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Effective mean value estimates for complex multiplicative functions

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1. *Introduction*

Quantitative estimates for finite mean values

$$x^{-1} \sum_{n \leqslant x} g(n) \tag{1}$$

of multiplicative functions are highly applicable tools in analytic and probabilistic number theory. Extending a result of Hall [**4**], Halberstam and Richert [**3**] proved a useful inequality valid for real, non-negative g satisfying for instance a Wirsing type condition, viz $0 \leqslant g(p^\nu) \leqslant \lambda_1 \lambda_2^{\nu-1}$ $(\nu = 1, 2, \ldots)$ for all primes p, with constants $\lambda_1 \geqslant 0$, $0 \leqslant \lambda_2 < 2$. Their upper bound is sharp to within a factor $(1+o(1))$, but even a weaker and easier to prove estimate, such as

$$x^{-1} \sum_{n \leqslant x} g(n) \ll \exp\left\{-\sum_{p \leqslant x} \frac{1-g(p)}{p}\right\} \tag{2}$$

(where the implied constants depend on λ_1 and λ_2), may become a surprisingly strong device. For instance, setting $g(p) = 1 \pm \epsilon$, where ϵ is an arbitrarily small positive number, provides immediately a proof of the famous Hardy–Ramanujan theorem on the normal order of the number of prime factors of an integer. This example, and many others, are discussed in detail in our book [**5**] where we make extensive use of (2) for various problems connected with the structure of the set of divisors of a normal number.

Hildebrand [**6, 7**] refined the Halberstam–Richert inequality and obtained corresponding sharp lower bounds. In another direction, an inequality of Tenenbaum([**10**], theorem III·4·7) states that, when $g : \mathbb{Z}^+ \to [-1, 1]$, the estimate (2) remains valid provided the p-sum is multiplied by $\frac{1}{8}$. The proof rests on Montgomery's effective version of Halász' mean value theorem (see Lemma 1 below) and the possibility of deducing a result of this sort had been indicated by Montgomery [**9**]. A slightly weaker version appears in Elliott [**1**], who uses a different method.

It is natural to ask whether there is a similar inequality under the weaker hypothesis $|g| \leqslant 1$, perhaps with a smaller constant and of course $\Re e\, g(p)$ appearing in the exponential. The example $g(n) = n^i$ shows that this is not the case, for the left-hand side of (2) is then $\Omega(1)$ whereas

$$\sum_{p \leqslant x} \frac{1-\cos(\log p)}{p} = \log_2 x + O(1).$$

(Here, and in the sequel, we let $\log_k$ denote the k-th fold iterated logarithm.) Nevertheless, we do have a result of this type if we impose an extra condition on g. At first sight this might appear artificial – in fact it is quite practical.

For $0 \leqslant \delta \leqslant 1$, $0 \leqslant \phi < \pi$, we let $\mathscr{E}(\delta, \phi)$ be the set of complex numbers z with

$$(\mathfrak{Im}\,(e^{-i\phi}z))^2 \leqslant \delta^2\{1-(\mathfrak{Re}\,(e^{-i\phi}z))^2\}, \tag{3}$$

and we denote by $\mathscr{G}(\delta, \phi)$ the class of all multiplicative functions g such that $|g(n)| \leqslant 1$ for all n and $g(p) \in \mathscr{E}(\delta, \phi)$ for all primes p. We shall also make use of the linear mapping W defined by

$$W(z) = e^{i\phi}\{\mathfrak{Re}\,(e^{-i\phi}z) + i\delta\mathfrak{Im}\,(e^{-i\phi}z)\}. \tag{4}$$

We have the following result.

THEOREM. *For $0 \leqslant \delta \leqslant 1$, $0 \leqslant \phi < \pi$, the integral equation*

$$\frac{1}{2\pi}\int_0^{2\pi} |W(e^{i\theta}-K)|\,d\theta = 1-K \tag{5}$$

has a unique solution $K = K(\delta, \phi) \geqslant 0$, which is, for fixed ϕ, a decreasing function of δ such that $K > 0$ if and only if $0 \leqslant \delta < 1$. We have uniformly for $x \geqslant 1$ and $g \in \mathscr{G}(\delta, \phi)$

$$\sum_{n \leqslant x} g(n) \ll x\exp\left\{-K(\delta, \phi) \sum_{p \leqslant x} \frac{1-\mathfrak{Re}\, g(p)}{p}\right\}. \tag{6}$$

Moreover the constant $K(\delta, \phi)$ is sharp: given $(\delta, \phi) \in [0, 1] \times [0, \pi[$ and $x \geqslant 3$, there is a $g \in \mathscr{G}(\delta, \phi)$ such that

(i) $$\sum_{p \leqslant x} \frac{1-\mathfrak{Re}\, g(p)}{p} \to +\infty \quad (x \to +\infty)$$

(ii) $$\left|\sum_{n \leqslant x} g(n)\right| \gg x\exp\left\{-K(\delta, \phi) \sum_{p \leqslant x} \frac{1-\mathfrak{Re}\, g(p)}{p}\right\}.$$

We prove the statements on $K(\delta, \phi)$, as well as some extra remarks, in Section 4 which also contains a table of $K(\delta, 0)$ computed by M. G. J. van der Burg.

Condition (i) is trivially realized unless $\phi = 0$ or $\delta = 1$ – indeed

$$\mathfrak{Re}\, g(p) \leqslant \sqrt{(\cos^2\phi + \delta^2 \sin^2\phi)}$$

for $g \in \mathscr{G}(\delta, \phi)$. When $\delta = 0$, $\phi = \frac{1}{2}\pi$, we have $W(e^{i\theta} - K) = i\sin\theta$ for every K. Hence $K(0, \frac{1}{2}\pi) = 1-2/\pi$, and the theorem implies that

$$\max_{g \in \mathscr{G}(0, \pi/2)} \left|\sum_{n \leqslant x} g(n)\right| \asymp x(\log x)^{(2/\pi)-1}.$$

The case $\delta = \phi = 0$ of the theorem corresponds to real functions and (6) provides then the sharp form of Tenenbaum's inequality. It is easily checked that $K(0,0) = 0{\cdot}32867\ldots = -\cos\phi_0$, where ϕ_0 is the unique root in $(0,\pi)$ of the equation $\sin\phi - \phi\cos\phi = \frac{1}{2}\pi$.

Quantitative estimates of the form (6) already appear in Halász [2], with a different condition on the values $g(p)$. He notes in particular that, for completely multiplicative functions g with values ± 1, his main theorem yields a bound of the type (6) with a constant which can be taken as large as 0·07 but not 0·37.

Conditions (i) and (ii) guarantee that the right-hand side of (6) cannot be multiplied by a function having lower limit zero as $\sum_{p\leqslant x}(1-\Re e\, g(p))\,p^{-1}$ tends to infinity. Thus $K(\delta,\phi)$ is sharp in the strongest possible sense. We imagine that our lower bound method is partly close to the lower bound derivation of an unpublished result of Montgomery and Vaughan

$$\max_{n\geqslant 1}\Big|\sum_{d\leqslant x,\, d|n}\mu(d)\Big| \asymp x(\log x)^{(1/\pi)-1}. \tag{7}$$

For a fixed function g, the estimate (6) may lead to an unsharp bound. Consider for instance $g(n) := \tau(n,\theta)/\tau(n)$ where $\tau(n,\theta) := \sum_{d|n} d^{i\theta}$. This last function occurs in the theory of Hooley's Δ-function and in the proof of the Maier–Tenenbaum theorem [**8**] – both applications being developed in [**5**]. We have $g(p) = \frac{1}{2}(1+p^{i\theta})$, so $g(p)$ lies in the ellipse $\mathscr{E}(1/\sqrt{2},0)$ and (6) yields

$$\sum_{n\leqslant x} g(n) \ll \frac{x}{(|\theta|\log x)^K}$$

uniformly for $1/\log x \leqslant |\theta| \leqslant 1$ (say), with $K = K(1/\sqrt{2},0) = 0{\cdot}135,\ldots$, whereas it can be shown directly by contour integration that the correct exponent is $K = \frac{1}{2}$. This example raises the problem of finding other conditions on $g(p)$ (i.e. defining a different class of functions), for which there is still a sharp general inequality but with less possible fluctuations in the result.

2. *Proof of the upper bound result*

In this section, we prove (6). We require the following lemmas, the first of which is the inequality of Montgomery [**9**] referred to in the Introduction.

LEMMA 1. *Let* $g\in\mathscr{G}(1,0)$. *Set*

$$G(x) := \sum_{n\leqslant x} g(n), \quad F(s) := \sum_{n=1}^{\infty} g(n)\,n^{-s}. \tag{8}$$

For $\alpha > 0$, *let* $H(\alpha)$ *be defined by*

$$H(\alpha)^2 := \sum_{k\in\mathbb{Z}} (k^2+1)^{-1} \max_{|\tau-k|\leqslant\frac{1}{2}} |F(1+\alpha+i\tau)|^2. \tag{9}$$

Then

$$G(x) \ll \frac{x}{\log x}\int_{1/\log x}^{1} H(\alpha)\frac{d\alpha}{\alpha}. \tag{10}$$

Montgomery restricted his attention to completely multiplicative g. The more general case involves only technical changes. For details, see [**10**] chapter III·4.

LEMMA 2. *Let f be a periodic function of bounded variation over the period* $[0, 2\pi]$ *and having mean value*

$$\bar{f} := \frac{1}{2\pi}\int_0^{2\pi} f(v)\,dv.$$

Then for real τ, w *such that* $\tau \neq 0$, $0 < \omega \leqslant z$ *and every positive* c, *we have*

$$\sum_{w < p \leqslant z} \frac{1}{p} f(\tau \log p) = \bar{f}\log\left(\frac{\log z}{\log w}\right) + O\left(\frac{V(f)}{|\tau| \log w}\right) + O_c(\{M(f) + (1+|\tau|)\,V(f)\}\,e^{-c\sqrt{\log w}}) \tag{11}$$

where

$$M(f) := \sup_v |f(v)|, \quad V(f) := \int_0^{2\pi} |df(v)|.$$

This is lemma 30·1 of [**5**]. We state the following immediate consequence of Chebyshev's inequality $\pi(t) \ll t/\log t$ for convenience of reference.

LEMMA 3. *Let* $0 \leqslant \alpha \leqslant 1$. *Then*

$$\sum_{p \leqslant \exp(1/\alpha)} \frac{1-p^{-\alpha}}{p} + \sum_{p > \exp(1/\alpha)} p^{-1-\alpha} \ll 1. \tag{12}$$

Recall the definition of W in (5) and set

$$T(z) := e^{i\phi}\{\Re e\,(e^{-i\phi}z) + i\delta^{-1}\Im m\,(e^{-i\phi}z)\}, \tag{13}$$

where, by convention, the term involving δ^{-1} is to be deleted if $\delta = 0$. The functions T and W may be regarded as linear mappings on $\mathbb{C}$ considered as a real vector space.

LEMMA 4. *For* $\delta = 0$, *the restrictions of* T *and* W *to* $\mathscr{E}(\delta, \phi)$ *both coincide with the identity. For* $\delta \neq 0$, *the restrictions* $T: \mathscr{E}(\delta,\phi) \to \mathscr{E}(1,0)$ *and* $W: \mathscr{E}(1,0) \to \mathscr{E}(\delta,\phi)$ *are reciprocal isomorphisms. Moreover, when* $\delta \neq 0$, *we have*

$$\Re e\,(z_1 \overline{z_2}) = \Re e\,(T(z_1)\,\overline{W(z_2)}) \quad (z_1 \in \mathbb{C}, z_2 \in \mathbb{C}). \tag{14}$$

Proof. The case $\phi = 0$ is trivial and we suppose henceforth that $\delta > 0$. We could of course check the various statements by direct explicit computation, but it will be more satisfactory to make an algebraic reasoning. Let $R(\phi)$ denote the matrix of the rotation of angle ϕ and $D(t)$ the matrix of the dilation $(\xi, \eta) \mapsto (\xi, t\eta)$. Then $D(t)$ induces a one-to-one mapping from $\mathscr{E}(1,0)$ to $\mathscr{E}(\delta, 0)$ and $D(t)^{-1} = D(t^{-1})$. Moreover, T and W are respectively associated with $R(\phi)D(\delta^{-1})R(-\phi)$ and $R(\phi)D(\delta)R(-\phi)$. Since the unit disc $\mathscr{E}(1,0)$ is invariant under $R(\phi)$ and $R(\phi)^{-1} = R(-\phi)$, we immediately obtain the first assertion. Next, let Z_1, Z_2 be the column matrices associated to the complex numbers z_1, z_2. Using a dash to indicate transposition, we have

$$\begin{aligned}\Re e\,(T(z_1)\,\overline{W(z_2)}) &= (R(\phi)D(\delta^{-1})R(-\phi)\,Z_1)'R(\phi)D(\delta)R(-\phi)\,Z_2\\ &= Z_1' R(-\phi)'\,D(\delta^{-1})'R(\phi)'R(\phi)\,D(\delta)\,R(-\phi)\,Z_2\\ &= Z_1' Z_2 = \Re e\,(z_1 \overline{z_2}),\end{aligned}$$

since $R(\pm\phi)' = R(\mp\phi)$ and $D(\delta^{-1})$ is symmetric. This establishes Lemma 4.

We may now embark on the proof. We begin by establishing that, uniformly for $(\delta, \phi) \in [0,1] \times [0, \pi[, 0 < \alpha \leqslant 1, \tau \in \mathbb{R}$, we have

$$\sum_{p \leqslant \exp(1/\alpha)} \frac{1}{p} |W(p^{i\tau} - K)| \leqslant (1-K) \log \frac{1}{\alpha} + O(\log_2(|\tau|+3)). \tag{15}$$

We set

$$f(v) := |W(e^{iv} - K)|.$$

When $|\tau| \leqslant \alpha$, we have

$$f(\tau \log p) = |W(1-K) + O(\tau \log p)| \leqslant 1 - K + O(\tau \log p), \tag{16}$$

since $W(1-K) = (1-K)\, W(1)$ and $0 \leqslant K \leqslant 1$, $|W(1)| \leqslant 1$. Hence if $|\tau| \leqslant \alpha$, we have immediately

$$\sum_{p \leqslant \exp(1/\alpha)} \frac{1}{p} f(\tau \log p) \leqslant (1-K) \log \frac{1}{\alpha} + O(1). \tag{17}$$

Next, let $\alpha < |\tau| \leqslant 1$. We put $w := \exp\{1/|\tau|\}$. By (16), we have

$$\sum_{p \leqslant w} \frac{1}{p} f(\tau \log p) \leqslant (1-K) \log_2 w + O(1). \tag{18}$$

Now, we put $z := \exp(1/\alpha) > w$ and apply Lemma 2. We have $\bar{f} = 1 - K$ by (5), $M(f) \ll 1$, $V(f) \ll 1$, and (11) gives

$$\sum_{w < p \leqslant z} \frac{1}{p} f(\tau \log p) \leqslant (1-K) \left(\log \frac{1}{\alpha} - \log_2 w \right) + O(1), \tag{19}$$

so that (18) and (19) imply (17) in this case.

Suppose now that $|\tau| > 1$ but $\log^2(|\tau|+3) \leqslant 1/\alpha$. We select $c = 1$ in Lemma 2, $w := \exp\{\log^2(|\tau|+3)\}$, $z := \exp(1/\alpha)$. This gives (19) and we make the trivial estimate $\ll \log_2 w$ for the sum in (18). So we obtain (17) with $O(\log_2(|\tau|+3))$ in place of $O(1)$. Finally, if $\log^2(|\tau|+3) > 1/\alpha$, we estimate the whole sum trivially. Hence (15) holds in all cases.

Let us now consider $(\delta, \phi) \in [0,1] \times [0, \pi[$, and $g \in \mathscr{G}(\delta, \phi)$. For $\alpha \in [0,1]$, we define $\lambda = \lambda(\alpha)$ by

$$\sum_{p \leqslant \exp(1/\alpha)} \frac{1 - \Re e\, g(p)}{p} = \lambda \sum_{p \leqslant \exp(1/\alpha)} \frac{1}{p}, \tag{20}$$

so that $\lambda \in [0,2]$ since $|g(p)| \leqslant 1$. We claim that (20) implies

$$\Re e \sum_{p \leqslant \exp(1/\alpha)} g(p)\, p^{-1-i\tau} \leqslant (1 - K\lambda) \log \frac{1}{\alpha} + O(\log_2(|\tau|+3)) \tag{21}$$

uniformly for $\tau \in \mathbb{R}$. For this we need a device which forces us to consider elliptical sets in the theorem rather than some other family of ovals. For primes only, define the function $h(p)$ by

$$h(p) := T(g(p)). \tag{22}$$

By Lemma 4, we have $|h(p)| \leqslant 1$. Also (14) with $z_1 = g(p)$, $z_2 = 1$ gives

$$\Re e\, g(p) = \Re e \{h(p)\, \overline{W(1)}\}$$

and (20) may be written as

$$\Re e \sum_{p \leqslant \exp(1/\alpha)} \frac{1}{p} h(p)\, \overline{W(1)} = (1-\lambda)\log\frac{1}{\alpha} + O(1). \tag{20'}$$

Now we use (14) again to obtain

$$\Re e\, g(p)\, p^{-i\tau} = \Re e\{h(p)\, \overline{W(p^{i\tau}-K)} + Kh(p)\, \overline{W(1)}\}.$$

From this and (20′) we deduce that

$$\Re e \sum_{p \leqslant \exp(1/\alpha)} g(p)\, p^{-1-i\tau} = \Re e \sum_{p \leqslant \exp(1/\alpha)} \frac{1}{p} h(p)\, \overline{W(p^{i\tau}-K)} + (1-\lambda) K \log\frac{1}{\alpha} + O(1). \tag{23}$$

Using the inequality $\Re e\{h(p)\, \overline{W(p^{i\tau}-K)}\} \leqslant |W(p^{i\tau}-K)|$ and applying (15) yields immediately (21).

Now consider the function F defined by (8). We have

$$F(1+\alpha+i\tau) \ll \exp\left\{\Re e \sum_p \frac{g(p)}{p^{1+\alpha+i\tau}}\right\} \tag{24}$$

and, by Lemma 3, we may restrict the p-sum to the range $p \leqslant \exp(1/\alpha)$ and strike out the p^{α}. We recall (20) and deduce from (21) that

$$F(1+\alpha+i\tau) \ll \alpha^{K\lambda-1} \log^B(|\tau|+3) \tag{25}$$

uniformly for $0 < \alpha \leqslant 1$, $\tau \in \mathbb{R}$, where B is an absolute constant. Hence Montgomery's function $H(\alpha)$ defined by (9) satisfies

$$H(\alpha)^2 \ll \alpha^{2K\lambda-2} \sum_{k \in \mathbb{Z}} \frac{\log^{2B}(|k|+4)}{k^2+1} \tag{26}$$

and so

$$H(\alpha) \ll \alpha^{K\lambda-1}. \tag{27}$$

Now let $\Lambda \in [0, 2]$ be defined by the equation

$$\sum_{p \leqslant x} \frac{1-\Re e\, g(p)}{p} = \Lambda \sum_{p \leqslant x} \frac{1}{p} = \Lambda \log_2 x + O(1). \tag{28}$$

Then we have for $1/\log x \leqslant \alpha \leqslant 1$

$$\sum_{p \leqslant \exp(1/\alpha)} \frac{1-\Re e\, g(p)}{p} \geqslant \sum_{p \leqslant x} \frac{1-\Re e\, g(p)}{p} - \sum_{\exp(1/\alpha) < p \leqslant x} \frac{2}{p}$$

$$\geqslant (\Lambda-2)\log_2 x + 2\log\frac{1}{\alpha} + O(1). \tag{29}$$

We deduce from this and (20) that

$$\alpha^{\lambda} \ll \alpha^2 (\log x)^{2-\Lambda}. \tag{30}$$

Whence we obtain from (27)

$$H(\alpha) \ll \alpha^{2K-1} (\log x)^{(2-\Lambda)K}. \tag{31}$$

Inserting this in (10) and using the fact that $K \leqslant K(0, \pi/2) = 1 - 2/\pi < \frac{1}{2}$, to be proved in Section 4 – equation (60) –, we obtain

$$G(x) \ll \frac{x}{\log x}(\log x)^{1-2K+(2-\Lambda)K} = x(\log x)^{-\Lambda K}.$$

Recalling the definition (28) of Λ, we obtain (6).

3. *Completion of the proof of the theorem*

We now show that the constant $K = K(\delta, \phi)$ is sharp for all $(\delta, \phi) \in [0, 1] \times [0, \pi[$ by constructing for every $x \geqslant 1$ a function $g \in \mathscr{G}(\delta, \phi)$ which satisfies conditions (i) and (ii) of the theorem. We first introduce some further notation

$$P(n) := \max\{p : p \mid n\} \quad (n > 1), \quad P(1) := 1,$$

$$S(x) := \sum_{p \leqslant x} \frac{1 - \Re e\, g(p)}{p} \quad (g \in \mathscr{G}(1, 0)),$$

$$L(x) := \sum_{n \leqslant x} g(n) \log n \quad (g \in \mathscr{G}(1, 0)).$$

We need two lemmas, the first of which arises as an intermediate step in Montgomery's proof of Lemma 1 – see [9] or [**10**], p. 380.

LEMMA 5. *Let $g \in \mathscr{G}(1, 0)$. Then we have*

$$\int_1^y \frac{|G(t)|}{t^2}\, dt \ll \int_{1/\log y}^1 \frac{H(\alpha)}{\alpha}\, d\alpha \quad (y \geqslant 3). \tag{32}$$

LEMMA 6. *Let $w \geqslant 2$ and $g \in \mathscr{G}(1, 0)$ be supported on squarefree numbers and such that $g(p) = 0$ for $p > w$. Then we have*

$$G(t)\log t - L(t) \ll t \exp\left\{-\frac{\log t}{2\log w}\right\} \quad (t \geqslant 2, w \geqslant 2) \tag{33}$$

and

$$\int_{\sqrt{y}}^y \frac{|G(t)|}{t^2}\, dt \ll \frac{H(\alpha_0)}{\sqrt{(\alpha_0 \log y)}} \quad (y \geqslant 2) \tag{34}$$

where

$$\alpha_0 := \max\left(\frac{1}{\log w}, \frac{1}{\log y}\right).$$

Proof. The left-hand side of (33) is plainly

$$\ll \sum_{\substack{n \leqslant t \\ P(n) \leqslant w}} \log(t/n).$$

The required estimate hence follows, by partial summation, from the estimate

$$\sum_{\substack{n \leqslant t \\ P(n) \leqslant w}} 1 \ll t \exp\left\{-\frac{\log t}{2\log w}\right\}$$

proved in [**10**], theorem III·5·1.

The proof of (34) consists in a reappraisal of part of the proof of Lemma 1, where we take into account the extra hypothesis that $g(n) = 0$ whenever $P(n) > w$. In view of (33) and the estimate $H(\alpha) \gg 1$ (see [**9**] or [**10**], p. 375) it is enough to show that

$$\int_1^y \frac{|L(t)|}{t^2} dt \ll H(\alpha_0) \sqrt{\frac{\log y}{\alpha_0}}. \tag{35}$$

Now we write Cauchy's inequality

$$\int_1^y \frac{|L(t)|}{t^2} dt \leqslant \left(\int_1^y \frac{|L(t)|^2}{t^3} dt \int_1^y \frac{dt}{t} \right)^{\frac{1}{2}} \tag{36}$$

and apply the Plancherel identity with $\alpha := 1/\log y$

$$\int_1^{+\infty} \frac{|L(t)|^2}{t^{3+2\alpha}} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{F'(1+\alpha+i\tau)}{1+\alpha+i\tau} \right|^2 d\tau. \tag{37}$$

Since the integral involving $|L(t)|^2$ in (36) does not exceed e^2 times the left-hand side of (37), we obtain

$$\int_1^y \frac{|L(t)|^2}{t^3} dt \ll \sum_{k \in \mathbf{Z}} \frac{1}{k^2+1} \max_{|\tau-k| \leqslant \frac{1}{2}} |F(1+\alpha+i\tau)|^2 \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \left| \frac{F'}{F}(1+\alpha+i\tau) \right|^2 d\tau. \tag{38}$$

By a lemma of Montgomery [**9**] the last τ-integral is

$$\leqslant 3 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{p \leqslant w} \frac{\log p}{p^{1+\alpha+i\tau}} \right|^2 d\tau. \tag{39}$$

But one can easily prove by standard contour integration (see e.g. [**10**], pp. 418–20) that for $s = 1+\alpha+i\tau$, $|\tau| \leqslant 1$,

$$\sum_{p \leqslant w} \frac{\log p}{p^s} = -\frac{\zeta'}{\zeta}(s) + \frac{w^{1-s}}{1-s} + O(1) = \frac{1-w^{1-s}}{s-1} + O(1)$$

$$\ll \min\left(\frac{1}{|s-1|}, \quad \log w \right) \asymp \frac{1}{\alpha_0+|\tau|}.$$

This shows that the integral in (39) is $\ll \alpha_0^{-1}$. We also notice that

$$\left| \frac{F(1+\alpha+i\tau)}{F(1+\alpha_0+i\tau)} \right| \ll \exp\left\{ \sum_{p \leqslant w} \frac{1-p^{\alpha-\alpha_0}}{p} \right\} \ll 1.$$

Collecting the above estimates and inserting in (38), we obtain

$$\int_1^y \frac{|L(t)|}{t^3} dt \ll \frac{H(\alpha_0)^2}{\alpha_0}.$$

In view of (36), this implies the required bound in (35).

We are now in a position to embark on the final part of the argument. We may restrict ourselves to the case $\delta < 1$ since $K(1, \phi) = 0$ and, as we remarked in the introduction, $g(n) = n^i$ provides the desired counter-example for $\delta = 1$. In the remainder of this section we hence let (δ, ϕ) be fixed in $[0, 1[\times [0, \pi[$. The variable

x being given, we introduce a parameter $w = w(x)$ to be chosen explicitly later. Let g_1 be supported on squarefree numbers and $g_1(p) = 0$ for $p > w$. We put

$$g_1(p) = \frac{W^2(p^i - K)}{|W(p^i - K)|} \quad (p \leqslant w), \tag{40}$$

where, by convention, the right-hand side is to be interpreted as 0 when the denominator vanishes – this case being possible only when $\delta = 0$. Since $g_1(p) \in W(\mathscr{E}(1,0))$, we have $g_1 \in \mathscr{G}(\delta, \phi)$ by Lemma 4.

We first show that

$$S_1(w) = \sum_{p \leqslant w} \frac{1 - \Re e\, g_1(p)}{p} \to +\infty \quad (w \to +\infty). \tag{41}$$

This is trivially satisfied when $\phi \neq 0$, since then $(\Re e\, g_1(p))^2 \leqslant \cos^2\phi + \delta^2 \sin^2\phi < 1$ for all p. For $\phi = 0$, let $f_1(\theta)$ be the periodic function of θ defined by $f_1(\theta) = 1$ if $\cos\theta < K$, $f_1(\theta) = 0$ otherwise. Then

$$1 - \Re e\, g_1(p) = 1 - \frac{\cos(\log p) - K}{|\cos(\log p) - K - i\delta \sin(\log p)|} \geqslant f_1(\log p)$$

for all p and Lemma 2 implies (41) since

$$\overline{f_1} = 1 - \frac{1}{\pi} \operatorname{Arc}\cos K \geqslant \tfrac{1}{2}.$$

Next, we prove that, with an obvious notation,

$$\int_1^{+\infty} \frac{|G_1(t)|}{t^2}\, dt \gg e^{-KS_1(w)} \log w. \tag{42}$$

For this, we notice on the one hand that

$$\sum_{n=1}^{\infty} g_1(n)\, n^{-1-i} = (1+i) \int_1^{+\infty} G_1(t)\, t^{-2-i}\, dt, \tag{43}$$

and on the other hand that

$$\left| \sum_{n=1}^{\infty} g_1(n)\, n^{-1-i} \right| \asymp \exp\left\{ \sum_{p \leqslant w} \frac{\Re e\,(g_1(p)\, p^{-i})}{p} \right\}. \tag{44}$$

But, by Lemma 4, we have for all p

$$\begin{aligned} \Re e\,(g_1(p)\, p^{-i}) &= \Re e\{T(g_1(p))\, \overline{W(p^i)}\} = \Re e \left\{ \frac{W(p^i - K)\, \overline{W(p^i)}}{|W(p^i - K)|} \right\} \\ &= |W(p^i - K)| + K\, \Re e\{T(g_1(p))\, \overline{W(1)}\} \\ &= |W(p^i - K)| + K\, \Re e\, g_1(p). \end{aligned} \tag{45}$$

Hence it follows from Lemma 2 and (5) that

$$\begin{aligned} \sum_{p \leqslant w} \frac{\Re e\,(g_1(p)\, p^{-i})}{p} &= (1-K) \log_2 w + K \sum_{p \leqslant w} \frac{\Re e\, g_1(p)}{p} + O(1) \\ &= \log_2 w - K S_1(w) + O(1). \end{aligned} \tag{46}$$

Together with (43) and (44), this yields (42).

We now show that (42) remains valid when the t-range in the integral is restricted to $w^{c_0} \leqslant t \leqslant w^{c_1}$ where c_0 and c_1 are suitable absolute constants, that is

$$\int_{w^{c_0}}^{w^{c_1}} \frac{|G_1(t)|}{t^2}\,dt \gg e^{-KS_1(w)} \log w. \tag{47}$$

We need to show that the contributions of the ranges $1 \leqslant t < w^{c_0}$ and $t > w^{c_1}$ are relatively small compared to the value of the whole integral in (42). For the first range, we apply (32) and (31). We obtain, with $y = w^{c_0}$, $c_0 < 1$,

$$\begin{aligned}\int_1^y \frac{|G_1(t)|}{t^2}\,dt &\ll \int_{1/\log y}^1 \alpha^{2K-2} (\log w)^{K(2-\Lambda)}\, d\alpha \\ &\ll (\log y)^{1-2K} (\log w)^{(2-\Lambda)K} \asymp c_0^{1-2K} e^{-KS_1}(w) \log w.\end{aligned}$$

Since $K \leqslant 1-2/\pi < \frac{1}{2}$ – see Section 4 – we obtain that, for suitable c_0,

$$\int_1^{w^{c_0}} \frac{|G_1(t)|}{t^2}\,dt \leqslant \frac{1}{3}\int_1^{+\infty} \frac{|G_1(t)|}{t^2}\,dt. \tag{48}$$

For the second range, we apply (34) to g_1 with $y = y_k = w^{c_1 2^k}$, $k = 1, 2, \ldots$, and sum the resulting estimates. This gives, with $\alpha_0 = 1/\log w$,

$$\begin{aligned}\int_{w^{c_1}}^{+\infty} \frac{|G_1(t)|}{t^2}\,dt &\ll H(\alpha_0) \sum_{k=1}^{+\infty} (c_1 2^k)^{-\frac{1}{2}} \ll \frac{H(\alpha_0)}{\sqrt{c_1}} \\ &\ll \frac{1}{\sqrt{c_1}} e^{-KS_1(w)} \log w\end{aligned}$$

where we used (31) in the last stage. Thus, for suitable c_1, we obtain in view of (42)

$$\int_{w^{c_1}}^{+\infty} \frac{|G_1(t)|}{t^2}\,dt \leqslant \frac{1}{3}\int_1^{+\infty} \frac{|G_1(t)|}{t^2}\,dt. \tag{49}$$

The estimate (47) now follows from (42), (48) and (49).

The constants c_0 and c_1 being fixed, with $c_0 < 1 < c_1$, let $\mathscr{A}(w,\gamma)$ denote, for positive γ, the set of all t such that

$$w^{c_0} \leqslant t \leqslant w^{c_1}, \quad |G_1(t)| > \gamma t\, e^{-KS_1(w)}. \tag{50}$$

Since, by (6), we have

$$|G_1(t)| \leqslant c_2\, t\, e^{-KS_1(w)}$$

for all t in $[w^{c_0}, w^{c_1}]$ and suitable absolute c_2, we have

$$\int_{w^{c_0}}^{w^{c_1}} \frac{|G_1(t)|}{t^2}\,dt \leqslant \gamma\, e^{-KS_1(w)} \int_{\overline{\mathscr{A}(w,\gamma)}} \frac{dt}{t} + c_2\, e^{-KS_1(w)} \int_{\mathscr{A}(w,\gamma)} \frac{dt}{t}. \tag{51}$$

But the first t-integral on the right does not exceed $c_1 \log w$, and we may hence infer from (51) and (47) that there exists an absolute constant $\gamma > 0$ such that

$$\int_{\mathscr{A}(w,\gamma)} \frac{dt}{t} \gg \log w \tag{52}$$

and hence
$$\int_{\mathscr{A}(w,\gamma)} \frac{dt}{t\log t} \gg 1. \tag{53}$$

We now reach the last part of the proof. We fix x large and put $w = x^{1/2c_1}$. Since $|G_1(t)-G_1(s)| \leqslant 1+s-t$ for $1 \leqslant t \leqslant s$, we plainly have

$$t\in\mathscr{A}(w,\gamma) \quad \text{implies} \quad \left[t, t\left(1+\frac{1}{\log^2 x}\right)\right] \subseteq \mathscr{A}(w, \tfrac{1}{2}\gamma).$$

Thus there is a finite sequence $t_1, t_2, \dots, t_r$ with $t_{j+1}/t_j \geqslant (1+1/\log^2 x)$ for all j, such that

$$\mathscr{A}(w,\gamma) \subseteq \bigcup_{j=1}^{r}\left[t_j, t_j\left(1+\frac{1}{\log^2 x}\right)\right] \subseteq \mathscr{A}(w,\tfrac{1}{2}\gamma). \tag{54}$$

In particular it follows from (53) that

$$\frac{r}{\log^3 x} \gg \sum_{j=1}^{r}\int_{t_j}^{t_j(1+1/\log^2 x)} \frac{dt}{t\log t} \gg 1. \tag{55}$$

However, by the prime number theorem, we have for every $j = 1, 2, \dots, r$

$$\sum_{t_j \leqslant x/p \leqslant t_j(1+1/\log^2 x)} \frac{1}{p} \gg \frac{1}{(\log x)^3}. \tag{56}$$

It therefore follows from (54), (55) and (56) that

$$\sum_{x/p \in \mathscr{A}(w,\frac{1}{2}\gamma)} \frac{1}{p} \gg 1. \tag{57}$$

We split the above sum into four parts corresponding to the different possibilities of signs for $\Re e\{e^{i\phi}G_1(x/p)\}$ and $\Re e\{e^{i\phi}G_1(x/p)\} - \Im m\{e^{i\phi}G_1(x/p)\}$. One of these is at least $\frac{1}{4}$ of the whole sum and we admit for instance that it corresponds to $\Re e\,\{e^{i\phi}G_1(x/p)\} \geqslant \max(0, \Im m\{e^{i\phi}G_1(x/p)\})$. Thus we obtain

$$\sum_p{}^{(*)} \frac{1}{p} \gg 1 \tag{58}$$

where the asterisk denotes that summation is restricted to those p such that

$$(*)\begin{cases} w^{c_0} < \dfrac{x}{p} \leqslant w^{c_1} = \sqrt{x} \\[2ex] \Re e\left\{e^{i\phi}G_1\left(\dfrac{x}{p}\right)\right\} \geqslant \frac{1}{8}\gamma\dfrac{x}{p}e^{-KS_1(w)} \gg \dfrac{x}{p}e^{-KS_1(x)}. \end{cases}$$

We now define a multiplicative function g_2, supported on squarefree numbers, by

$$\begin{aligned} g_2(p) &= g_1(p), && \text{for } p \leqslant w, \\ g_2(p) &= e^{i\phi}, && \text{for } p \text{ counted in (58)}, \\ g_2(p) &= 0, && \text{otherwise.} \end{aligned}$$

Plainly, $g_2 \in \mathscr{G}(\delta,\phi)$. Since condition (∗) implies $p > \sqrt{x}$, we have

$$G_2(x) := \sum_{n \leqslant x} g_2(n) = \sum_{n \leqslant x} g_1(n) + {\sum_p}^{(*)} e^{i\phi} \sum_{n \leqslant x/p} g_1(n)$$

$$= G_1(x) + {\sum_p}^{(*)} e^{i\phi} G_1\left(\frac{x}{p}\right).$$

From condition (∗) and (58) it follows that

$$\Re e\,(G_2(x) - G_1(x)) \gg x\,e^{-KS_1(x)} {\sum}^{(*)} \frac{1}{p}$$

$$\gg x\,e^{-KS_1(x)}.$$

Hence either g_1 or g_2 fulfils conditions (i) and (ii) of the theorem, and the proof is thereby completed.

4. *On the function* $K(\delta,\phi)$

For $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \delta \leqslant 1$, $0 \leqslant K \leqslant 1$ define

$$G(K) = G(K,\delta,\phi) := K + \frac{1}{2\pi}\int_0^{2\pi} |W(e^{i\theta} - K)|\,d\theta. \tag{59}$$

For $\delta = 1$, we have $W(e^{i\theta}-K) = e^{i\theta}-K$ and $K = 0$ is the only root of the equation $G(K) = 1$. We hence assume from this point on that $\delta < 1$. We have with this hypothesis

$$|W(e^{i\theta})|^2 = \cos^2(\theta-\phi) + \delta^2 \sin^2(\theta-\phi) < 1 \quad (\theta \not\equiv \phi (\mathrm{mod}\,\pi)),$$

hence $G(0) < 1 < G(1)$. Moreover, the triangle inequality implies that, as a function of K, $|W(e^{i\theta}-K)| = |W(e^{i\theta}) - KW(1)|$ satisfies a Lipchitz condition with modulus $|W(1)|$, so $\partial G/\partial K > 0$ unless perhaps when $\phi = 0$. In this latter case, the Lipchitz condition is strict unless $\theta \equiv 0\,(\mathrm{mod}\,\pi)$, whence again $\partial G/\partial K > 0$. We may therefore define $K(\delta,\phi)$ to be the unique solution of the equation

$$G(K,\delta,\phi) = 1.$$

We also observe that, for any fixed z,

$$|W(z)| = \sqrt{(\Re e\,(z\,e^{i\phi})^2 + \delta^2\,\Im m\,(z\,e^{i\phi})^2)}$$

in an increasing function of δ, hence $\partial G/\partial\delta \geqslant 0$ and $K(\delta,\phi)$ is a decreasing function of δ.

Our main purpose in this section is to show that for all δ,ϕ we have

$$K(\delta,\phi) \leqslant K(0,\tfrac{1}{2}\pi) = 1 - 2/\pi \tag{60}$$

and

$$\tfrac{1}{4}(1-\delta^2) \leqslant K(\delta,\phi) \leqslant \tfrac{1}{2}(1-\delta^2). \tag{61}$$

We begin with (60). Since $K(\delta,\phi) \leqslant K(0,\phi)$, we only have to prove that, for any fixed $K, G(K,0,\phi)$ is minimal when $\phi = \pi/2$. But

$$G(K,0,\phi) = K + \frac{1}{2\pi}\int_0^{2\pi} |\cos\theta - K\cos\phi|\,d\theta$$

$$= K + \frac{2}{\pi}\left\{\sin\beta + \left(\frac{\pi}{2} - \beta\right)\cos\beta\right\}$$

with $\beta := \operatorname{Arccos}(K\cos\phi)$. The function of β between curly brackets attains its minimum value 1 when $\beta = \frac{1}{2}\pi$, which corresponds to $\phi = \frac{1}{2}\pi$, and this is all we need.

Next, we show (61). To this end, define, for $r \geqslant 0$,

$$M(r) = \frac{1}{2\pi}\int_0^{2\pi} |W(e^{i\theta}-K)|^r \, d\theta. \tag{62}$$

We have

$$M(2) = \tfrac{1}{2}(1+\delta^2)+K^2(\cos^2\phi+\delta^2\sin^2\phi) \leqslant \tfrac{1}{2}(1+\delta^2)+K^2. \tag{63}$$

When $K = K(\delta,\phi)$ we have $M(1) = 1-K$ and by Cauchy's inequality, $M(2) \geqslant (1-K)^2$. It follows from (63) that

$$\tfrac{1}{2}(1+\delta^2) \geqslant 1-2K$$

which is equivalent to the lower bound in (61). To obtain the upper bound, it will be sufficient to show that

$$G(\tfrac{1}{2}(1-\delta^2),\delta,\phi) \geqslant 1 \tag{64}$$

or equivalently

$$M(1) \geqslant \tfrac{1}{2}(1+\delta^2). \tag{65}$$

By Hölder's inequality, $M(2) \leqslant M(1)^{2/3}M(4)^{1/3}$ and so (65) will follow from

$$M(2)^3 \geqslant \tfrac{1}{4}(1+\delta^2)^2 M(4). \tag{66}$$

Let us set

$$t := (1-\delta^2)^2\sin^2\phi. \tag{67}$$

From (63), on substituting $K = \frac{1}{2}(1-\delta^2)$, we have

$$M(2) = \tfrac{3}{4}+\tfrac{1}{4}\delta^4-\tfrac{1}{4}(1-\delta^2)\,t. \tag{68}$$

To compute $M(4)$, define

$$\begin{aligned} F(\theta) &= |W(e^{i\theta}-K)|^2 \\ &= M(2)-2K(\cos\phi\cos(\theta-\phi)-\delta^2\sin\phi\sin(\theta-\phi))+\tfrac{1}{2}(1-\delta^2)\cos 2(\theta-\phi). \end{aligned} \tag{69}$$

We have, from (62) and (69),

$$\begin{aligned} M(4) &= \frac{1}{2\pi}\int_0^{2\pi} F(\theta)^2\, d\theta \\ &= M(2)^2+2K^2(\cos^2\phi+\delta^4\sin^2\phi)+\tfrac{1}{8}(1-\delta^2)^2, \end{aligned} \tag{70}$$

and we substitute $K = \frac{1}{2}(1-\delta^2)$ and simplify, employing (67), to obtain

$$M(4) = M(2)^2+\tfrac{5}{8}(1-\delta^2)^2-\tfrac{1}{2}(1-\delta^4)\,t. \tag{71}$$

Therefore (66) is equivalent to

$$M(2)^2(M(2)-\tfrac{1}{4}(1+\delta^2)^2) \geqslant \tfrac{1}{4}(1-\delta^2)^2\{\tfrac{5}{8}(1-\delta^2)^2-\tfrac{1}{2}(1-\delta^4)\,t\}$$

or

$$M(2)^2(2-t) \geqslant (1-\delta^2)^2\{\tfrac{5}{8}(1-\delta^2)-\tfrac{1}{2}(1-\delta^2)\,t\}. \tag{72}$$

We substitute the right-hand side of (68) for $M(2)$ and arrange the desired inequality as a cubic in t; we require that (after multiplication through by 8),

$$\begin{aligned} &\{(3+\delta^4)^2-5(1+\delta^2)(1-\delta^4)\}-\{\tfrac{13}{2}-18\delta^2-7\delta^4-6\delta^6+\tfrac{1}{2}\delta^8\}\,t \\ &\qquad +\{(1-\delta^2)(4-\delta^2+\delta^4)\}\,t^2-\tfrac{1}{2}(1-\delta^2)^2t^3 \geqslant 0. \end{aligned} \tag{73}$$

Table 1

δ	$K(\delta,0)$	$(1-\delta^2)/K(\delta,0)$
0	0·32867...	3·042...
0·05	0·326...	3·057...
0·1	0·320...	3·088...
0·15	0·312...	3·126...
0·2	0·302...	3·170...
0·25	0·291...	3·218...
0·3	0·278...	3·267...
0·35	0·264...	3·318...
0·4	0·249...	3·370...
0·45	0·232...	3·423...
0·5	0·215...	3·477...
0·55	0·197...	3·530...
0·6	0·178...	3·583...
0·65	0·158...	3·637...
0·7	0·138...	3·690...
0·75	0·116...	3·742...
0·8	0·094...	3·795...
0·85	0·072...	3·847...
0·9	0·048...	3·898...
0·95	0·024...	3·950...
1	0	4

Since $t \leqslant 1$ the sum of the last two terms on the left is at least $\frac{1}{2}(1-\delta^2)(7-\delta^2+2\delta^4)t^2$ and so it will be sufficient to show that

$$\{8-10\delta^2+22\delta^4+10\delta^6+2\delta^8\}-\{13-36\delta^2-14\delta^4-12\delta^6+\delta^8\}t$$
$$+\{(1-\delta^2)(7-\delta^2+2\delta^4)\}t^2 \geqslant 0 \quad (74)$$

and we notice that when $\delta^2 \geqslant 1/3$ this is trivially true because all three coefficients are positive, and $0 \leqslant t \leqslant 1$. Denoting the above quadratic by at^2+bt+c, we have

$$b^2-4ac = -55-400\delta^2-100\delta^4+1304\delta^6+1034\delta^8+384\delta^{10}+172\delta^{12}-8\delta^{14}+\delta^{16}. \quad (75)$$

This is negative when $\delta^2 < 1/3$. and the quadratic is positive for all t. It follows that (64), and hence the right-hand inequality in (61), hold.

We append the table of $K(\delta,0)$ (Table 1) kindly provided by M. G. J. van der Burg.

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