[7] H. Halberstam & K.F. Roth, *Sequences*, Oxford, 1966, 2nd. ed. Springer, 1983.

[8] R.R. Hall & G. Tenenbaum, On the average and normal orders of Hooley's Δ-function, *J. London Math. Soc.* (2) **25** (1982), 392-406.

[9] R.R. Hall & G. Tenenbaum, The average orders of Hooley's Δ_r-functions, *Mathematika* **31** (1984), 98–109.

[10] R.R. Hall & G. Tenenbaum, The average orders of Hooley's Δ_r-functions, II, *Compositio Math.* **60** (1986), 163–186.

[11] D. Koukoulopoulos & T. Tao, An upper bound on the mean value of the Erdős–Hooley Delta function, *Proc. London Math. Soc. (3)* **127**, no. 6 (2023), 1865–1885.

[12] R.R. Hall & G. Tenenbaum, *Divisors*, Cambridge tracts in mathematics, no 90, Cambridge University Press (1988).

[13] C. Hooley, On a new technique and its applications to the theory of numbers, *Proc. London Math. Soc.* (3) **38** (1979), 115–151.

[14] G. Tenenbaum, Sur la concentration moyenne des diviseurs, *Comment. Math. Helvetici* **60** (1985), 411-428.

[15] G. Tenenbaum, Fonctions Δ de Hooley et applications, Séminaire de théorie des nombres, Paris 1984-85, Prog. Math. 63 (1986), 225-239.

[16] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, 3rd ed., Graduate Studies in Mathematics 163, Amer. Math. Soc. (2015).

[17] G. Tenenbaum, Fonctions multiplicatives, sommes d'exponentielles, et loi des grands nombres, *Indag. Math.* **27** (2016), 590-600.

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