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AN INGHAM–MÜNTZ TYPE THEOREM AND SIMULTANEOUS OBSERVATION PROBLEMS

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Abstract. We establish a theorem combining the estimates of Ingham and Müntz–Szász. Moreover, we allow complex exponents instead of purely imaginary exponents for the Ingham type part or purely real exponents for the Müntz–Szász part. A very special case of this theorem allows us to prove the simultaneous observability of some string–heat and beam–heat systems.

1. Introduction.

Non-harmonic Fourier series proved to be very useful in control theory of partial differential equations [8], [9], [21]. Although less general than the Hilbert Uniqueness Method (HUM) of J.-L. Lions [17], [18], [13] or the method based on microlocal analysis [4], in many cases the other methods fail.

In the case of reversible linear evolutionary systems these methods are often based on various generalizations of a classical theorem of Ingham [11], itself a generalization of Parseval's equality, see, e.g., [10], [14], [16] and their references. See also [7] for a generalization allowing for complex exponents.

For parabolic systems an equally powerful method is based on the Müntz–Szász generalization [19], [24], [6] of the Weierstrass approximation theorem, see, e.g., [22].

In this paper we establish a theorem combining the estimates of Ingham and Müntz–Szász. Moreover, we allow complex exponents instead of purely imaginary exponents for the Ingham type part or purely real exponents for the Müntz–Szász part.

In formulating our theorem we use henceforth Vinogradov's notation: $f(t) \ll g(t)$ $(t \in E)$ means that the real or complex quantities $f(t)$ and $g(t)$ satisfy

$$|f(t)| \leqslant C_E |g(t)|$$

for all $t \in E$ where C_E is a constant depending at most on the set E and possibly on various parameters to be specified.

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Theorem 1.1. *Consider four real sequences*

$$\Lambda := (\lambda_n)_{n\in\mathbb{Z}}, \quad \mathcal{E} := (\varepsilon_n)_{n\in\mathbb{Z}}, \quad \mathcal{M} := (\mu_k)_{k\in\mathbb{N}}, \quad \mathcal{H} := (\eta_k)_{k\in\mathbb{N}}$$

and corresponding complex sequences $(z_n)_{n\in\mathbb{Z}}$, $(w_k)_{k\in\mathbb{N}}$, *defined by the formulae*

$$z_n := \lambda_n + i\varepsilon_n \quad (n \in \mathbb{Z}), \qquad w_k := \mu_k + i\eta_k \quad (k \in \mathbb{N}).$$

Let $\gamma > 0$, *and assume that the the following conditions hold for some* $\alpha > 1$ *and* $p \in \mathbb{N}^*$:

$$\gamma_1 := \inf_{n\in\mathbb{Z}}\{\lambda_{n+1} - \lambda_n\} > 0, \quad \inf_{n\in\mathbb{Z}}\left\{\frac{\lambda_{n+p} - \lambda_n}{p}\right\} \geqslant \gamma, \tag{1.1}$$

$$\inf_{k\in\mathbb{N}}\{\mu_{k+1} - \mu_k\} > 0 \tag{1.2}$$

$$\mu_k > 0 \quad (k \geqslant 0), \qquad \sum_{|\mu_k-\mu|\leqslant t} 1 \ll t^{1/\alpha} \quad (\mu > 0,\ t \geqslant 1), \tag{1.3}$$

$$\varepsilon_n \ll 1 \quad (n \in \mathbb{Z}), \qquad \eta_k \ll 1 \quad (k \in \mathbb{N}), \tag{1.4}$$

$$\inf_{n\in\mathbb{Z},\ k\in\mathbb{N}} |iz_n \pm w_k| > 0. \tag{1.5}$$

Then the following estimate holds for all $T > 2\pi/\gamma$ *and all square summable sequences* $(a_n)_{n\in\mathbb{Z}}$ *and* $(b_k)_{k\in\mathbb{N}}$:

$$\int_0^T \Big|\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{iz_n t} + \sum_{k\in\mathbb{N}} b_k \mathrm{e}^{-w_k t}\Big|^2 \,\mathrm{d}t \gg \sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{k\in\mathbb{N}} |b_k|^2 \mathrm{e}^{-\mu_k T}. \tag{1.6}$$

Here, the implied constant depends at most on α, γ_1, γ, p, T, *and on the implicit constants in the assumptions.*

In the second part of this paper we apply Theorem 1.1 to some observability problems

Simultaneous observability of string–string, string–beam and beam–beam systems have been investigated in [2], [3], and [23] by applying some weakened Ingham type theorems. A very special case of Theorem 1.1 allows us to prove the simultaneous observability of some string–heat and beam–heat systems. We note that for a different kind of wave-heat systems observability estimates have been obtained by different approaches in [1], [20], [26], [27].

Let us consider a vibrating string of length ℓ_1 and a heated rod of length ℓ_2, both with homogeneous Dirichlet boundary conditions. We assume that they have a common endpoint, where we may observe only the cumulative action of them during some time T. A natural question is whether this observation allows us to determine the unknown initial data for both equations.

We may model this problem in the following way. For some given real number κ we consider the following two independent problems:

$$\begin{cases} u_{tt} + 2\kappa u_t - u_{xx} = 0 \quad \text{in} \quad (0, \ell_1) \times (0, \infty), \\ u(0, t) = u(\ell_1, t) = 0 \quad \text{for} \quad t \in [0, \infty), \\ u(x, 0) = \varrho_0(x) \quad \text{and} \quad u_t(x, 0) = \varrho_1(x) \quad \text{for} \quad x \in (0, \ell_1), \end{cases} \tag{1.7}$$

and

$$\begin{cases} v_t - v_{xx} = 0 \quad \text{in} \quad (0, \ell_2) \times (0, \infty), \\ v(0, t) = v(\ell_2, t) = 0 \quad \text{for} \quad t \in [0, \infty), \\ v(x, 0) = \sigma_0(x) \quad \text{for} \quad x \in (0, \ell_2). \end{cases} \tag{1.8}$$

It is well-known that for any given initial data

$$\varrho_0 \in H_0^1(0,\ell_1), \quad \varrho_1 \in L^2(0,\ell_1) \quad \text{and} \quad \sigma_0 \in L^2(0,\ell_2),$$

problem (1.7) has a unique solution satisfying

$$u \in \mathcal{C}([0,\infty); H_0^1(0,\ell_1)) \cap \mathcal{C}^1([0,\infty); L^2(0,\ell_1))$$

and problem (1.8) has a unique solution satisfying

$$v \in \mathcal{C}([0,\infty); L^2(0,\ell_2)).$$

Furthermore, the Fourier series representation of the solutions shows that for any fixed $T > 0$ the linear maps

$$(\varrho_0,\varrho_1) \mapsto u_x(0,\cdot)|_{(0,T)} \quad \text{and} \quad \sigma_0 \mapsto v_x(0,\cdot)|_{(0,T)}$$

are well defined and continuous from $H_0^1(0,\ell_1) \times L^2(0,\ell_1)$ to $L^2(0,T)$ and from $L^2(0,\ell_2)$ to $L^2(0,T)$, respectively.

We ask whether the linear map

$$(\varrho_0,\varrho_1,\sigma_0) \mapsto (u_x + v_x)(0,\cdot)|_{(0,T)} \tag{1.9}$$

is one-to-one on $H_0^1(0,\ell_1) \times L^2(0,\ell_1) \times L^2(0,\ell_2)$.

Since there is a finite propagation speed for the wave equation, this cannot hold unless T is sufficiently large, more precisely unless $T \geqslant 2\ell_1$; see, e.g., [13, Remark 3.6] for a simple proof even in higher dimension.

In order to formulate our result we expand the initial data into Fourier series:

$$\varrho_0(x) := \sum_{n\geqslant 1} a_n \sin(n\pi x/\ell_1), \quad \varrho_1(x) := \sum_{n\geqslant 1} b_n \sin(n\pi x/\ell_1),$$
$$\sigma_0(x) := \sum_{n\geqslant 1} c_n \sin(n\pi x/\ell_2).$$

Proposition 1.2. *If $|\kappa| < \pi/\ell_1$ and $T > 2\ell_1$, then the linear map (1.9) is one-to-one.*

More precisely, there exists a positive constant c_T such that the solutions of (1.7) and (1.8) satisfy the following estimate:

$$\int_0^T |u_x(0,t) + v_x(0,t)|^2 \, \mathrm{d}t \geqslant c_T \sum_{n\geqslant 1} \left(n^2 |a_n|^2 + |b_n|^2 + \mathrm{e}^{-n^2\pi^2 T/\ell_2^2} n^2 |c_n|^2 \right).$$

Next we investigate the observability problem when the string is replaced by a hinged beam, modelled by the following system:

$$\begin{cases} u_{tt} + 2\kappa u_t + u_{xxxx} = 0 \quad \text{in} \quad (0,\ell_1)\times(0,\infty), \\ u(0,t) = u_{xx}(0,t) = 0 \quad \text{for} \quad t\in[0,\infty), \\ u(\ell_1,t) = u_{xx}(\ell_1,t) = 0 \quad \text{for} \quad t\in[0,\infty), \\ u(x,0) = \varrho_0(x) \quad \text{and} \quad u_t(x,0) = \varrho_1(x) \quad \text{for} \quad x\in(0,\ell_1) \end{cases} \tag{1.10}$$

We recall that for any given initial data $\varrho_0 \in H_0^1(0,\ell_1)$ and $\varrho_1 \in H^{-1}(0,\ell_1)$ the system (1.10) has a unique solution satisfying

$$u \in \mathcal{C}([0,\infty); H_0^1(0,\ell_1)) \cap \mathcal{C}^1([0,\infty); H^{-1}(0,\ell_1))$$

Furthermore, for any fixed $T > 0$ the linear map

$$(\varrho_0,\varrho_1) \mapsto u_x(0,\cdot)|_{(0,T)}$$

is well defined and continuous from $H_0^1(0,\ell_1) \times H^{-1}(0,\ell_1)$ to $L^2(0,T)$.

We ask whether the linear map

$$(\varrho_0, \varrho_1, \sigma_0) \mapsto (u_x + v_x)(0, \cdot)|_{(0,T)} \tag{1.11}$$

is one-to-one on $H_0^1(0, \ell_1) \times H^{-1}(0, \ell_1) \times L^2(0, \ell_2)$. Since the propagation speed is infinite for both our beam and heat conduction model, we may expect observability for arbitrarily small $T > 0$. Indeed, we obtain the following result.

Proposition 1.3. *If $|\kappa| < \pi/\ell_1$, then the linear map (1.11) is one-to-one for any fixed $T > 0$.*

More precisely, there exists a positive constant c_T such that the solutions of (1.10) and (1.8) satisfy the following estimate:

$$\int_0^T |u_x(0,t) + v_x(0,t)|^2 \, \mathrm{d}t \geqslant c_T \sum_{n \geqslant 1} \left(n^2 |a_n|^2 + n^{-2} |b_n|^2 + \mathrm{e}^{-n^2\pi^2 T/\ell_2^2} n^2 |c_n|^2 \right).$$

Our next applications illustrate the flexibility provided by Theorem 1.1 regarding the complex sequences of the frequencies. We fix two real or complex numbers α, β and we consider the following coupled wave–heat system on some bounded interval $(0, \ell)$:

$$\begin{cases} u_{tt} - u_{xx} + \alpha v = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty), \\ v_t - v_{xx} + \beta u = 0 \quad \text{in} \quad (0,\ell) \times (0,\infty), \\ u(0,t) = u(\ell,t) = v(0,t) = v(\ell,t) = 0 \quad \text{for} \quad t \in [0,\infty), \\ u(x,0) = \varrho_0(x), \quad u_t(x,0) = \varrho_1(x) \quad \text{for} \quad x \in (0,\ell), \\ v(x,0) = \sigma_0(x) \quad \text{for} \quad x \in (0,\ell). \end{cases} \tag{1.12}$$

Since the parameters α, β represent a bounded perturbation of the uncoupled system, the problem is well posed. More precisely, given any initial data

$$(\varrho_0, \varrho_1, \sigma_0) \in H_0^1(0,\ell) \times L^2(0,\ell) \times L^2(0,\ell),$$

the system has a unique solution satisfying

$$u \in \mathcal{C}([0,\infty); H_0^1(0,\ell)) \cap \mathcal{C}^1([0,\infty); L^2(0,\ell))$$

and

$$v \in \mathcal{C}([0,\infty); L^2(0,\ell)).$$

Given $T > 0$, we may then ask whether the linear maps

$$(\varrho_0, \varrho_1, \sigma_0) \mapsto u_x(0, \cdot)|_{(0,T)} \tag{1.13}$$

and

$$(\varrho_0, \varrho_1, \sigma_0) \mapsto v_x(0, \cdot)|_{(0,T)} \tag{1.14}$$

are one-to-one.

Since we only observe one of the two unknown functions, these properties cannot hold in the uncoupling case $\alpha = \beta = 0$.

We shall prove the following results, where we use the Fourier coefficients of the initial data defined by changing ℓ_1 and ℓ_2 to ℓ in the above formulae.

Proposition 1.4. *Consider the solutions of the system (1.12) and assume that we have $0 < |\alpha\beta| \leqslant \pi^3/(8\ell^3)$. Then, the linear maps (1.13) and (1.14) are one-to-one for any fixed $T > 2\ell$.*

More precisely, there exists a positive constant $c_T = c_T(\alpha, \beta)$ such that the solutions of (1.12) satisfy the estimates

$$\int_0^T |u_x(0,t)|^2 \, \mathrm{d}t \geqslant c_T \sum_{n\in\mathbb{N}} n^{-6} e^{-\pi^2 n^2 T/\ell^2} \left(n^2 |a_n|^2 + |b_n|^2 + |c_n|^2\right)$$

and

$$\int_0^T |v_x(0,t)|^2 \, \mathrm{d}t \geqslant c_T \sum_{n\in\mathbb{N}} n^{-4} e^{-\pi^2 n^2 T/\ell^2} \left(n^2 |a_n|^2 + |b_n|^2 + |c_n|^2\right).$$

Given an interior point $x_0 \in (0, \ell)$, we may also ask whether the linear maps

$$(\varrho_0, \varrho_1, \sigma_0) \mapsto u(x_0, \cdot)|_{(0,T)} \quad \text{and} \quad (\varrho_0, \varrho_1, \sigma_0) \mapsto v(x_0, \cdot)|_{(0,T)}$$

are one-to-one.

These problems may be solved by a simple adaptation of the proof of Proposition 1.4, combined with some Diophantine approximation results as, e.g., in [2], [3] or [14]. We leave the details for the interested reader.

The same questions may be asked for the following coupled beam-heat system:

$$\begin{cases} u_{tt} + u_{xxxx} + \alpha v = 0 & \text{in} \quad (0,\ell) \times (0,\infty), \\ v_t - v_{xx} + \beta u = 0 & \text{in} \quad (0,\ell) \times (0,\infty), \\ u(0,t) = u(\ell,t) = u_{xx}(0,t) = u_{xx}(\ell,t) = 0 & \text{for} \quad t \in [0,\infty), \\ v(0,t) = v(\ell,t) = 0 & \text{for} \quad t \in [0,\infty), \\ u(x,0) = \varrho_0(x), \quad u_t(x,0) = \varrho_1(x) & \text{for} \quad x \in (0,\ell), \\ v(x,0) = \sigma_0(x) & \text{for} \quad x \in (0,\ell). \end{cases} \tag{1.15}$$

Since the parameters α, β represent a bounded perturbation of the uncoupled system, for any given initial data

$$(\varrho_0, \varrho_1, \sigma_0) \in H_0^1(0,\ell) \times H^{-1}(0,\ell) \times L^2(0,\ell),$$

the system has a unique solution satisfying

$$u \in \mathcal{C}([0,\infty); H_0^1(0,\ell)) \cap \mathcal{C}^1([0,\infty); H^{-1}(0,\ell))$$

and $v \in \mathcal{C}([0,\infty); L^2(0,\ell))$.

Proposition 1.5. *Consider the solutions of the system (1.15) and assume that we have $0 < |\alpha\beta| \leqslant \pi^6/\{6\ell^5(\pi + \ell)\}$. Then the linear maps (1.13) and (1.14) are one-to-one for any fixed $T > 0$.*

More precisely, there exists a positive constant $c_T = c_T(\alpha, \beta)$ such that the solutions of (1.15) satisfy the estimates

$$\int_0^T |u_x(0,t)|^2 \, \mathrm{d}t \geqslant c_T \sum_{n\in\mathbb{N}} n^{-6} \mathrm{e}^{-\pi^2 n^2 T/\ell^2} \left(n^4 |a_n|^2 + |b_n|^2 + |c_n|^2\right)$$

and

$$\int_0^T |v_x(0,t)|^2 \, \mathrm{d}t \geqslant c_T \sum_{n\in\mathbb{N}} n^{-6} \mathrm{e}^{-\pi^2 n^2 T/\ell^2} \left(n^4 |a_n|^2 + |b_n|^2 + |c_n|^2\right).$$

The next two sections are devoted to the proof of Theorem 1.1. The remainder of the paper is devoted to the proof of Propositions 1.2–1.5.

2. **A lemma from complex analysis.** The following result is a variant (and actually an extension) of Proposition 2 in [12]. We provide a very simple proof, analogous to that of Lemma 3.3 in [25].

We systematically write a complex number as $z = x + iy$ and let

$$\widehat{h}(z) := \int_{\mathbb{R}} h(t)\mathrm{e}^{-itz}\,\mathrm{d}t$$

denote the Fourier transform of a function h, extended to suitable complex values of the variable z.

Lemma 2.1. *Let $0 < \beta < 1$ and $\varepsilon > 0$. There exists a function $h \in \mathcal{C}^\infty(\mathbb{R})$ such that $\operatorname{supp} h \subset [-\varepsilon, \varepsilon]$, $\widehat{h}(0) = 1$, and*

$$\widehat{h}(z) \ll \mathrm{e}^{-\varepsilon|z|^\beta + \varepsilon|y|} \quad (z \in \mathbb{C}), \quad \widehat{h}(z) \gg \mathrm{e}^{\varepsilon|y|/2} \quad (z \in \mathbb{C},\ |x| \ll 1). \tag{2.1}$$

Here the implicit constants depend at most upon β and ε.

Proof. Let $p \in \mathbb{N}^*$. Put

$$H(t) := \begin{cases} \exp\Big\{ -\Big(\dfrac{1}{1-t^2}\Big)^p \Big\} & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geqslant 1, \end{cases} \qquad L := \Big(\int_{-1}^1 H(t)\,\mathrm{d}t \Big)^{-1}.$$

We shall see that, for a sufficiently large p, the function $t \mapsto h(t) := (L/\varepsilon)H(t/\varepsilon)$ meets our requirements. We have

$$\widehat{h}(z) = L \int_{-1}^1 H(t)\mathrm{e}^{-i\varepsilon tz}\,\mathrm{d}t,$$

and so, for any integer $j \geqslant 0$ and all $z = x + iy \in \mathbb{C}$,

$$|\widehat{h}(z)| \leqslant \frac{2L\|H^{(j)}\|_\infty\,\mathrm{e}^{\varepsilon|y|}}{(\varepsilon|z|)^j}. \tag{2.2}$$

In order to estimate $\|H^{(j)}\|_\infty$, we consider some $t \in [0, 1]$ and put $\varrho := 1 - t$. For $\delta \in]0, \frac{1}{2}]$, $w := t + \delta\varrho\mathrm{e}^{i\vartheta}$, $-\pi < \vartheta \leqslant \pi$, we have

$$\begin{aligned} \frac{2}{1-w^2} = \frac{1}{1-w} + \frac{1}{1+w} &= \frac{1}{\varrho(1-\delta\mathrm{e}^{i\vartheta})} + \frac{1}{2-\varrho+\delta\varrho\mathrm{e}^{i\vartheta}} \\ &= \frac{1+O(\delta)}{\varrho} + \frac{1+O(\delta)}{2-\varrho} = \frac{2\{1+O(\delta)\}}{\varrho(2-\varrho)}. \end{aligned}$$

This implies, for each fixed p,

$$\Re\Big\{\Big(\frac{1}{1-w^2}\Big)^p\Big\} = \frac{1+O(\delta)}{\varrho^p(2-\varrho)^p} \geqslant \frac{1}{(3\varrho)^p},$$

up to selecting $\delta = \delta_p$ sufficiently small. Cauchy's formula then yields

$$|H^{(j)}(t)| \leqslant \frac{j!\,\mathrm{e}^{-1/(3\varrho)^p}}{(\delta\varrho)^j} \quad (0 \leqslant t \leqslant 1),$$

a bound clearly also valid, with now $\varrho := 1 + t$, for $-1 \leqslant t \leqslant 0$ by symmetry. Taking the supremum in ϱ, assumed at $\varrho = \frac{1}{3}(p/j)^{1/p}$, we get, for some suitable constant K_p,

$$\|H^{(j)}\|_\infty \leqslant K_p^j j^{j(p+1)/p} \qquad (j \geqslant 1).$$

Inserting into (2.2) and choosing j equal to some approximate optimum, for instance

$$j := \Big\lfloor \mathrm{e}^{-1}\Big(\frac{|\varepsilon z|}{K_p}\Big)^{p/(p+1)} \Big\rfloor$$

when $|z|$ is sufficiently large, we obtain the expected upper bound in (2.1) by selecting p sufficiently large.

The lower bound in (2.1) follows immediately from the formula

$$\Re\,\widehat{h}(z) = 2L \int_0^1 H(t)\cos(\varepsilon xt)\cosh(\varepsilon yt)\,\mathrm{d}t. \qquad \square$$

3. **Proof of Theorem 1.1.** The basic idea of the proof is the construction of a suitable biorthogonal sequence by using complex analysis tools.

Let $g \in (2\gamma_1/3, \gamma)$. By definition, each interval of length pg contains at most p values of the sequence Λ. Up to modifying Λ by inserting some new points, we may assume that each interval $J_{\mathbb{R}} := [rpg, (r+1)pg)$, $r \in \mathbb{Z}$, contains exactly p terms from Λ and that

$$\inf_{n\in\mathbb{Z}} \{\lambda_{n+1} - \lambda_n\} \geqslant \gamma_1/3 > 0. \tag{3.1}$$

Indeed, this may be performed in two steps. First, for each n, we define m_n by $m_n\gamma_1/3 \leqslant \lambda_{n+1} - \lambda_n < (m_n+1)\gamma_1/3$, and we add the points $\lambda_n + j\gamma_1/3$, $j = 1, \ldots, m_n - 1$. Then we get a sequence with gaps between $\gamma_1/3$ and $2\gamma_1/3$, and hence each of the disjoint intervals J_r contains at least $pg/(2\gamma_1/3) > p$ elements of the sequence Λ. We conclude by deleting as many points as necessary to reach the required goal.

For fixed p, we thus have

$$z_n = ng + O(1) \quad (n \in \mathbb{Z}), \qquad \inf_{n\in\mathbb{Z}} \Re\,\{z_{n+1} - z_n\} \geqslant \gamma_1/3.$$

We put

$$\Phi_m(z) := \prod_{\substack{n\in\mathbb{Z}\\ n\neq m}} \left(1 - \frac{z - z_m}{z_n - z_m}\right) \prod_{k\geqslant 0} \frac{1 - iz/w_k}{1 - iz_m/w_k} \qquad (m \in \mathbb{Z},\ z \in \mathbb{C}),$$

$$\Psi_j(z) := \prod_{n\in\mathbb{Z}} \frac{1 - z/z_n}{1 - iw_j/z_n} \prod_{\substack{k\geqslant 0\\ k\neq j}} \left(1 - \frac{iz - w_j}{w_k - w_j}\right) \qquad (j \in \mathbb{N},\ z \in \mathbb{C}).$$

The convergence of the infinite products on the right is immediate, since each term is $1 + O(1/k^{\alpha})$. That of the infinite products on the left follows from the above alteration of the sequence Λ, as explained in [5, lemma 7], provided that these products be defined as limits as $R \to \infty$ of the finite products for $|\lambda_n| \leqslant R$.

We immediately see that

$$\begin{aligned}
&\Phi_m(z_n) = \Phi_m(-iw_k) = 0 \quad (m, n \in \mathbb{Z},\ n \neq m,\ k \in \mathbb{N}),\\
&\Phi_m(z_m) = 1 \qquad (m \in \mathbb{Z}),\\
&\Psi_j(z_n) = \Psi_j(-iw_k) = 0 \quad (n \in \mathbb{Z},\ k, j \in \mathbb{N},\ k \neq j),\\
&\Psi_j(-iw_j) = 1 \qquad (j \in \mathbb{N}).
\end{aligned} \tag{3.2}$$

As a first step, we observe that, still writing $z = x + iy$, we have for some suitable c_1 depending only on p and on the implicit constants of our statement,

$$\prod_{\substack{n\in\mathbb{Z}\\ n\neq m}} \left(1 - \frac{z - z_m}{z_n - z_m}\right) \ll \mathrm{e}^{\pi|y|/g}(1 + |z - z_m|)^{c_1} \qquad (z \in \mathbb{C}). \tag{3.3}$$

Indeed, with an obvious reindexing, we may write the product in the form

$$\prod_{n\geqslant 1} T_n T_{-n}$$

with $Z := (z - z_m)/g$ and

$$T_n := 1 - \frac{Z}{n + O(1)} = 1 - \frac{Z}{n} + O\Big(\frac{|Z|}{n^2}\Big) \quad (n \in \mathbb{Z}^*).$$

Write $Z = X + iY$, with, say, $X \geqslant 0$ (the case $X \leqslant 0$ may be dealt with symmetrically) and let $q := \lfloor X \rfloor$. For $n \geqslant 1$, $n \neq q, (q+1)$, we have

$$T_n T_{-n} = \Big(1 - \frac{Z^2}{n^2}\Big)\Big\{1 + O\Big(\frac{X + |Y|}{n|n - X| + n|Y|}\Big)\Big\}.$$

We claim that the product, say P_q, over $n \geqslant 1$, $n \neq q, (q+1)$, of the terms inside curly brackets satisfies the upper bound $P_q \ll (1 + |Z|)^b$ for some constant b depending on our parameters.

First, if $X \leqslant 1$, then

$$P_q \leqslant \exp\Big\{O\Big(\sum_{n \geqslant 1} \frac{|Y| + 1}{n^2 + n|Y|}\Big)\Big\} \ll (1 + |Y|)^{O(1)},$$

since the last sum does not exceed

$$\sum_{1 \leqslant n \leqslant 1 + |Y|} \frac{1}{n} + \sum_{n > 1 + |Y|} \frac{1 + |Y|}{n^2} \leqslant \log(1 + |Y|) + O(1).$$

If $X \geqslant 1$, we consider in turn the ranges $1 \leqslant n \leqslant \frac{1}{2}X$, $\frac{1}{2}X < n \leqslant \frac{3}{2}X$, and $n > \frac{3}{2}X$. We have

$$\sum_{n \leqslant X/2} \frac{X + |Y|}{n|n - X| + n|Y|} \leqslant \sum_{n \leqslant X/2} \frac{2}{n} \ll \log(1 + X),$$

$$\sum_{\substack{X/2 < n \leqslant 3X/2 \\ n \neq q, q+1}} \frac{X + |Y|}{n|n - X| + n|Y|} \ll \sum_{1 \leqslant h \leqslant X} \frac{1 + |Y|/X}{h + |Y|} \ll \log\Big(1 + \frac{X}{|Y| + 1}\Big) \ll 1,$$

and

$$\begin{aligned} \sum_{n > 3X/2} \frac{X + |Y|}{n|n - X| + n|Y|} &\ll \sum_{n > X} \frac{X + |Y|}{n^2 + n|Y|} \\ &\ll \sum_{X < n \leqslant X + |Y|} \frac{1}{n} + \sum_{n > X + |Y|} \frac{X + |Y|}{n^2} \\ &\ll \log(1 + |Y|) + O(1). \end{aligned}$$

Combining these estimates we obtain the claimed upper bound.

Invoking Euler's infinite product formula for $\sin(\pi Z)$, and observing that

$$\frac{\sin \pi Z}{\pi Z} \ll \begin{cases} (1 + |Z|^2)\Big(1 - \dfrac{Z^2}{q^2}\Big)\Big(1 - \dfrac{Z^2}{(q+1)^2}\Big)\mathrm{e}^{\pi|Y|} & (q \geqslant 1) \\ (1 + |Z|^2)\big(1 - Z^2\big)\mathrm{e}^{\pi|Y|} & (q = 0), \end{cases}$$

we readily get (3.3). Indeed, $T_q T_{-q}$ and $T_{q+1} T_{-q-1}$ are both bounded above by fixed a power of $1 + |Z|$.

Now, we have

$$\frac{1 - iz/w_k}{1 - iz_m/w_k} = 1 - \frac{i(z - z_m)}{w_k - iz_m} \qquad (m \in \mathbb{Z},\, k \in \mathbb{N})$$

and

$$\begin{aligned}\sum_{k\geqslant 0} \log\Big(1 + \frac{|z - z_m|}{|w_k - iz_m|}\Big) &\ll \sum_{k\geqslant 0} \int_0^{|z-z_m|} \frac{\mathrm{d}t}{t + \mu_k} \\ &\ll \int_0^{|z-z_m|} \sum_{k\geqslant 0} \frac{1}{t + (k+1)^\alpha}\,\mathrm{d}t \ll |z - z_m|^{1/\alpha}\end{aligned}$$

for $z \in \mathbb{C}$, where we used the fact that the last inner sum is trivially $\ll t^{1/\alpha - 1}$. Hence there exists a constant $K = K(\Lambda, \mathcal{H})$ such that

$$\Phi_m(z) \ll \mathrm{e}^{\pi|y|/g + K|z - z_m|^{1/\alpha}} \qquad (z \in \mathbb{C}).$$

Next, we give ourselves a parameter $\varepsilon > 0$ and recall the definition of the function h from Lemma 2.1. We then put

$$F_m(z) := \Phi_m(z)\widehat{h}(z - z_m) \qquad (m \in \mathbb{Z},\, z \in \mathbb{C}).$$

Select $\beta := 2/(1 + \alpha)$ in Lemma 2.1. For any integer $m \in \mathbb{Z}$, we have

$$\begin{gathered}F_m(z_m) = 1, \quad F_m(z_n) = F_m(-iw_k) = 0 \quad (n \in \mathbb{Z},\, n \neq m,\, k \in \mathbb{N}) \\ F_m(z) \ll \mathrm{e}^{T_\varepsilon|z|} \qquad (z \in \mathbb{C}), \qquad F_m(x) \ll \mathrm{e}^{-\frac{1}{2}\varepsilon|x - z_m|^\beta} \quad (x \in \mathbb{R}),\end{gathered}$$

with $T_\varepsilon := 2\varepsilon + \pi/g$. In the last two upper bounds, implicit constants only depend on α, γ_1, γ, ε, p, and the implicit constants in the statement.

Since F_m belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and has exponential type at most T, we infer from the Paley–Wiener theorem that it is the Fourier transform of some function φ_m with support included in $[-T_\varepsilon, T_\varepsilon]$, i.e.,

$$F_m(z) = \int_{-T_\varepsilon}^{T_\varepsilon} \varphi_m(t)\mathrm{e}^{-izt}\,\mathrm{d}t \qquad (z \in \mathbb{C}).$$

In order to obtain an analogous result for Ψ_j, we first observe that relation (3.3) with $z_m = 0$ enables us to write

$$\prod_{n\in\mathbb{Z}} \frac{1 - z/z_n}{1 + iw_j/z_n} \ll \mathrm{e}^{\pi|y|/g}(1 + |z|)^{c_1} \qquad (j \in \mathbb{N},\, z \in \mathbb{C}).$$

Indeed, it may be readily checked that the infinite product of the denominators converges and is bounded from below independently of j: this follows from the estimates

$$\begin{aligned}\theta_n := 1 + \frac{iw_j}{z_n} &= \Big(1 + \frac{iw_j}{n}\Big)\Big\{1 + O\Big(\frac{w_j}{n(|n| + |w_j|)}\Big)\Big\} \quad (n \in \mathbb{Z}^*), \\ \vartheta_n\vartheta_{-n} &= \left(1 + \frac{w_j^2}{n^2}\right)\Big\{1 + O\Big(\frac{w_j}{n(|n| + |w_j|)}\Big)\Big\} \quad (n \in \mathbb{N}^*),\end{aligned}$$

where we used the fact that, since $\mu_j > 0$ for all j and η_j remains bounded, we have $|n + iw_j| = |n - \eta_j + i\mu_j| \gg |n| + |w_j|$ for $n \in \mathbb{Z}^*$.

Furthermore, we have, uniformly with respect to $z \in \mathbb{C}$,

$$\begin{aligned} P(j,z) :&= \sum_{\substack{k\geqslant 0\\ k\neq j}} \log\Big(1+\frac{|iz-w_j|}{|w_k-w_j|}\Big) = \sum_{\substack{k\geqslant 0\\ k\neq j}} \int_0^{|iz-w_j|} \frac{\mathrm{d}t}{t+|w_k-w_j|} \\ &= \int_0^{|iz-w_j|} \sum_{\substack{k\geqslant 0\\ k\neq j}} \int_{|w_k-w_j|}^{\infty} \frac{\mathrm{d}s}{(t+s)^2}\,\mathrm{d}t \ll \int_0^{|iz-w_j|} \int_\gamma^\infty \frac{M_j(s)\,\mathrm{d}s}{(s+t)^2}\,\mathrm{d}t, \end{aligned}$$

where we have put $M_j(s) := \sum_{|w_k-w_j|\leqslant s} 1 \ll s^{1/\alpha}$. Hence the inner integral is $\ll t^{1/\alpha-1}$ and so

$$P(j,z) \ll |iz-w_j|^{1/\alpha}.$$

Therefore, there exists a constant $C = C(\mathcal{M},\mathcal{E})$ such that

$$\Psi_j(z) \ll (1+|z|)^{c_1}\mathrm{e}^{\pi|y|/g+C|iz-w_j|^{1/\alpha}} \ll (1+\mu_j)^{c_1}\mathrm{e}^{\pi|y|/g+2C|iz-w_j|^{1/\alpha}}$$

for all $z \in \mathbb{C}$. Let us then put

$$G_j(z) := \Psi_j(z)\widehat{h}(z)/\widehat{h}(-iw_j) \qquad (m \in \mathbb{Z},\, z \in \mathbb{C}).$$

For any integer $j \in \mathbb{N}$, we have

$$\begin{gathered} G_j(-iw_j) = 1, \quad G_j(z_n) = G_j(-iw_k) = 0 \quad (n \in \mathbb{Z},\, k \in \mathbb{N},\ k \neq j) \\ G_j(z) \ll \mathrm{e}^{T_\varepsilon|z|} \qquad (z \in \mathbb{C}), \qquad G_j(x) \ll \mathrm{e}^{-\frac13\varepsilon\{|x|^\beta+\mu_j\}} \quad (x \in \mathbb{R}), \end{gathered}$$

with $T_\varepsilon := 2\varepsilon + \pi/g$, and where, as previously, implied constants only depend on α, γ_1, γ, ε, p, and the implicit constants in the statement.

Since, for each $j \in \mathbb{N}$, the function G_j belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and has exponential type at most T_ε, it is the Fourier transform of a function ψ_j supported in $[-T_\varepsilon, T_\varepsilon]$, i.e.,

$$G_j(z) = \int_{-T_\varepsilon}^{T_\varepsilon} \psi_j(t)\mathrm{e}^{-izt}\,\mathrm{d}t \qquad (j \in \mathbb{N},\, z \in \mathbb{C}).$$

Now let us consider the functions

$$\begin{aligned} f(t) &:= \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{iz_nt} + \sum_{k\geqslant 0} b_k \mathrm{e}^{-w_kt}, \\ g(t) &:= \sum_{n\in\mathbb{Z}} a_n \varphi_n(t) + \sum_{k\geqslant 0} b_k \psi_k(t) \end{aligned}$$

for $t \in \mathbb{R}$. We have, employing the Cauchy–Schwarz inequality,

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{k\geqslant 0} |b_k|^2 &= \int_{-T_\varepsilon}^{T_\varepsilon} \overline{f(t)}g(t)\,\mathrm{d}t \\ &\leqslant \Big(\int_{-T_\varepsilon}^{T_\varepsilon} |f(t)|^2\,\mathrm{d}t\Big)^{1/2} \Big(\int_{-T_\varepsilon}^{T_\varepsilon} |g(t)|^2\,\mathrm{d}t\Big)^{1/2}. \end{aligned}$$

Moreover, by Plancherel's formula, still with the notation $\beta := 2/(1+\alpha)$, we have

$$\begin{aligned}\int_{-T_\varepsilon}^{T_\varepsilon} |g(t)|^2\,\mathrm{d}t &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{g}(x)|^2\,\mathrm{d}x \\ &\ll \sum_{m,n} |a_m a_n| \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}\varepsilon\{|x-z_n|^\beta+|x-z_m|^\beta\}}\,\mathrm{d}x \\ &\qquad + \sum_{j,k} |b_j b_k| \mathrm{e}^{-\frac{1}{3}\varepsilon(\mu_j+\mu_k)} \int_{\mathbb{R}} \mathrm{e}^{-\frac{2}{3}\varepsilon|x|^\beta}\,\mathrm{d}x \\ &\ll \sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{k\geqslant 0} |b_k|^2,\end{aligned}$$

where we used the fact that the penultimate integral is

$$\ll \mathrm{e}^{-B\varepsilon|z_n-z_m|^\beta} \ll \mathrm{e}^{-B\varepsilon\gamma_1|n-m|^\beta/3^\beta}$$

for some constant B depending only on β.

Combining the above two inequalities we obtain that

$$\int_{-T_\varepsilon}^{T_\varepsilon} \Big|\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{iz_n t} + \sum_{k\in\mathbb{N}} b_k \mathrm{e}^{-w_k t}\Big|^2\,\mathrm{d}t \gg \sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{k\in\mathbb{N}} |b_k|^2.$$

Replacing a_n by $a_n\mathrm{e}^{iz_nT_\varepsilon}$ and b_k by $b_k\mathrm{e}^{-w_kT_\varepsilon}$, we get

$$\int_0^{2T_\varepsilon} \Big|\sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{iz_n t} + \sum_{k\in\mathbb{N}} b_k \mathrm{e}^{-w_k t}\Big|^2\,\mathrm{d}t \gg \sum_{n\in\mathbb{Z}} |a_n|^2 + \sum_{k\in\mathbb{N}} |b_k|^2\mathrm{e}^{-\mu_k T_\varepsilon}.$$

Now (1.6) follows easily. Indeed, given $T > 2\pi/\gamma$ arbitrarily, we may choose $g \in (2\gamma_1/3, \gamma)$ such that $T > 2\pi/g$, and then select $\varepsilon > 0$ such that $T = 2T_\varepsilon$.

4. **Proof of Propositions 1.2 and 1.3.** In order to simplify the formulae, we consider only the case $\ell_1 = \ell_2 = \pi$. The proofs may be extended without any difficulty to the general case.

Proof of Proposition 1.2. Using the Fourier series

$$\varrho_1(x) = \sum_{n\geqslant 1} a_n \sin nx, \quad \varrho_1(x) = \sum_{n\geqslant 1} b_n \sin nx, \quad \sigma_0(x) = \sum_{n\geqslant 1} c_n \sin nx \tag{4.1}$$

of the initial data and writing $n_\kappa := \sqrt{n^2-\kappa^2}$, we may write the solutions as

$$u(x,t) = \sum_{n\geqslant 1} \mathrm{e}^{-\kappa t}\left(a_n \cos(tn_\kappa) + b_n \frac{\sin(tn_\kappa)}{n_\kappa}\right)\sin nx$$

and

$$v(x,t) = \sum_{n\geqslant 1} c_n \mathrm{e}^{-n^2 t} \sin nx,$$

whence

$$\begin{aligned}&u_x(0,t) + v_x(0,t) \\ &\quad = \sum_{n\geqslant 1}\left\{\left(\frac{na_n}{2} + \frac{nb_n}{2in_\kappa}\right)\mathrm{e}^{(-\kappa+in_\kappa)t} + \left(\frac{na_n}{2} - \frac{nb_n}{2in_\kappa}\right)\mathrm{e}^{(-\kappa-in_\kappa)t} + nc_n\mathrm{e}^{-n^2t}\right\}.\end{aligned}$$

The proposition follows by applying Theorem 1.1 with $z_0 := 0$, and with

$$z_{-n} := -n_\kappa + i\kappa, \quad z_n := n_\kappa + i\kappa, \quad w_n := n^2$$

for $n \in \mathbb{N}^*$. Indeed, the assumptions of the theorem are satisfied with $\gamma = 1$ and $\alpha = 2$. □

Proof of Proposition 1.3. Using the same Fourier series (4.1) again, but redefining $n_\kappa := \sqrt{n^4 - \kappa^2}$ we may write the solutions as

$$u(x,t) = \sum_{n\geqslant 1} \mathrm{e}^{-\kappa t} \left(a_n \cos(n_\kappa t) + b_n \frac{\sin(n_\kappa t)}{n_\kappa} \right) \sin nx,$$

and

$$v(x,t) = \sum_{n\geqslant 1} c_n \mathrm{e}^{-n^2 t} \sin nx,$$

whence

$$\begin{aligned} &u_x(0,t) + v_x(0,t) \\ &\quad = \sum_{n\geqslant 1} \left\{ \left(\frac{na_n}{2} + \frac{nb_n}{2in_\kappa} \right) \mathrm{e}^{(-\kappa + in_\kappa)t} + \left(\frac{na_n}{2} - \frac{nb_n}{2in_\kappa} \right) \mathrm{e}^{(-\kappa - in_\kappa)t} + nc_n \mathrm{e}^{-n^2 t} \right\}. \end{aligned}$$

The proposition follows by applying Theorem 1.1 with $z_0 := 0$, and

$$z_{-n} := -n_\kappa + i\kappa, \quad z_n := n_\kappa + i\kappa, \quad w_n := n^2$$

for $n \in \mathbb{N}^*$. Indeed, the assumptions of the theorem are satisfied with arbitrarily large γ and $\alpha = 2$. □

5. **Proof of Propositions 1.4 and 1.5.** First we prove Proposition 1.4. Writing $n_\ell := \pi n/\ell$ $(n \geqslant 1)$, we may expand the solutions of (1.12) into Fourier series

$$u(x,t) = \sum_{n\geqslant 1} u_n(t) \sin(n_\ell x), \quad v(x,t) = \sum_{n\geqslant 1} v_n(t) \sin(n_\ell x),$$

where the functions $u_n(t)$, $v_n(t)$ are solutions, for each n, of the linear initial value problem

$$\begin{cases} u_n'' + n_\ell^2 u_n + \alpha v_n = 0 & \text{in} \quad (0,\infty), \\ v_n' + n_\ell^2 v_n + \beta u_n = 0 & \text{in} \quad (0,\infty), \\ u_n(0) = a_n, \quad u_n'(0) = b_n, \quad v_n(0) = c_n. \end{cases} \tag{5.1}$$

Lemma 5.1. *If $|\alpha\beta| \leqslant \pi^3/(8\ell^3)$, then, for each positive integer n, the characteristic equation*

$$z^3 + n_\ell^2 z^2 + n_\ell^2 z + (n_\ell^4 - \alpha\beta) = 0$$

of (5.1) has three distinct complex roots iz_{-n}, iz_n and $-w_n$, satisfying

$$\max\left\{ |z_{-n} + n_\ell|, |z_n - n_\ell|, \left|w_n - n_\ell^2\right| \right\} < \pi/(2\ell).$$

Moreover, the triplets of roots corresponding to distinct values of n are disjoint.

Proof. Put $s_\ell := \pi/\ell$. Rewriting the equation in the form

$$f(z) := (z - in_\ell)(z + in_\ell)(z + n_\ell^2) = \alpha\beta,$$

it is sufficient by Rouché's theorem to show that $|f(z)| > s_\ell^3/8$ on each of the three circles of radius $s_\ell/2$, centered at in_ℓ, $-in_\ell$ and $-n_\ell^2$.

If $|z - in_\ell| = \frac{1}{2}s_\ell$, then

$$|z + in_\ell| \geqslant 2n_\ell - \tfrac{1}{2}s_\ell \geqslant \tfrac{3}{2}s_\ell$$

and

$$\left|z + n_\ell^2\right| \geqslant \left|n_\ell^2 + in_\ell\right| - \tfrac{1}{2}s_\ell > n_\ell - \tfrac{1}{2}s_\ell \geqslant \tfrac{1}{2}s_\ell,$$

so that

$$|f(z)| > \tfrac{3}{8}s_\ell^3.$$

Similarly, if $|z + in_\ell| = \frac{1}{2}s_\ell$, then $|f(z)| > \frac{3}{8}s_\ell^3$.

Finally, if $\left|z + n_\ell^2\right| = s_\ell/2$, then

$$|z \pm in_\ell| \geqslant \left|n_\ell^2 \mp in_\ell\right| - \tfrac{1}{2}s_\ell > n_\ell - \tfrac{1}{2}s_\ell \geqslant \tfrac{1}{2}s_\ell,$$

so that

$$|f(z)| > \tfrac{1}{8}s_\ell^3. \qquad \square$$

In the remaining of the proof, we assume $\ell = \pi$ for notational simplicity. The proof of the general case is the same: we only have to change the coefficients n to n_ℓ everywhere.

It follows from the lemma that the above sequences (z_n) and (w_n) satisfy the hypotheses of Theorem 1.1, and that, for each $n \geqslant 1$, we have

$$\begin{aligned} u_n(t) &= \alpha_{-n}\mathrm{e}^{iz_{-n}t} + \alpha_n\mathrm{e}^{iz_nt} + \beta_n\mathrm{e}^{-w_nt},\\ v_n(t) &= \gamma_{-n}\mathrm{e}^{iz_{-n}t} + \gamma_n\mathrm{e}^{iz_nt} + \delta_n\mathrm{e}^{-w_nt}, \end{aligned}$$

with suitable complex coefficients α_n, β_n, γ_n, δ_n. Substituting these expressions into the equations (5.1) we may express these coefficients through a_n, b_n and c_n:

$$\begin{aligned} \alpha_{-n}(-z_{-n}^2 + n^2) + \alpha\gamma_{-n} &= 0,\\ \alpha_n(-z_n^2 + n^2) + \alpha\gamma_n &= 0,\\ \beta_n(w_n^2 + n^2) + \alpha\delta_n &= 0,\\ \gamma_{-n}(iz_{-n} + n^2) + \beta\alpha_{-n} &= 0,\\ \gamma_n(iz_n + n^2) + \beta\alpha_n &= 0,\\ \delta_n(-w_n + n^2) + \beta\beta_n &= 0,\\ \alpha_{-n} + \alpha_n + \beta_n &= a_n,\\ iz_{-n}\alpha_{-n} + iz_n\alpha_n - w_n\beta_n &= b_n,\\ \gamma_{-n} + \gamma_n + \delta_n &= c_n. \end{aligned}$$

Expressing γ_{-n}, γ_n, and δ_n from the first three equations and substituting their expressions into the last equation, the last three equations become

$$\begin{aligned} a_n &= \alpha_{-n} + \alpha_n + \beta_n,\\ b_n &= iz_{-n}\alpha_{-n} + iz_n\alpha_n - w_n\beta_n,\\ \alpha c_n &= (z_{-n}^2 - n^2)\alpha_{-n} + (z_n^2 - n^2)\alpha_n - (w_n^2 + n^2)\beta_n. \end{aligned}$$

Since $|z_{\pm n}| \ll n$, $|w_n| \ll n^2$ and

$$\left|z_{\pm n}^2 - n^2\right| = |z_{\pm n} - n| \cdot |z_{\pm n} + n| \ll n,$$

it follows that

$$n^2\left|a_n\right|^2 + \left|b_n\right|^2 + \left|c_n\right|^2 \ll n^2\left|\alpha_n\right|^2 + n^2\left|\alpha_{-n}\right|^2 + n^8\left|\beta_n\right|^2. \tag{5.2}$$

Since

$$u_x(0,t) = \sum_{n\in\mathbb{Z}^*} n\alpha_n\mathrm{e}^{iz_nt} + \sum_{k\in\mathbb{N}} k\beta_k\mathrm{e}^{-w_kt},$$

we deduce from Theorem 1.1, for each $T > 2\pi$, the estimate

$$\int_0^T |u_x(0,t)|^2 \ dt \gg \sum_{n\in\mathbb{Z}^*} n^2 |\alpha_n|^2 + \sum_{k\in\mathbb{N}} k^2 |\beta_k|^2 \mathrm{e}^{-\mu_k T},$$

or equivalently

$$\int_0^T |u_x(0,t)|^2 \ dt \gg \sum_{n\in\mathbb{N}} n^2 \left(|\alpha_n|^2 + |\alpha_{-n}|^2 + |\beta_n|^2 \mathrm{e}^{-\mu_n T} \right).$$

Combining this with (5.2), the first estimate of Proposition 1.4 follows:

$$\begin{aligned}\int_0^T |u_x(0,t)|^2 \ dt &\gg \sum_{n\in\mathbb{N}} n^{-6} \mathrm{e}^{-\mu_n T} \left(n^2 |\alpha_n|^2 + n^2 |\alpha_{-n}|^2 + n^8 |\beta_n|^2 \right) \\ &\gg \sum_{n\in\mathbb{N}} n^{-6} \mathrm{e}^{-\mu_n T} \left(n^2 |a_n|^2 + |b_n|^2 + |c_n|^2 \right).\end{aligned}$$

The proof of the second estimate is similar. Considering the same linear system of nine equations as above, now we start by expressing α_n, α_{-n} and β_n from the three middle equations, and we substitute the results into the last three equations to obtain

$$\begin{aligned}\beta a_n &= -\gamma_{-n}(iz_{-n} + n^2) - \gamma_n(iz_n + n^2) - \delta_n(-w_n + n^2), \\ \beta b_n &= -\gamma_{-n} iz_{-n}(iz_{-n} + n^2) - \gamma_n iz_n(iz_n + n^2) + \delta_n w_n(-w_n + n^2), \\ c_n &= \gamma_{-n} + \gamma_n + \delta_n.\end{aligned}$$

Similarly to above, using also the relation $\left|w_n - n^2\right| \ll 1$, we infer that

$$n^2 |a_n|^2 + |b_n|^2 + |c_n|^2 \ll n^6 |\gamma_n|^2 + n^6 |\gamma_{-n}|^2 + n^4 |\delta_n|^2. \tag{5.3}$$

Since

$$v_x(0,t) = \sum_{n\in\mathbb{Z}^*} n\gamma_n \mathrm{e}^{iz_n t} + \sum_{k\in\mathbb{N}} k\delta_k \mathrm{e}^{-w_k t},$$

we deduce from Theorem 1.1, for each $T > 2\pi$, the validity of the estimate

$$\int_0^T |v_x(0,t)|^2 \ dt \gg \sum_{n\in\mathbb{N}} n^2 \left(|\gamma_n|^2 + |\gamma_{-n}|^2 + |\delta_n|^2 \mathrm{e}^{-\mu_n T} \right).$$

Combining this with (5.3) the second estimate of Proposition 1.4 follows:

$$\begin{aligned}\int_0^T |v_x(0,t)|^2 \ dt &\gg \sum_{n\in\mathbb{N}} n^{-4} \mathrm{e}^{-\mu_n T} \left(n^6 |\gamma_n|^2 + n^6 |\gamma_{-n}|^2 + n^4 |\delta_n|^2 \right) \\ &\gg \sum_{n\in\mathbb{N}} n^{-4} \mathrm{e}^{-\mu_n T} \left(n^2 |a_n|^2 + |b_n|^2 + |c_n|^2 \right).\end{aligned}$$

Now we turn to the proof of Proposition 1.5. We retain the notations $s_\ell := \pi/\ell$ and $n_\ell := ns_\ell$. Expanding the solutions of (1.15) into Fourier series

$$u(x,t) = \sum_{n\geqslant 1} u_n(t) \sin(n_\ell x), \quad v(x,t) = \sum_{n\geqslant 1} v_n(t) \sin(n_\ell x),$$

we see that, for each n, the functions $u_n(t)$, $v_n(t)$ are solutions of the linear initial value problem

$$\begin{cases} u_n'' + n_\ell^4 u_n + \alpha v_n = 0 \quad \text{in} \quad (0,\infty), \\ v_n' + n_\ell^2 v_n + \beta u_n = 0 \quad \text{in} \quad (0,\infty), \\ u_n(0) = a_n, \quad u_n'(0) = b_n, \quad v_n(0) = c_n. \end{cases} \tag{5.4}$$

The following proof is a variant of Lemma 5.1.

Lemma 5.2. *If $|\alpha\beta| < s_\ell^6/(6+6s_\ell)$, then, for each positive integer n, the characteristic equation*

$$z^3 + n_\ell^2 z^2 + n_\ell^4 z + (n_\ell^6 - \alpha\beta) = 0$$

of (5.4) has three distinct real or complex roots iz_{-n}, iz_n, and $-w_n$, satisfying

$$\max\left\{\left|z_{-n} + n_\ell^2\right|, \left|z_n - n_\ell^2\right|, \left|w_n - n_\ell^2\right|\right\} < s_\ell^2/(1+s_\ell).$$

Moreover, the triplets of roots corresponding to distinct values of n are disjoint.

Proof. Put $\varepsilon := s_\ell^2/(1+s_\ell)$ and observe that $|m_\ell^2 - n_\ell^2| > 2\varepsilon$ for $m \geqslant 1$, $n \geqslant 1$, $m \neq m$. Rewriting the equation in the form

$$f(z) := (z - in_\ell^2)(z + in_\ell^2)(z + n_\ell^2) = \alpha\beta,$$

we infer from Rouché's theorem that it is sufficient to show that the lower bound $|f(z)| > s_\ell^4\varepsilon/6$ holds on each of the three circles of radius ε centered at in_ℓ^2, $-in_\ell^2$ and $-n_\ell^2$.

If $\left|z - in_\ell^2\right| = \varepsilon$, then $\left|z + in_\ell^2\right| \geqslant 2n_\ell^2 - \varepsilon > s_\ell^2$ and

$$\left|z + n_\ell^2\right| \geqslant \left|n_\ell^2 + in_\ell^2\right| - s_\ell^2 > (\sqrt{2}-1)s_\ell^2,$$

so that $|f(z)| > (\sqrt{2}-1)\varepsilon s_\ell^4 > \varepsilon s_\ell^4/6$.

Similarly, if $|z + in_\ell| = \varepsilon$, then $|f(z)| > \varepsilon s_\ell^4/6$.

Finally, if $\left|z + n_\ell^2\right| = \varepsilon$, then

$$\left|z \pm in_\ell^2\right| \geqslant \left|n_\ell^2 \mp in_\ell^2\right| - \varepsilon \geqslant \sqrt{2}n_\ell^2 - s_\ell^2 \geqslant (\sqrt{2}-1)s_\ell^2,$$

so that $|f(z)| \geqslant (\sqrt{2}-1)^2\varepsilon s_\ell^4 > \varepsilon s_\ell^4/6$. □

For the completion of the proof, let us assume again for notational simplicity that $\ell = \pi$.

It follows from the lemma that the above sequences (z_n) and (w_n) satisfy the hypotheses of Theorem 1.1 for any fixed $T > 0$, and that, for each $n \geqslant 1$, we have

$$u_n(t) = \alpha_{-n}e^{iz_{-n}t} + \alpha_n e^{iz_n t} + \beta_n e^{-w_n t},$$
$$v_n(t) = \gamma_{-n}e^{iz_{-n}t} + \gamma_n e^{iz_n t} + \delta_n e^{-w_n t},$$

with suitable complex coefficients α_n, β_n, γ_n, δ_n. Substituting these expressions into the equations (5.4) we may express these coefficients in terms of a_n, b_n and c_n:

$$\begin{aligned}
\alpha_{-n}(-z_{-n}^2+n^4)+\alpha\gamma_{-n}&=0,\\
\alpha_n(-z_n^2+n^4)+\alpha\gamma_n&=0,\\
\beta_n(w_n^2+n^4)+\alpha\delta_n&=0,\\
\gamma_{-n}(iz_{-n}+n^2)+\beta\alpha_{-n}&=0,\\
\gamma_n(iz_n+n^2)+\beta\alpha_n&=0,\\
\delta_n(-w_n+n^2)+\beta\beta_n&=0,\\
\alpha_{-n}+\alpha_n+\beta_n&=a_n,\\
iz_{-n}\alpha_{-n}+iz_n\alpha_n-w_n\beta_n&=b_n,\\
\gamma_{-n}+\gamma_n+\delta_n&=c_n.
\end{aligned}$$

Adapting the proof of the previous proposition we now obtain:

$$n^4|a_n|^2+|b_n|^2+|c_n|^2\ll n^4|\alpha_n|^2+n^4|\alpha_{-n}|^2+n^8|\beta_n|^2; \tag{5.5}$$

$$n^4|a_n|^2+|b_n|^2+|c_n|^2\ll n^8|\gamma_n|^2+n^8|\gamma_{-n}|^2+n^4|\delta_n|^2. \tag{5.6}$$

Since

$$u_x(0,t)=\sum_{n\in\mathbb{Z}^*}n\alpha_n\mathrm{e}^{iz_nt}+\sum_{k\in\mathbb{N}}k\beta_k\mathrm{e}^{-w_kt},$$

we deduce from Theorem 1.1, for each $T>0$, the validity of the estimate

$$\int_0^T|u_x(0,t)|^2\ dt\gg\sum_{n\in\mathbb{N}}n^2\left(|\alpha_n|^2+|\alpha_{-n}|^2+|\beta_n|^2\mathrm{e}^{-\mu_nT}\right).$$

Combining this with (5.5) the first estimate of Proposition 1.5 follows:

$$\begin{aligned}
\int_0^T|u_x(0,t)|^2\ dt&\gg\sum_{n\in\mathbb{N}}n^{-6}\mathrm{e}^{-\mu_nT}\left(n^4|\alpha_n|^2+n^4|\alpha_{-n}|^2+n^8|\beta_n|^2\right)\\
&\gg\sum_{n\in\mathbb{N}}n^{-6}\mathrm{e}^{-\mu_nT}\left(n^4|a_n|^2+|b_n|^2+|c_n|^2\right).
\end{aligned}$$

Similarly, since

$$v_x(0,t)=\sum_{n\in\mathbb{Z}^*}n\gamma_n\mathrm{e}^{iz_nt}+\sum_{k\in\mathbb{N}}k\delta_k\mathrm{e}^{-w_kt},$$

applying Theorem 1.1 we conclude for each $T>0$ the estimate

$$\int_0^T|v_x(0,t)|^2\ dt\gg\sum_{n\in\mathbb{N}}n^2\left(|\gamma_n|^2+|\gamma_{-n}|^2+|\delta_n|^2\mathrm{e}^{-\mu_nT}\right).$$

Combining this with (5.6) the second estimate of Proposition 1.5 follows:

$$\begin{aligned}
\int_0^T|v_x(0,t)|^2\ dt&\gg\sum_{n\in\mathbb{N}}n^{-6}\mathrm{e}^{-\mu_nT}\left(n^8|\gamma_n|^2+n^8|\gamma_{-n}|^2+n^4|\delta_n|^2\right)\\
&\gg\sum_{n\in\mathbb{N}}n^{-6}\mathrm{e}^{-\mu_nT}\left(n^4|a_n|^2+|b_n|^2+|c_n|^2\right).
\end{aligned}$$

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