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On a class of differential–difference equations arising in number theory

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1. Introduction

Functions defined by differential-difference equations of the type

$$uf'(u) + af(u) + bf(u-1) = 0, \tag{1.1}$$

where a and b are constants, arise frequently in number theory. Probably the best known example is the Dickman function $\varrho(u)$, which is defined for $u \geqslant 0$ as the unique continuous solution of the system

$$\begin{aligned} \varrho(u) &= 1 \qquad (0 \leqslant u \leqslant 1), \\ u\varrho'(u) + \varrho(u-1) &= 0 \qquad (u > 1). \end{aligned}$$

It is defined arithmetically as the asymptotic proportion of integers n having no prime factors greater than $n^{1/u}$ [Di][dB3], and it may also be given a probabilistic interpretation [He][Wh]. De Bruijn [dB4] showed that, as $u \to \infty$,

$$\varrho(u) \sim \frac{e^{\gamma}}{\sqrt{2\pi u}} \exp\left\{ -u\xi + \int_0^{\xi} \frac{e^t - 1}{t}\, \mathrm{d}t \right\}, \tag{1.2}$$

where γ denotes Euler's constant and $\xi = \xi(u)$ is the unique positive solution to the equation

$$e^{\xi} = 1 + u\xi. \tag{1.3}$$

The function $\xi(u)$ is non-elementary, but standard techniques of asymptotic analysis (see [dB6, §2.4]) show that for all sufficiently large u, $\xi(u)$ is represented by the convergent series

$$\xi(u) = \log u + \log_2 u + \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} c_{mk} \left(\frac{1}{\log u} \right)^m \left(\frac{1 + u\log_2 u}{u \log u} \right)^k,$$

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where $\log_k$ denotes the k-fold iterated logarithm and

$$c_{mk} := \binom{m+k}{m} \operatorname{Res}\left\{ \frac{z^m}{(e^z-1)^{m+k}}\Big(\frac{ze^z}{e^z-1} - \frac{m}{m+k}\Big); 0 \right\}.$$

Computing the first few coefficients c_{mk} yields the estimate

$$\text{(1.4)}\quad \xi(u) = \log u + \log_2 u + \frac{\log_2 u}{\log u} - \frac{\frac{1}{2}\log_2^2 u - \log_2 u}{(\log u)^2} + O\Big(\frac{(\log_2 u)^2}{(\log u)^3}\Big) \quad (u \geqslant u_0).$$

A slightly weaker estimate is given in [dB4].

Another well-known example of a function satisfying (1.1) is the Buchstab function $\omega(u)$, which is defined for $u \geqslant 1$ as the unique continuous solution of the system

$$\omega(u) = \frac{1}{u} \quad (1 \leqslant u \leqslant 2),$$
$$u\omega'(u) + \omega(u) - \omega(u-1) = 0 \quad (u > 2).$$

This function arises when estimating the frequency of integers n whose smallest prime factor is $\geqslant n^{1/u}$ [Bu][dB2]. It is easy to show (see, e.g., [dB2]) that, as $u \to \infty$, $\omega(u)$ converges exponentially to $e^{-\gamma}$, where γ is Euler's constant. The finer behavior of the difference $\omega(u) - e^{-\gamma}$ has recently been investigated by several authors [Ma] [CG] [Hi] ; it turns out that $\omega(u) - e^{-\gamma}$ behaves essentially like a periodic function with period 2, dampened by an exponentially decaying factor. Oscillation results of this type play an important role in proving irregularity results on the distribution of primes [Ma] [FGHM] [CG] [Hi].

Functions satisfying (1.1) when $a + b$ is an integer arise in sieve theory, and in this context have been investigated by various authors ; see, for example, [AO] [JR] [Iw] [GR] [DHR] [Wh]. Alladi [Al] and Hensley [He] obtained asymptotic formulae of the type (1.2) for a particular class of such functions satisfying (1.1) with $(a, b) = (-\kappa + 1, \kappa)$, where κ is any positive number. Wheeler [Wh] gave similar formulae for the parameter pairs $(a, b) = (-\kappa + n, \kappa)$, for any positive real number κ and positive integer n. He also obtained an asymptotic formula, of a rather different shape, for the case when $a + b$ is not an integer. The first author [Hi] recently extended the results of Alladi and Hensley to the cases $(a, b) = (-z, z)$ and $(a, b) = (-z + 1, z)$, where z is any non-zero complex number with $\operatorname{Re} z > -1$ or $z = -1$. The latter result covers, in particular, the Buchstab function.

The equation (1.1) with general coefficients a and b appears to have been first studied by Iwaniec [Iw] in connection with his work on sieve theory, and a more systematic investigation has been undertaken recently by Wheeler [Wh]. Iwaniec and Wheeler both make extensive use of the "adjoint equation"

$$\text{(1.5)}\qquad ug'(u) + (1 - a)g(u) - bg(u + 1) = 0,$$

which, in a sense, is easier to deal with that (1.1). Since solutions to the equations (1.1) and (1.5) are connected via a simple integral relation (see section 3), one can derive asymptotic information for a given solution to (1.1) by studying the asymptotic behavior of a suitable solution to (1.5).

Despite the considerable amount of literature on functions satisfying equations of the type (1.1), most results deal only with *particular* solutions to this equation which arise naturally in some number-theoretic context. The object of the present paper is to describe, for any given pair of complex coefficients (a, b) with $b \neq 0$, the structure and asymptotic behavior of the *general* solution to (1.1).

The problem of describing the general solution to (1.1) amounts to describing the solutions $f(u) = f(u; \varphi)$ to systems of the type

$$f(u) = \varphi(u) \qquad (u_0 - 1 \leqslant u \leqslant u_0), \tag{1.6}$$

$$uf'(u) + af(u) + bf(u-1) = 0 \qquad (u > u_0), \tag{1.7}$$

where u_0 is any positive real number and $\varphi(u)$ is any given continuous function on $[u_0 - 1, u_0]$. Indeed, rewriting (1.7) in the equivalent form

$$(f(u)u^a)' = -bu^{a-1}f(u-1) \qquad (u > u_0), \tag{1.7$'$}$$

it is easily seen this system has a unique continuous solution $f(u) = f(u; \varphi)$ for $u \geqslant u_0 - 1$; conversely any function $f(u)$ satisfying (1.1) for sufficiently large u is a solution to this system with a suitable choice of u_0 and with $\varphi = f$ as initial function on $[u_0 - 1, u_0]$. We shall construct a set of "fundamental" solutions $F(u) = F(u; a, b)$ and $F_n(u) = F_n(u; a, b)$ $(n \in \mathbb{Z})$, which span the solution space of (1.1) in the sense that any solution $f(u; \varphi)$ of the system (1.6)-(1.7) can be expressed as a convergent series

$$f(u) = \alpha F(u) + \sum_{n \in \mathbb{Z}} \alpha_n F_n(u) \tag{1.8}$$

with suitable coefficients α and α_n depending on the initial function φ.

Series expansions of solutions of differential difference equations have been obtained in the literature for various special equations. A complete theory exists for the case of a retarded differential difference equation with constant (or asymptotically constant) coefficients : the general solution of such an equation can be expanded into a series of exponential solutions e^{su}, where s runs through the roots of the associated characteristic function (see [BC]). De Bruijn [dB5] gave an expansion quite similar to (1.8) for the solutions to the differential difference equation $f'(x) = e^{ax+b}f(x-1)$. His approach, however, is different from the one we shall use. While preparing the present paper, the authors were made aware of a thesis by Beenakker [Be], who applies de Bruijn's method to the equation (1.1) in the case $a = 0$.

The functions F and F_n in (1.8) can be estimated rather precisely. Using such estimates, we will be able to describe the asymptotic behavior of the general solution to (1.1) and its dependence on the initial function $\varphi(u)$. For example, we will show (see Corollary 6) that if $f(u)$ is a real-valued function which satisfies the differential-difference equation of the Dickman function (i. e. (1.1) with $(a,b) = (0,1)$) for sufficiently large u and is of order $o(1/u)$ as $u \to \infty$, then there exists a constant c such that the function $\big(f(u)/\varrho(u)\big) - c$ is either identically zero for sufficiently large u or oscillates in a rather regular fashion with asymptotically constant differences between consecutive sign changes and an exponentially decreasing amplitude.

The plan for the remainder of this paper is as follows. In Section 2, we shall define the functions F and F_n and formulate asymptotic estimates for these functions (Theorem 1). In Section 3, we will state our principal result (Theorem 2), which gives an expansion of the form (1.8) for the general solution to the system (1.6), (1.7), and we shall derive several corollaries from this result. We shall prove Theorem 1 in Section 4 and Theorem 2 in the final two sections.

Notation and conventions. Throughout this paper we fix a positive number $\varepsilon < 1$ and complex numbers a and b satisfying

$$|a| \leqslant \frac{1}{\varepsilon}, \quad \varepsilon \leqslant |b| \leqslant \frac{1}{\varepsilon}. \tag{1.9}$$

We define real numbers h, κ and ϑ by

$$h = a + b, \quad b = \kappa e^{i\vartheta} \quad (\kappa > 0,\ -\pi < \vartheta \leqslant \pi). \tag{1.10}$$

We shall use the notations $f = O(g)$ and $f \ll g$ interchangeably to mean that $|f| \leqslant cg$ holds with some constant c in the range under consideration, and we write $f \asymp g$ if $f \ll g$ and $g \ll f$ hold. The constants here may depend at most on ε, unless specified otherwise. By $w_0(n)$, $u_0(\varepsilon)$, etc., we shall denote generic sufficiently large constants depending on the indicated parameters.

We denote the sets of positive and negative integers, excluding zero, by $\mathbb{Z}^+$ and $\mathbb{Z}^-$, respectively, and the sets of positive and negative reals by $\mathbb{R}^+$ and $\mathbb{R}^-$.

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2. Fundamental Solutions

With ϑ defined by (1.10), we consider the angular path

$$\mathcal{C} = [-\infty + (\vartheta - \pi)i,\ (\vartheta - \pi)i,\ 0],$$

where, here and in the sequel, we use the notation $[z_1, z_2, \ldots, z_k]$ to denote the contour going from z_1 to z_k via $z_2, \ldots, z_{k-1}$ by straight line segments. We set

$$\lambda(s) = \lambda(s; a, b) := e^{bI(-s)} s^h, \tag{2.1}$$

where

$$I(s) := \int_0^s \frac{e^z - 1}{z}\, \mathrm{d}z$$

is the exponential integral arising in de Bruijn's formula (1.2), h is given by (1.10), and s^h is defined using the (unique) continuous branch of arg(s) in $\mathbb{C} \setminus \mathcal{C}$ which is equal to 0 for positive values of s. Thus $\lambda(s)$ is an entire funtion if $h \in \mathbb{Z}^+ \cup \{0\}$, analytic in $\mathbb{C} \setminus \{0\}$ if $h \in \mathbb{Z}^-$, and analytic in $\mathbb{C} \setminus \mathcal{C}$ with a branch cut at $\mathcal{C}$ if $h \notin \mathbb{Z}$.

For any integer n we define

$$F_n(u) = F_n(u; a, b) := \frac{1}{2\pi i} \int_{\Gamma_n} \lambda(s) e^{us} \frac{\mathrm{d}s}{s}, \tag{2.2}$$

where, in the above notation,

$$\Gamma_n = \Gamma + i\vartheta_n, \quad \Gamma = [-\infty - i\pi+,\ -1 - i\pi+,\ -1 + i\pi-,\ -\infty + i\pi-]$$

with

$$\vartheta_n = \vartheta + 2n\pi. \tag{2.3}$$

Note that the upper horizontal portion of Γ_{-1} is taken below the branch cut $\mathcal{C}$, whereas the lower horizontal portion of Γ_0 is taken above $\mathcal{C}$. We further set

$$F(u) = F(u; a, b) = \begin{cases} \dfrac{\Gamma(1-h)}{2\pi i} \displaystyle\int_{\Gamma^*} \lambda(s) e^{us} \frac{ds}{s} & \text{if } h \notin \mathbb{Z}^+, \\ -\dfrac{e^{\pi i h}}{\Gamma(h)} \displaystyle\int_{\Gamma^0} \lambda(s) e^{us} \frac{\mathrm{d}s}{s} & \text{if } h \in \mathbb{Z}^+, \end{cases} \tag{2.4}$$

where Γ^* is a positively oriented Hankel contour around the branch cut $\mathcal{C}$, and Γ^0 is a path going from $-\infty + i(\vartheta - \pi)-$ to 0 below $\mathcal{C}$, as indicated in the figure.

The estimate

$$I(s) \sim \frac{e^s}{s} \quad (\operatorname{Re} s \to \infty, \quad |\operatorname{Im} s| \ll 1), \tag{2.5}$$

which follows easily by partial integration from the definition of $I(s)$ (cf. Lemma 2 below), implies that the function $e^{bI(-s)}$ tends to zero at faster than exponential rate as $\operatorname{Re} s \to -\infty$ in a strip $|\operatorname{Im} s - (\vartheta_n + \pi)| \leqslant \frac{1}{2}\pi - \delta$, for any fixed $\delta > 0$ and integer n. Since the infinite portions of the contours Γ_n, Γ^*, and Γ^0 fall into one of these strips, and the integrand in the second integral of (2.4) is bounded near $s = 0$ for $h \in \mathbb{Z}^+$, all three integrals in (2.2) and (2.4) are absolutely convergent for any real or complex value of u, and the functions F_n and F are entire functions of u. Using integration by parts as in Iwaniec [Iw ; p.18] it is easily checked that each of the functions F_n and F is indeed a solution to (1.1). We shall, however, not make use of this fact.

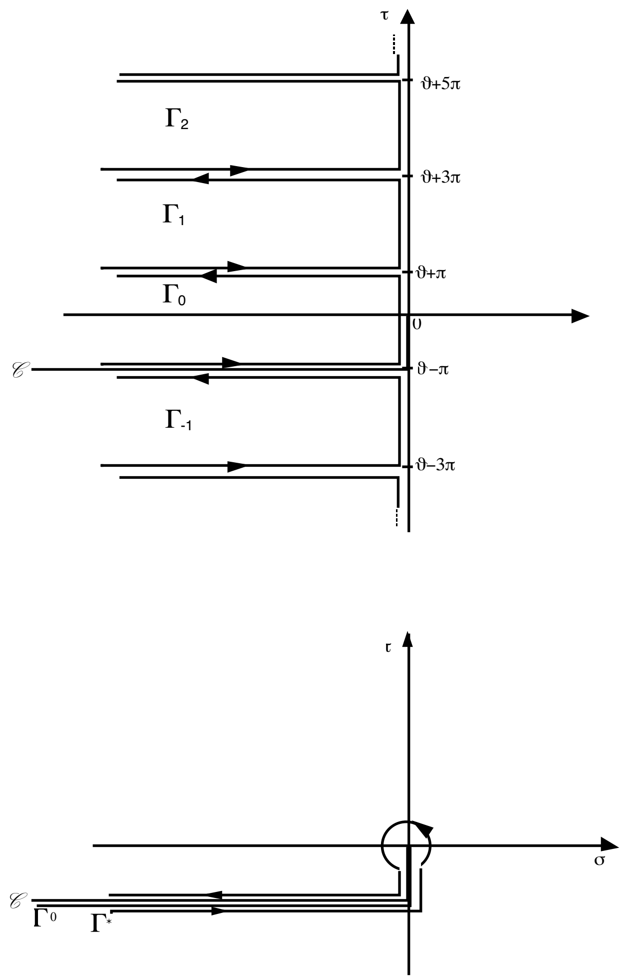

Figure.— *Contours.*

We define a similar set of fundamental solutions G_n and G to the adjoint equation (1.5) by

$$G_n(u) = G_n(u; a, b) := \int_{\Delta_n} \frac{e^{-us}}{\lambda(s)}\ \mathrm{d}s\,, \tag{2.6}$$

$$\text{(2.7)}\qquad G(u) = G(u; a, b) := \begin{cases} \dfrac{e^{\pi i h}\Gamma(h)}{2\pi i} \displaystyle\int_{\Delta^*} \dfrac{e^{-us}}{\lambda_0(s)}\,\mathrm{d}s, & \text{if } h \notin \mathbb{Z}^- \cup \{0\}\,, \\ \dfrac{1}{\Gamma(1-h)} \displaystyle\int_{\Delta^0} \dfrac{e^{-us}}{\lambda_0(s)}\,\mathrm{d}s & \text{if } h \in \mathbb{Z}^- \cup \{0\}\,, \end{cases}$$

where $\lambda_0(s)$ is defined by the same formula as $\lambda(s)$, namely

$$\lambda_0(s) := e^{bI(-s)}s^h,$$

but with s^h defined by taking $\arg(s) \in (0, 2\pi)$, so that $\lambda_0(s)$ is analytic in $\mathbb{C} \setminus \mathbb{R}^+ \cup \{0\}$. Here Δ_n is defined for $n \neq 0$ as the horizontal path $[-\infty + i\vartheta_n, +\infty + i\vartheta_n]$, and for $n = 0$ as the path

$$\Delta_0 := [-\infty + i\vartheta, -1 + i\vartheta, -1 + i\pi, 1 + i\pi, 1 + i\vartheta, +\infty + i\vartheta],$$

which avoids the branchcut $\mathcal{C}$. Thus Δ_n lies above $\mathcal{C}$ if $n \geqslant 0$, and below $\mathcal{C}$ if $n < 0$. The contour Δ^* is defined as a negatively oriented Hankel contour around $\mathbb{R}^+$, and Δ^0 a contour going from 0 to $+\infty$ above the real axis. As in the case of the functions F_n and F, it is easily seen that the integrals in (2.6) and (2.7) are absolutely convergent for any real or complex number u with positive real part, and that the functions $G(u)$ and $G_n(u)$ defined in this way are analytic in $\operatorname{Re} u > 0$ and satisfy (1.5) in this half plane.

We note that the functions $G(u)$ coincide, up to a constant factor, with the function $q(s)$ introduced by Iwaniec [Iw ; p.181] (see also [Wh]).

Remarks. (i) The second formula on the right of (2.4) may be obtained as a limiting case of the first. Indeed, if $\operatorname{Re} h > 0$ then the integral over Γ^0 in (2.4) is absolutely convergent, depends continuously on h, and we have

$$\int_{\Gamma^*} e^{us}\lambda(s)\frac{\mathrm{d}s}{s} = (1 - e^{2\pi i h})\int_{\Gamma^0} e^{us}\lambda(s)\frac{\mathrm{d}s}{s}.$$

For $\operatorname{Re} h > 0$ and $h \notin \mathbb{Z}^+$ the first formula in (2.4) therefore becomes

$$\begin{aligned} \frac{\Gamma(1-h)}{2\pi i}\int_{\Gamma^*} e^{us}\lambda(s)\frac{\mathrm{d}s}{s} &= \frac{\Gamma(1-h)(1-e^{2\pi i h})}{2\pi i}\int_{\Gamma^0} e^{us}\lambda(s)\frac{\mathrm{d}s}{s} \\ &= -\frac{\Gamma(1-h)e^{\pi i h}\sin\pi h}{\pi}\int_{\Gamma^0} e^{us}\lambda(s)\frac{\mathrm{d}s}{s} \\ &= -\frac{e^{\pi i h}}{\Gamma(h)}\int_{\Gamma^0} e^{us}\lambda(s)\frac{\mathrm{d}s}{s} \end{aligned}$$

by the reflection formula for the Gamma function, and thus coincides with the second formula. It can be shown similarly that the second expression on the right hand side of (2.7) is well defined for $\operatorname{Re} h < 1$ and coincides with the first whenever $\operatorname{Re} h < 1$ and $h \notin \mathbb{Z}^- \cup \{0\}$.

(ii) It is easily checked that the functions $F_n(u;a,b)$ and $G_n(u;a,b)$ are continuous in a and b, and in fact uniformly continuous in any compact u-interval in $(0,+\infty)$. In view of the previous remark, the same is true for the functions $F(u;a,b)$ and $G(u;a,b)$.

(iii) We have

$$F_n'(u;a,b) = F_n(u;a+1,b) \quad (n \in \mathbb{Z}), \tag{2.8}$$

$$G_n'(u;a,b) = -G_n(u;a-1,b) \quad (n \in \mathbb{Z}), \tag{2.9}$$

where the derivatives are taken with respect to u. This follows immediately from the definitions of F_n and G_n.

(iv) If a and b are real then we have the relations

$$F_{-n} = \begin{cases} \overline{F_n} & \text{if } b > 0, \\ \overline{F_{n-1}} & \text{if } b < 0, \end{cases} \tag{2.10}$$

and

$$G_{-n} = \begin{cases} \overline{G_n} & \text{if } b > 0, \\ \overline{G_{n-1}} & \text{if } b < 0, \end{cases} \tag{2.11}$$

for any positive integer n. To see the first relation in (2.10), note that in the case $b > 0$ (so that $\vartheta = 0$) the paths Γ_n and Γ_{-n} lie symmetrically with respect to the real axis, and that replacing $s \in \Gamma_{-n}$ by $\overline{s} \in \Gamma_n$ changes the integrand $e^{us}\lambda(s)/s = e^{us+bI(-s)}s^{h-1}$ into its complex conjugate if a and b are real. A similar argument gives the other relations.

In Theorem 1 below we will give asymptotic estimates for the fundamental solutions $F_n(u)$ and $F(u)$. The estimates for F_n are of the same form as de Bruijn's estimate (1.2) for the Dickman function, but with the quantity $\xi = \xi(u)$ replaced by an appropriate *complex* solution to the equation

$$e^{\zeta} = 1 + w\zeta \tag{2.12}$$

(with $w = u/b$) in a certain region of the complex plane depending on n. To this end we shall first establish the following result, which proves the existence and uniqueness of such a solution for sufficiently large $|w|$ and gives an asymptotic estimate for ζ.

Lemma 1. *Let n be a fixed integer. For $u > 1$, let $\xi(u)$ denote the unique positive solution to* (1.3). *There exists a constant $w_0 = w_0(n)$ such that for all complex numbers $w = \varrho e^{i\varphi}$ satisfying*

$$\varrho \geqslant w_0(n), \qquad -\pi \leqslant \varphi < \pi,$$

the equation (2.12) has a unique solution ζ_n in the disk

$$|\zeta_n - (\xi + i\varphi_n)| \leqslant \pi \tag{2.13}$$

with $\xi = \xi(\varrho)$ and $\varphi_n = \varphi - 2n\pi$. Moreover, we have

$$\zeta_n = \xi + \frac{\varphi_n^2}{2\xi^2} + i\frac{\xi\varphi_n}{\xi - 1} + O\big(\frac{1}{\xi^3}\big), \tag{2.14}$$

$$\frac{\mathrm{d}\zeta_n}{\mathrm{d}w} = \frac{\zeta_n}{w(\zeta_n - 1)}\Big(1 + O\big(\frac{1}{w\xi}\big)\Big). \tag{2.15}$$

Setting

$$\Phi(u,s) = \Phi(u,s;a,b) = \frac{\exp\{-us + bI(s)\}\, s^{h-1}}{\sqrt{2\pi u(1-1/s)}} \tag{2.16}$$

we then have the following result.

Theorem 1.
(i) *For any fixed non-zero integer n and $u \geqslant u_0(\varepsilon, n)$, we have*

$$F_n(u;a,b) = \big(1 + O(\frac{1}{u})\big)\Phi(u,\zeta_n;a,b) \tag{2.17}$$

where $\zeta_n = \zeta_n(u/b)$ is defined as in Lemma 1 with $(\varrho, \varphi) = (u/\kappa, -\vartheta)$ and the implied constant depends at most on ε and n.
(ii) *For any fixed positive integer K and $u \geqslant u_0(\varepsilon)$ we have*

$$F(u;a,b) = u^{-h}\Big\{1 + \sum_{k=1}^{K} \frac{b_k\Gamma(1-h)}{\Gamma(1-h-k)}u^{-k} + O(u^{-K-1})\Big\} \tag{2.18}$$

where b_k $(k = 1, 2, \ldots)$ is the k-th Taylor coefficient of $\exp\{bI(-s)\}$ and the implied constant depends at most on ε and K. Moreover, if $h \in \mathbb{Z}^- \cup \{0\}$, then $F(u;a,b)$ is a polynomial of degree $-h$ whose coefficients are given by (2.18).

It is easy to obtain more explicit estimates for the functions on the right hand side of (2.17). For example, if we put

$$T(s) = 1 + \frac{1}{s} + \frac{2}{s^2}, \tag{2.19}$$

it is easily checked (cf. Lemma 2) that

$$I(s) = \frac{e^s - 1}{s}T(s) + O\big(\frac{e^\sigma}{\sigma^4}\big) \qquad (\sigma \geqslant 1).$$

In particular, taking in turn $s=\zeta_n$ and $s=\xi=\xi(u/\kappa)$ and estimating the error term by Lemma 1, we obtain

$$bI(\zeta_n)=uT(\zeta_n)+O\big(\frac{u}{\xi^3}\big),$$
$$\kappa I(\xi)=uT(\xi)+O\big(\frac{u}{\xi^3}\big),$$

so that

$$\Phi(u,\zeta_n;a,b)=\exp\left\{-u\left(\zeta_n-T(\zeta_n)+O\big(\frac{1}{\xi^3}\big)\right)\right\}$$
$$\Phi(u,\xi;a,b)=\exp\left\{-u\left(\xi-T(\xi)+O\big(\frac{1}{\xi^3}\big)\right)\right\}.$$

Applying the mean value theorem and (2.14) to estimate $T(\zeta_n)-T(\xi)$, it readily follows that

$$\frac{\Phi(u,\zeta_n;a,b)}{\Phi(u,\xi;a,\kappa)}=\exp\Big\{-\frac{u\vartheta_n^2}{2\xi^2}+iu\vartheta_nT(\xi)+O\big(\frac{u}{\xi^3}\big)\Big\}, \tag{2.20}$$

where ϑ_n is as in (2.3). (Observe that, in the notation of Lemma 1, $\vartheta_n=-\varphi_n$, with $\varphi=-\vartheta$.) Combining these estimates with (2.17) and (2.14), we obtain the following corollary.

Corollary 1. *For any fixed integer n and $u\geqslant u_0(\varepsilon,n)$, we have*

$$F_n(u)=F_0(u)\exp\Big\{-u\left(\frac{\vartheta_n^2-\vartheta^2}{2\xi^2}-2n\pi T(\xi)i+O\big(\frac{1}{\xi^3}\big)\right)\Big\} \tag{2.21}$$

where ϑ_n is as in (2.3). Moreover,

$$F_0(u)=\exp\Big\{-u\Big(\xi+\frac{\vartheta^2}{2\xi^2}-(1+i\vartheta)T(\xi)+O\big(\frac{1}{\xi^3}\big)\Big)\Big\}. \tag{2.22}$$

Using the expansion (1.4) for $\xi(u/\kappa)$, one can replace the expressions on the right of (2.21) and (2.22) by a slightly more complicated expression involving only elementary functions.

Analogous results could be proved for the functions $G_n(u)$ and $G(u)$; for example, it can be shown that

$$G_n(u;a,b)=\Big(1+O\Big(\frac{1}{u}\Big)\Big)\Psi(u,\zeta_n;a,b), \tag{2.23}$$

where

$$\Psi(u,s)=\Psi(u,s;a,b)=\frac{\sqrt{2\pi}\exp\{us-bI(s)\}\,s^{-h}}{\sqrt{u(1-1/s)}}=\big(\Phi(u,s;a,b)su(1-1/s)\big)^{-1}$$

Thus, apart from the factor $\zeta_n u$, the functions F_n and G_n are essentially reciprocals of each other. The proof of (2.23) is very similar to that of part (i) of Theorem 1, and we shall not give it here. We note only that, for any fixed $\delta > 0$,

$$|G_n(u;a,b)| \ll 1, \quad |G(u;a,b)| \ll 1 \quad (\delta \leqslant u \leqslant 1/\delta), \tag{2.24}$$

where the implied constant depends at most on ε and δ. This follows easily from the definition of G_n.

3. The main result

We assume that ε, a and b satisfy (1.9), and we fix a positive number u_0 in the range

$$\varepsilon \leqslant u_0 \leqslant \frac{1}{\varepsilon} \tag{3.1}$$

Given two functions f and g defined on $[u_0 - 1, u_0]$ and $[u_0, u_0 + 1]$, respectively, we set

$$< f, g >= u_0 f(u_0) g(u_0) - b \int_{u_0-1}^{u_0} f(u) g(u+1) \, \mathrm{d}u.$$

This "scalar product" is independent of u_0 if f and g are solutions to (1.1) and (1.5), respectively ; cf. [Wh, p. 508].

Our principal result is as follows.

Theorem 2. *Let $\varphi(u)$ be a continuous function on $[u_0 - 1, u_0]$, and let $f(u) = f(u;\varphi)$ be the unique continuous solution to* (1.6) *and* (1.7). *Then we have*

$$f(u) = \alpha F(u) + \sum_{n\in\mathbb{Z}} \alpha_n F_n(u) \quad (u > u_0 + 1), \tag{3.2}$$

where

$$\alpha :=< \varphi, G >, \quad \alpha_n :=< \varphi, G_n >, \tag{3.3}$$

and the series in (3.2) *is uniformly convergent for $u \geqslant u_0 + 1 + \delta$, for any fixed $\delta > 0$. Moreover, for any fixed nonnegative integer N and $u \geqslant u_0(\varepsilon, N)$ we have*

$$f(u) = \alpha F(u) + \sum_{|n|\leqslant N} \alpha_n F_n(u) + R_N(u) \tag{3.4}$$

with

$$R_N(u) \ll \|\varphi\| \, |F_0(u)| \exp\left\{ -\frac{u}{2\xi^2} \Big(\big(2(N+1)\pi - |\vartheta|\big)^2 - \vartheta^2 \Big) + O\Big(\frac{u}{\xi^3}\Big) \right\}, \tag{3.5}$$

where $\xi = \xi(u/\kappa)$ is defined by (1.3), *$\|\varphi\| = \max_{u_0-1\leqslant u\leqslant u_0} |\varphi(u)|$, and the implied constants depend at most on N and the constant ε in* (1.9) *and* (3.1).

For real solutions to (1.1), i.e., solutions $f(u) = f(u;\varphi)$ in the cases

$$a \in \mathbb{R}, \quad b > 0, \quad \varphi \text{ real-valued,} \tag{3.6}$$

or

$$a \in \mathbb{R}, \quad b < 0, \quad \varphi \text{ real-valued,} \tag{3.7}$$

the relation (3.4) may be simplified somewhat. In the first case, (3.4) reduces to

$$f(u) = \alpha F(u) + \alpha_0 F_0(u) + 2\,\mathrm{Re} \sum_{n=1}^{N} \alpha_n F_n(u) + R_N(u), \tag{3.4$'$}$$

where

$$R_N(u) \ll \|\varphi\|\,|F_0(u)| \exp\left\{ -\frac{2(N+1)^2\pi^2 u}{\xi^2} \, \Big(1 + O\Big(\frac{1}{\xi}\Big)\Big)\right\}; \tag{3.5$'$}$$

in the second case, we may replace (3.4) for $N \geqslant 1$ by

$$f(u) = \alpha F(u) + 2\,\mathrm{Re} \sum_{n=0}^{N-1} \alpha_n F_n(u) + R_N(u), \tag{3.4$''$}$$

where

$$R_N(u) \ll \|\varphi\|\,|F_0(u)| \exp\left\{ -\frac{2N(N+1)\pi^2 u}{\xi^2} \Big(1 + O\Big(\frac{1}{\xi}\Big)\Big)\right\}. \tag{3.5$''$}$$

The former estimate follows immediately from (3.4), (3.5), and the relation $\alpha_{-n}F_{-n}(u) = \overline{\alpha_n F_n(u)}$ for $n \geqslant 1$, which is a consequence of (2.10), (2.11), and (3.3). The latter estimate is obtained from (3.4) and (3.5), evaluating the term $\alpha_N F_N(u)$ trivially by means of Corollary 1 and (2.24) (which implies $|\alpha_n| \ll \|\varphi\|$), and using the relation $\alpha_{-n}F_{-n}(u) = \overline{\alpha_{n-1}F_{n-1}(u)}$, which follows as before, for any positive integer n, from (2.10), (2.11), and (3.3).

In the remainder of this section we shall derive a number of corollaries from Theorems 1 and 2, which give less precise, but simpler and more explicit estimates for $f(u)$. If we combine (3.4) and (3.5) (resp. (3.4$''$) and (3.5$''$)) with the estimates of Theorem 1 and Corollary 1, we obtain the following result.

Corollary 2. *Under the hypotheses of Theorem 2 we have either*

$$f(u) = u^{-h}\Big(\alpha + O\Big(\frac{\|\varphi\|}{u}\Big)\Big) \tag{3.8}$$

with $\alpha \neq 0$, or

$$f(u) = o\Big(u^{-\mathrm{Re}\,h}\Big) \quad (u \to +\infty). \tag{3.9}$$

In the latter case $f(u)$ satisfies

$$f(u)=\Phi(u,\zeta_0)\Big(\alpha_0+O\Big(\frac{\|\varphi\|}{u}\Big)\Big) \tag{3.10}$$

if $-\pi+\delta\leqslant\vartheta\leqslant\pi-\delta$ for some $\delta>0$, where the O-constant depends at most on ε and δ, and

$$f(u)=2\,Re\,\alpha_0\Phi(u,\zeta_0)+O\Big(\frac{\|\varphi\|}{u}\,|\Phi(u,\zeta_0)|\Big), \tag{3.11}$$

if (3.7) *is satisfied.*

The estimates (3.10) and (3.11) generalize and refine de Bruijn's asymptotic formula (1.2) for the Dickman function, and contain the above-mentioned asymptotic estimates of Alladi [Al], Hensley [He], and the first author [Hi] for particular solutions to (1.1).

In the case when both α and α_0 are zero, Corollary 2 yields only an upper bound for $f(u)$. The next corollary, which follows by a similar argument from Theorems 1 and 2, gives a refined estimate for $f(u)$ in this case. Here, as in the remaining corollaries, we shall concentrate on the case of real, exponentially decreasing solutions to (1.1), i.e. solutions satisfying the second alternative (3.9) of Corollary 2 and either (3.6) or (3.7). The results could be extended to complex solutions, but the statements would then become more complicated.

Corollary 3. *Let the hypotheses and notations of Theorem* 2 *be in force and suppose that f is not identically zero and satisfies* (3.9). *Suppose further that a and b are real, $b\neq 0$, and that φ is real valued. Put*

$$k:=\min\{n\geqslant 0:\ \alpha_n\neq 0\}, \tag{3.12}$$

and suppose furthermore that $\vartheta_k\neq 0$ (i.e. either $k>0$ or $k=0$ and $b<0$). Then we have for $u\geqslant u_0(\varepsilon,k)$

$$f(u)=2\,Re\,\alpha_k\Phi(u,\zeta_k)+O\Big(\frac{\|\varphi\|}{u}\,|\Phi(u,\zeta_k)|\Big), \tag{3.13}$$

where the O-constant depends at most on ε and k.

The local behavior of the right-hand expressions in (3.11) and (3.13) is easy to analyze. Indeed, applying (2.14)–(2.16) with $(\varrho,\varphi)=(u/\kappa,-\vartheta)$, we get

$$\begin{aligned}\frac{\frac{\mathrm{d}}{\mathrm{d}u}\Phi(u,\zeta_n)}{\Phi(u,\zeta_n)}&=-\zeta_n+O\left(\frac{|\zeta_n'(u/b)|}{|\zeta_n|}+\frac{1}{u}\right)\\&=-\xi+\frac{\xi}{\xi-1}\vartheta_n i+O\Big(\frac{1}{\xi^2}\Big)\end{aligned}$$

with $\xi=\xi(|u/b|)=\xi(u/\kappa)$ and ϑ_n is as in (2.3). Since, by (2.15), $\xi\big((u+t)/\kappa\big)=\xi(u/\kappa)+O(|t|/u)$ for $|t|\leqslant u/2$, this implies

$$\frac{\Phi\big(u+t,\zeta_n\big((u+t)/b\big)\big)}{\Phi(u,\zeta_n(u/b))}=\exp\left\{-t\xi+t\vartheta_n\frac{\xi}{\xi-1}i+O\Big(\frac{|t|}{\xi^2}\Big)\right\}$$

with $\xi=\xi(u/\kappa)$, uniformly for $|t|\ll u/\xi^2$. Using this estimate in (3.11) and (3.13), we obtain the following result.

Corollary 4. *Let the hypotheses of Corollary 3 be in force. Then we have, for* $u \geqslant u_0(\varepsilon, k)$ *and* $|t| \ll \xi^2$,

$$f(u+t) = \varrho_u e^{-t\xi} \left\{ \cos\left(\tau_u + \vartheta_k t \frac{\xi}{\xi-1}\right) + O\left(\frac{|t|}{\xi^2} + \frac{\|\varphi\|}{|\alpha_k|u}\right)\right\}, \tag{3.14}$$

where $\xi = \xi(u/\kappa)$, k *is defined by* (3.12),

$$\varrho_u := 2|\alpha_k \Phi(u, \zeta_k)|, \quad \tau_u := \arg\left(\alpha_k \Phi(u, \zeta_k)\right), \tag{3.15}$$

and ϑ_k *is given by* (2.3), *i.e.*

$$\vartheta_k := \vartheta + 2k\pi = \begin{cases} 2k\pi & \text{if (3.6) holds,} \\ (2k+1)\pi & \text{if (3.7) holds.} \end{cases} \tag{3.16}$$

The implied constant here depends at most on ε *and* k.

This result can be used to determine asymptotically the location of the zeros of $f(u)$ when $\vartheta_k \neq 0$. Indeed, from (3.14) and the continuity of f it follows that $f(u)$ has a zero in every interval of length $(\pi/\vartheta_k)(1-1/\xi)+O(1/\xi^2)$. Moreover, if $u < u'$ are two consecutive zeros of $f(u)$ then either $u' - u = O\left(1/u\right)$ or

$$u' - u = \frac{\pi}{\vartheta_k}\left(1 - \frac{1}{\xi}\right) + O\left(\frac{1}{\xi^2}\right). \tag{3.17}$$

If the first alternative holds, then $f'(u+t) = 0$ for some t with $|t| \ll 1/u$. But by (1.1) and (3.14), applied to $f(u+t)$, $f(u-1+t)$, and $f(u)(=0)$ we have in this range

$$\begin{aligned} -(u+t)f'(u+t) &= af(u+t) + bf(u+t-1) \\ &= be^{-(t-1)\xi}\varrho_u \left\{ \cos\left(\tau_u + \vartheta_k(t-1)\frac{\xi}{\xi-1}\right) + O\left(\frac{1}{u}\right)\right\} \\ &= be^{-(t-1)\xi}\varrho_u \left\{ \pm\frac{\vartheta_k}{\xi} + O\left(\frac{1}{\xi^2}\right)\right\}, \end{aligned}$$

whence $f'(u+t)$ is non-zero for sufficiently large u if $\vartheta_k \neq 0$. Thus any two consecutive zeros $u < u'$ with u sufficiently large must satisfy (3.17). This observation together with Corollaries 3 and 4 easily leads to the following result.

Corollary 5. *Let the hypotheses of Corollary 3 be in force. Then there exists a number* $u_1 \geqslant u_0$ *such that* $f(u)$ *has only countably many zeros,* $\lambda_1 < \lambda_2 < \cdots$, *exceeding* u_1. *Moreover, we have*

$$\lambda_{n+1} - \lambda_n = \frac{\pi}{|\vartheta_k|}\Big(1 + O\Big(\frac{1}{\log n}\Big)\Big) \quad (n \to +\infty), \tag{3.18}$$

$$\begin{aligned} \max_{\lambda_n \leqslant u \leqslant \lambda_{n+2}} f(u) &= |\Phi(\lambda_n, \zeta_k(\lambda_n))| e^{O(\xi)}, \\ \min_{\lambda_n \leqslant u \leqslant \lambda_{n+2}} f(u) &= -|\Phi(\lambda_n, \zeta_k(\lambda_n)| e^{O(\xi)}, \end{aligned} \tag{3.19}$$

where k *and* ϑ_k *are given by* (3.12) *and* (3.16), $\xi = \xi(\lambda_n)$, *and the implied constants may depend on the function* f.

The above results show that the behavior of the general solution to the equation (1.1) is to a large degree independent of the particular initial function $\varphi(u)$. For, example, the asymptotic formula (3.13) depends on φ only via the coefficient α_k. To further illustrate this phenomenon, we consider the case $(a,b)=(0,1)$ of (3.6). Suppose $f(u)$ is a real-valued solution to (1.1) with $(a,b)=(0,1)$ satisfying $f(u)=o(1/u)$. We shall compare the asymptotic behavior of this solution to that of a particular solution, namely the Dickman function $\varrho(u)$. Set $c=\alpha_0(f)e^{-\gamma}$ and $g=f-c\varrho$, where $\alpha_0(f)$ is defined as in Theorem 2 with respect to f. The function g is again a solution to (1.1) and thus has an expansion of the form (3.2) with coefficients $\alpha(g)=\alpha(f)-c\alpha(\varrho)$ and $\alpha_n(g)=\alpha_n(f)-c\alpha_n(\varrho)$. In particular, since $\alpha(f)=\alpha(\varrho)=0$ and $c\alpha_0(\varrho)=ce^{\gamma}=\alpha_0(f)$, by the hypothesis $f(u)=o(1/u)$, Corollary 2, and (1.2), we have $\alpha(g)=\alpha_0(g)=0$. Thus, if g is not identically zero, we can apply Corollary 5 to $g(u)$, getting (3.18) and (3.19) for some positive integer k. Since $\vartheta_k=2k\pi$, we deduce from (3.18) that $\lambda_n=(n/2k)\left(1+O\left(1/\log n\right)\right)$. Furthermore, (2.20) and (1.2) yield in this case

$$\begin{aligned}|\Phi(\lambda_n,\zeta_k(\lambda_n);0,1)| &= \Phi(\lambda_n,\xi(\lambda_n);0,1)\exp\left\{-\frac{2\pi^2k^2\lambda_n}{\xi(\lambda_n)^2}\left(1+O\left(\frac{1}{\xi(\lambda_n)}\right)\right)\right\}\\ &= \varrho(\lambda_n)\exp\left\{-\frac{2\pi^2k^2\lambda_n}{\xi(\lambda_n)^2}\left(1+O\left(\frac{1}{\xi(\lambda_n)}\right)\right)\right\}\\ &= \varrho(\lambda_n)\exp\left\{-\frac{\pi^2kn}{\xi(\lambda_n)^2}\left(1+O\left(\frac{1}{\xi(\lambda_n)}\right)\right)\right\}.\end{aligned}$$

We thus arrive at the following result.

Corollary 6. *Let $f(u)$ be a real-valued continuous function on $[u_0,+\infty)$ satisfying (1.1) with $(a,b)=(0,1)$, and $f(u)=o(1/u)$ as $u\to+\infty$. Then there exists a constant c such that $f(u)=\varrho(u)(c+h(u))$, where the function $h(u)$ is either identically zero for all sufficiently large u, or has infinitely many zeros $\lambda_1<\lambda_2<\cdots$ satisfying (3.18) with $\mu_k=2k\pi$ for some positive integer k and*

$$\max_{\lambda_n\leqslant u\leqslant\lambda_{n+2}} h(u)=\exp\left\{\frac{-\pi^2kn}{\log^2(n\log n)}\left(1+O\left(\frac{1}{\log n}\right)\right)\right\}, \tag{3.20}$$

$$\min_{\lambda_n\leqslant u\leqslant\lambda_{n+2}} h(u)=-\exp\left\{\frac{-\pi^2kn}{\log^2(n\log n)}\left(1+O\left(\frac{1}{\log n}\right)\right)\right\}, \tag{3.21}$$

as $n\to+\infty$. The O-constants here may depend on f.

Previously, de Bruijn [dB] and Alladi [Al] had obtained estimates of the type

$$f(u)=\varrho(u)(c+O(u^{-1/4+\varepsilon}))$$

and

$$f(u)=\varrho(u)\Big(c+O\Big(\exp\left\{-u^{1/4-\varepsilon}\right\}\Big)\Big),$$

respectively, for functions satisfying the differential difference equation of the Dickman function under slightly stronger hypotheses. The above result not only improves the error term in these estimates to its best-possible form, namely

$$\exp\left\{-\frac{2\pi^2 u}{\log^2 u}(1+o(1))\right\},$$

but also exhibits the finer behavior of the difference $f(u)-c\varrho(u)$ by showing that the function $f(u)$ oscillates around $c\varrho(u)$ like a periodic function dampened by an exponentially decaying factor.

§ 4. Proof of Lemma 1 and Theorem 1.

Proof of Lemma 1.

Put $\zeta=\xi+i\varphi_n+v$, where $\xi=\xi(\varrho)$. Multiplying both sides of (2.12) (with $w=\varrho e^{i\varphi}$) by $e^{-\xi-i\varphi_n}=\dfrac{e^{-i\varphi}}{1+\varrho\xi}$, we obtain

$$e^v-\frac{e^{-i\varphi}}{1+\varrho\xi}=\frac{\varrho}{1+\varrho\xi}(\xi+i\varphi_n+v).$$

Setting

$$\mu:=\frac{\varrho}{1+\varrho\xi},\qquad \nu:=\frac{1-e^{i\varphi}}{w}+i\varphi_n,$$

this may be written as

$$e^v-1-\mu(v+\nu)=0. \tag{4.1}$$

Since $|\mu|\sim 1/\xi\sim 1/\log\varrho$ and $\nu\sim i\varphi_n$ as $\varrho\to+\infty$, we have for $\varrho\geqslant w_0(n)$

$$|e^z-1|>|\mu(z+\nu)|\qquad (|z|=\pi).$$

By Rouché's theorem, it follows that equation (4.1) has a unique solution in the disk $|v|\leqslant\pi$, which is given by Cauchy's formula

$$v=\frac{1}{2\pi i}\oint_{|z|=\pi}\frac{e^z-\mu}{e^z-1-\mu(z+\nu)}z\,dz.$$

Expanding the integrand as a power series in μ, we readily obtain

$$v=\sum_{k=1}^{+\infty}c_k(\nu)\mu^k \tag{4.2}$$

where $c_k(\nu)$ is the coefficient of z^{k-1} in the Taylor expansion of

$$H_{k,\nu}(z):=\Big(\frac{z}{e^z-1}\Big)^{k+1}(z+\nu)^{k-1}\big(e^z(z+\nu-1)+1\big).$$

Now, observe that $H_{k,\nu}(z)$ is regular for $|z| < 2\pi$ and that, for some absolute positive constant C,

$$|H_{k,\nu}(z)| \leqslant \Big\{C(|\nu|+1)\Big\}^{k+1} \qquad (|z| \leqslant \pi).$$

Cauchy's formula hence yields that

$$c_k(\nu) \leqslant \Big\{C_1(|n|+1)\Big\}^{k+1} \qquad (k = 1, 2, \ldots), \tag{4.3}$$

so the Taylor expansion (4.2) is certainly convergent for $\varrho \geqslant w_0(n)$ with a suitable constant $w_0(n)$.

This puts us in a position to complete the proof immediately. Indeed, (2.14) follows from (4.2) and (4.3) on noticing that

$$c_1(\nu) = \nu, \quad c_2(\nu) = \nu - \tfrac{1}{2}\nu^2,$$

and (2.15) is obtained by differentiating (2.12) with respect to w, using (2.12) and (2.14) in the weak form $\zeta_n = \xi + O(1)$.

For the proof of Theorem 1, we shall need a preliminary result. We suppose, here and in the remainder of this section, that ε, a, and b satisfy (1.9) and (1.10).

Lemma 2. *Let $s \in \mathbb{C} \smallsetminus \mathcal{C}$. Then we have*

$$|\lambda(s)| \asymp |s^h|(1+|s|)^{-\operatorname{Re} b} \quad (\sigma \geqslant -1), \tag{4.4}$$

$$|\lambda(s)| \asymp |s^a| \exp\Big\{ Re\big(b\, I_1(-s)\big)\Big\} \quad (\sigma \leqslant -1), \tag{4.5}$$

$$|\lambda(s)| = |s^a| \exp\left\{ Re\Big(b\frac{e^{-s}}{-s}T(-s)\Big) + O\left(\frac{e^{|\sigma|}}{|s|^4}+1\right)\right\} \quad (\sigma \leqslant -1). \tag{4.6}$$

where

$$I_1(s) := \int_1^s \frac{e^z}{z}\,\mathrm{d}z$$

and $T(s)$ is defined by (2.19).

Proof. If $|s| \leqslant 1$, the estimate (4.4) follows trivially from the bound $I(-s) \ll 1$. If $\sigma \geqslant -1$, $|s| > 1$, then we have

$$\begin{aligned} I(-s) &= I(-1) + \int_{-1}^{-s} \frac{e^v - 1}{v}\,\mathrm{d}v = I(-1) - \log s + \int_1^s e^{-v}\frac{\mathrm{d}v}{v} \\ &= I(-1) - \log s + \frac{1}{e} - \frac{e^{-s}}{s} + \int_1^s e^{-v}\frac{\mathrm{d}v}{v^2} \\ &= -\log s + O(1), \end{aligned}$$

which again yields (4.4).

The validity of (4.5) and (4.6) follows from the formula

$$I(-s)=I(1)+\int_1^{-s}\frac{e^v-1}{v}\,\mathrm{d}v=I(1)-\log(-s)+I_1(-s)\qquad(\sigma\leqslant-1),$$

and the estimate

$$I_1(s)=\frac{e^s}{s}T(s)+O\left(\frac{e^\sigma}{|s|^4}+1\right)\quad(\sigma\geqslant1),$$

which follows by integrating over the path $[1,1+i\tau,s]$, using repeated integration by parts.

Proof of Theorem 1.

We first prove the estimate (2.17) for $F_n(u)$. Set $\zeta_n=\xi_n+i\eta_n$. In the representation (2.2) for $F_n(u)$, we shift the vertical part of the path Γ_n to the line $\operatorname{Re}s=\xi_n$, so that it passes through $-\zeta_n(u/b)$. Our determination of $\arg s$ guarantees that the integrand is regular in the intermediate region, and that the value of the integral is not affected. The contributions of the infinite horizontal branches of the new contour may easily be estimated. Indeed, we have by (4.6)

$$\lambda(s)\ll\exp\Big\{-\tfrac12\kappa\frac{e^{\xi_n+t}}{\xi_n+t}\Big\}\qquad\Big(s=-\xi_n-t+i(\vartheta_n\pm\pi),\,t>0\Big),$$

and we obtain that the corresponding contributions to $F_n(u)$ are

$$\ll\exp\{-u\xi_n\}\ll|\Phi(u,\zeta_n)|e^{-\frac12u}$$

by (2.14) and (2.20).

It remains to estimate the contribution of the vertical part of the contour, namely the segment $[-\xi_n+i(\vartheta_n-\pi),-\xi_n+i(\vartheta_n+\pi)]$. This can be done by the saddle point method, writing $\lambda(s)e^{us}s^{-1}=\exp\psi(s)$ and expanding $\psi(s)$ into a Taylor series around $s=-\zeta_n$, which is a saddle point for the function $\lambda(s)s^{-h}$. The argument is almost identical to that in [Hi] where (2.17) was (essentially) proved when $n=h=0$, $\operatorname{Re}b>-1$. We therefore omit the details.

We now prove (2.18). First consider the case when $h\notin\mathbb{Z}^+$. We write $\Gamma^*=\Gamma_1^*+\Gamma_2^*$, where Γ_1^* is the part of the contour Γ^* which lies in the half-plane $\sigma\geqslant-2$. For $s\in\Gamma_1^*$, we plainly have

$$e^{bI(-s)}=\sum_{k=0}^{+\infty}b_ks^k=\sum_{k=0}^{K}b_ks^k+O(|s|^{K+1})$$

with $b_0=1$. Next, we replace Γ_1^* by a Hankel contour Γ_1' around the path $[-2+(\vartheta-\pi)i,-2,0]$. This does not modify the integral, and we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_1'}e^{bI(-s)}e^{us}s^{h-1}\,\mathrm{d}s=\sum_{k=0}^{K}b_k\frac{1}{2\pi i}\int_{\Gamma_1'}e^{us}s^{k+h-1}\,\mathrm{d}s+O\Big(\int_{\Gamma_1^*}\big|e^{us}s^{K+h}\big|\cdot|\,\mathrm{d}s|\Big).$$

From [Ten] (Cor. II.5.2.1) we have for $0 \leqslant k \leqslant K$

$$\frac{1}{2\pi i}\int_{\Gamma_1'} e^{us}s^{k+h-1}\,\mathrm{d}s = u^{-k-h}\Big\{\frac{1}{\Gamma(1-h-k)}+O(e^{-u})\Big\},$$

where the implied constant depends at most on K and ε. Moreover, selecting $r = 1/u$ as the radius of the circular part of Γ_1', we may write for $K > |h|$

$$\begin{aligned}\int_{\Gamma_1'} \big|e^{us}s^{K+h}\big|\cdot|\,\mathrm{d}s| &\ll \int_0^{|\vartheta-\pi|} e^{-2u}\,\mathrm{d}\tau + \Big|\int_{-2}^{-r} e^{u\sigma}\sigma^{K+h}\,\mathrm{d}\sigma\Big| + \int_0^{2\pi} e^{u\sigma}r^{K+h+1}\,\mathrm{d}\varphi\\ &\ll u^{-K-\,\mathrm{Re}\,h-1}\Gamma(K+\,\mathrm{Re}\,h+1).\end{aligned}$$

Hence the contribution of Γ_1^* to $F(u)$ yields the main term in the required formula (2.18). We now need an upper bound for the contribution of Γ_2^*. By (4.6), we may write

$$|\lambda(s)| \ll |s^a|\exp\Big\{-\tfrac{1}{2}\kappa\frac{e^{|\sigma|}}{|\sigma|}\Big\} \qquad (s\in\Gamma_2^*)$$

whence

$$\int_{\Gamma_2^*}\lambda(s)\frac{e^{us}}{s}\,\mathrm{d}s \ll e^{-u}.$$

This is plainly sufficient, and completes the proof of (2.18) when $h \notin \mathbb{Z}^+$. Indeed, if $-h \in \mathbb{Z}^+$, the contributions of the straight lines of the Hankel contour cancel and one only has to consider the contribution of a small circle around the origin ; the result then follows from Hankel's formula — which in this case reduces to Cauchy's formula.

When $h \in \mathbb{Z}^+$, we use the fact that $F(u)$ is a continuous function of h — see Remark (ii) in Section 2.

4. The Laplace transform of $f(u)$

Let $f(u)$ be as in Theorem 2. In this section we investigate the Laplace transform of $f(u)$ defined by

$$\widehat{f}(s) := \int_{u_0}^{+\infty} f(u)e^{-us}\,\mathrm{d}u. \tag{5.1}$$

Here s is a complex variable, and we denote its real and imaginary parts by σ and τ, respectively. We begin with the following lemma which shows that the integral in (5.1) is absolutely convergent in the half-plane $\sigma > 0$.

Lemma 3. *There exist positive constants c_1 and c_2 depending on f such that for all $u \geqslant u_0$*

$$|f(u)| < c_1 u^{c_2}. \tag{5.2}$$

Proof. Since, by the hypotheses of Theorem 2, $u_0 > 0$ and $f(u)$ is continuous on $[u_0, +\infty)$, (5.2) holds for $u \in [u_0, u_0+1]$ with suitable positive constants c_1 and c_2, where we may assume that

$$c_1 \geqslant 2|f(u_0+1)(u_0+1)^a|, \quad c_2 \geqslant 2|b| - \operatorname{Re} a. \tag{5.3}$$

Now suppose that $n \geqslant 1$ and that (5.2) is valid in the interval $[u_0, u_0+n]$ with constants c_1 and c_2 satisfying (5.3). Using (1.7′) we then obtain, for $u \in [u_0+n, u_0+n+1]$,

$$\begin{aligned} |f(u)u^a| &\leqslant |f(u_0+1)(u_0+1)^a| + |b| \int_{u_0+1}^{u} |f(v-1)v^{a-1}| \,\mathrm{d}v \\ &\leqslant \tfrac{1}{2}c_1 + c_1|b| \int_{u_0+1}^{u} (v-1)^{c_2} v^{\operatorname{Re} a - 1} \,\mathrm{d}v \\ &\leqslant \tfrac{1}{2}c_1 + c_1|b| \frac{u^{c_2+\operatorname{Re} a}}{c_2+\operatorname{Re} a} \\ &\leqslant \tfrac{1}{2}c_1\left(1+u^{c_2+\operatorname{Re} a}\right) \leqslant c_1 u^{c_2+\operatorname{Re} a}, \end{aligned}$$

and hence (5.2) for $u_0 \leqslant u \leqslant u_0+n+1$. By induction it follows that (5.2) holds for all $u \geqslant u_0$, and the lemma is proved.

We next give an explicit representation of $\widehat{f}(s)$ in terms of the parameters a and b and the initial function $\varphi(u)$. For $s \in \mathbb{C} \smallsetminus \mathcal{C}$ we set

$$Q(s) := \int_s^{+\infty} \frac{\Phi(z)}{\lambda(z)} \,\mathrm{d}z,$$

where

$$\Phi(s) := u_0\varphi(u_0)e^{-u_0 s} - b\int_{u_0-1}^{u_0} \varphi(u)e^{-(u+1)s} \,\mathrm{d}u,$$

$\lambda(s) = s^h e^{bI(-s)}$ is defined as in Section 2 and the integral may be taken over any path within the domain $\mathbb{C} \smallsetminus \mathcal{C}$ going from s to a point on the positive real axis and then to infinity along this axis. The integral along the infinite portion of this path is absolutely convergent, since

$$\begin{aligned} |\Phi(s)| &\leqslant |u_0\varphi(u_0)e^{-u_0 s}| + |b| \int_{u_0-1}^{u_0} \left|\varphi(u)e^{-(u+1)s}\right| \,\mathrm{d}u \\ &\ll u_0\|\varphi\|e^{-u_0\sigma} + \|\varphi\| \int_{u_0-1}^{u_0} e^{-(u+1)\sigma} \,\mathrm{d}u \ll \|\varphi\|e^{-u_0\sigma}\left(1+e^{-\sigma}\right) \end{aligned} \tag{5.4}$$

for all $s \in \mathbb{C}$, and by (4.4), $|\lambda(s)| \asymp |s^a|$ for $\sigma \geqslant 1$. Hence $Q(s)$ is well-defined and analytic in $\mathbb{C} \smallsetminus \mathcal{C}$.

Lemma 4. *For* $\operatorname{Re} s > 0$ *we have*

$$\widehat{f}(s) = \frac{1}{s}\lambda(s)Q(s). \tag{5.5}$$

Proof. An integration by parts yields

$$\widehat{f}(s) = \int_{u_0}^{+\infty} f(u)e^{-us}\,\mathrm{d}u = f(u_0)\frac{e^{-u_0 s}}{s} + \int_{u_0}^{+\infty} f'(u)\frac{e^{-us}}{s}\,\mathrm{d}u.$$

By (1.6) and (1.7) it follows that

$$\begin{aligned}
(s\widehat{f}(s))' &= -u_0 f(u_0)e^{-u_0 s} - \int_{u_0}^{+\infty} uf'(u)e^{-us}\,\mathrm{d}u \\
&= -u_0 f(u_0)e^{-u_0 s} + \int_{u_0}^{+\infty} (af(u) + bf(u-1))e^{-us}\,\mathrm{d}u \\
&= -u_0\varphi(u_0)e^{-u_0 s} + a\int_{u_0}^{+\infty} f(u)e^{-us}\,\mathrm{d}u + b\int_{u_0}^{+\infty} f(u)e^{-(u+1)s}\,\mathrm{d}u \\
&\qquad + b\int_{u_0-1}^{u_0} \varphi(u)e^{-(u+1)s}\,\mathrm{d}u \\
&= \left(a + be^{-s}\right)\widehat{f}(s) - \Phi(s).
\end{aligned}$$

This shows that $\widehat{f}(s)$ is a solution to the first order linear differential equation

$$\widehat{f}(s)' + \widehat{f}(s)\frac{1 - a - be^{-s}}{s} = -\frac{\Phi(s)}{s}.$$

The general solution to this equation is of the form

$$\widehat{f}(s) = e^{-J(s)}R(s) \tag{5.6}$$

with

$$J(s) = \int_1^s \frac{1 - a - be^{-z}}{z}\,\mathrm{d}z, \quad R'(s) = -\frac{\Phi(s)}{s}e^{J(s)}.$$

Writing

$$\begin{aligned}
J(s) &= (1 - a - b)\log s + b\int_1^s \frac{1 - e^{-z}}{z}\,\mathrm{d}z \\
&= (1 - h)\log s - b\int_{-1}^{-s} \frac{e^z - 1}{z}\,\mathrm{d}z \\
&= (1 - h)\log s - b\big(I(-s) - I(-1)\big),
\end{aligned}$$

we obtain

$$e^{-J(s)} = s^{h-1}e^{b(I(-s)-I(-1))} = \frac{e^{-bI(-1)}}{s}\lambda(s)$$

and

$$R'(s) = -\frac{\Phi(s)}{s}e^{J(s)} = -\frac{\Phi(s)}{\lambda(s)}e^{bI(-1)} = Q'(s)e^{bI(-1)}.$$

Thus (5.6) implies

$$\widehat{f}(s) = \frac{1}{s}\lambda(s)(Q(s)+A)$$

with some constant A. Now observe that, by (5.1), (5.2), (4.4) and (5.4), $\widehat{f}(s)$ and $Q(s)$ tend to zero exponentially as $s \to +\infty$ through positive values, whereas, by Lemma 2, $|\lambda(s)| \asymp |s^a|$ for $s \geqslant 1$. Thus the constant A must be zero, and (5.5) follows.

Equation (5.5) provides an analytic continuation of $\widehat{f}(s)$ into the domain $\mathbb{C} \smallsetminus \mathcal{C}$, and we shall henceforth regard $\widehat{f}(s)$ as being defined throughout this domain.

In the remainder of this section, we shall establish some auxiliary estimates for the proof of Theorem 2. The implied constants here are allowed to depend at most on the parameter ε in (1.9) and (3.1). We suppose throughout that $s \in \mathbb{C} \smallsetminus \mathcal{C}$, so that the functions $\lambda(s)$, $Q(s)$, and $\widehat{f}(s)$ are well-defined.

Lemma 5. *For $\sigma < -1$ and any integer n let $\tau_n = \tau_n(\sigma)$ be the unique solution to the equation*

$$\tau_n - \vartheta + \arctan\frac{\tau_n}{\sigma} = n\pi, \tag{5.7}$$

where the arctangent is taken as principal value. Then we have, for any $\sigma < -1$,

$$\frac{\partial}{\partial\tau}\log|\lambda(s)| \begin{cases} \geqslant O(1/|s|) & \text{if } \tau_{2n-1} \leqslant \tau \leqslant \tau_{2n}, \\ \leqslant O(1/|s|) & \text{if } \tau_{2n} \leqslant \tau \leqslant \tau_{2n+1}. \end{cases} \tag{5.8}$$

Moreover, for $\sigma \leqslant -2$ and any integer n, we have

$$|\tau_n'(\sigma)| \ll 1 \tag{5.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\log|\lambda(\sigma + i\tau_{2n}(\sigma))| = -\frac{\kappa}{|s|}\Big(e^{|\sigma|} + O(1)\Big). \tag{5.10}$$

Proof. We have

$$\log|\lambda(s)| = \operatorname{Re}\left(h\log s + b\int_0^{-s}\frac{e^z-1}{z}\,\mathrm{d}z\right),$$

whence

$$\begin{aligned}
\frac{\partial}{\partial\tau}\log|\lambda(s)| &= \operatorname{Re}\left(i\frac{h}{s} - ib\frac{e^{-s}-1}{-s}\right)\\
&= -\operatorname{Im} b\frac{e^{-s}}{s} + O\left(\frac{1}{|s|}\right)\\
&= -\kappa\frac{e^{-\sigma}}{|s|}\sin\psi(s) + O\left(\frac{1}{|s|}\right)
\end{aligned} \tag{5.11}$$

with

$$\psi(s) := \tau - \vartheta + \arctan\frac{\tau}{\sigma},$$

since for $\sigma < 0$, $-s = |s|\exp\{i\arctan(\tau/\sigma)\}$ with $|\arctan(\tau/\sigma)| < \frac{1}{2}\pi$. Similarly we obtain

$$\frac{\partial}{\partial\sigma}\log|\lambda(s)| = -\kappa\frac{e^{-\sigma}}{|s|}\cos\psi(s) + O\left(\frac{1}{|s|}\right). \tag{5.12}$$

The equation (5.7) is equivalent to $\psi(\sigma + i\tau_n) = n\pi$, and since

$$\frac{\partial}{\partial\tau}\psi(s) = 1 + \frac{\sigma}{\sigma^2+\tau^2} \geqslant 1 + \frac{1}{\sigma} > 0$$

for $\sigma < -1$, this equation has indeed a unique solution $\tau_n = \tau_n(\sigma)$ for any positive integer n. Moreover, the numbers τ_n form an increasing sequence with

$$\lim_{n\to-\infty}\tau_n = -\infty, \qquad \lim_{n\to+\infty}\tau_n = +\infty,$$

and we have $\psi(s) \in [n\pi, (n+1)\pi]$ if and only if $\tau \in [\tau_n, \tau_{n+1}]$. Hence the main term in (5.11) is $\geqslant 0$ if $\tau \in [\tau_{2n-1}, \tau_{2n}]$, and $\leqslant 0$ if $\tau \in [\tau_{2n}, \tau_{2n+1}]$. This proves (5.8).

Taking the derivative with respect to σ in the equation $\psi(\sigma + i\tau_n(\sigma)) = n\pi$, we obtain

$$\frac{-\tau_n(\sigma)}{\sigma^2+\tau_n(\sigma)^2} + \tau_n'(\sigma)\left(1 + \frac{1}{\sigma\Big(1+\big(\tau_n(\sigma)/\sigma\big)^2\Big)}\right) = 0,$$

whence, for $\sigma \leqslant -2$,

$$|\tau_n'(\sigma)| = \frac{|\tau_n(\sigma)|}{\sigma^2+\sigma+\tau_n(\sigma)^2} \ll 1, \tag{5.13}$$

which proves (5.9).

The estimate (5.10) follows from (5.11), (5.12), (5.13), and the relation

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\log|\lambda(\sigma+i\tau_{2n}(\sigma))| = \left(\frac{\partial}{\partial\sigma} + \tau_{2n}'(\sigma)\frac{\partial}{\partial\tau}\right)\log|\lambda(\sigma+i\tau_{2n}(\sigma))|.$$

Lemma 6. *We have*

$$|Q(s)| \ll \frac{\|\varphi\| e^{-u_0\sigma}}{|s^a|} \qquad (\sigma \geqslant -1,\ |s| \geqslant 1), \tag{5.14}$$

$$|Q(s)| \ll \|\varphi\| e^{(u_0+1)|\sigma|}\Big\{(|\tau|+1)^{-\operatorname{Re} a} + |\lambda(s)|^{-1}\Big\} \qquad (\sigma < -1). \tag{5.15}$$

Moreover, for a suitable positive constant c_3, any integer n, and s in the range

$$|\tau - \vartheta_n| < \pi, \quad \sigma \leqslant -c_3(|n|+1), \tag{5.16}$$

we have

$$Q(s) = \alpha_n + O\left(\frac{\|\varphi\| e^{(u_0+1)|\sigma|}}{|\lambda(s)|}\right), \tag{5.17}$$

where α_n is as defined in Theorem 2.

Proof. Suppose first that $\sigma \geqslant -1$. If $|\tau| \geqslant 1$, then writing $Q(s)$ as an integral over $\Phi(z)/\lambda(z)$ along the path $[s, s+T, \sigma+T, +\infty]$, and estimating the integrand by (4.4) and (5.4), we obtain

$$\begin{aligned}
|Q(s)| &\leqslant \lim_{T\to+\infty} \int_0^T \left|\frac{\Phi(s+t)}{\lambda(s+t)}\right| \,\mathrm{d}t + \lim_{T\to+\infty} \int_0^\tau \left|\frac{\Phi(\sigma+T+it)}{\lambda(\sigma+T+it)}\right| \,\mathrm{d}t \\
&\ll \|\varphi\| e^{-u_0\sigma} \left\{\int_0^{+\infty} \frac{e^{-u_0 t}}{|(s+t)^a|}\,\mathrm{d}t + \lim_{T\to+\infty} \int_0^\tau \frac{e^{-u_0(\sigma+T)}}{|(\sigma+T+it)^a|}\,\mathrm{d}t\right\} \\
&\ll \frac{\|\varphi\| e^{-u_0\sigma}}{|s^a|}.
\end{aligned}$$

If $|\tau| < 1$ (and $\sigma \geqslant -1$, $|s| \geqslant 1$), we get the same bound by integrating first from s to $\sigma+i$ (if $\tau \geqslant 0$) or $\sigma-i$ (if $\tau < 0$) and then along the path $[\sigma\pm i, \sigma+T\pm i, \sigma+T, +\infty]$, and letting T tend to infinity. This proves (5.14).

Now suppose that $\sigma < -1$. If $-2 \leqslant \sigma < -1$, we obtain (5.14), and hence (5.15) by integrating first over the path $[s, -1+i\tau]$ and then applying (5.14) with $-1+i\tau$ in place of s. We may therefore suppose that $\sigma < -2$. In this case we choose n such that $\tau_{2n-1}(\sigma) \leqslant \tau < \tau_{2n+1}(\sigma)$, where $\tau_n(\sigma)$ is defined as in Lemma 5, set $s_n(v) = v + i\tau_{2n}(v)$, and evaluate $Q(s)$ by integrating first over $[s, s_n(\sigma)]$ and then over the path $s_n(v)$, $\sigma \leqslant v \leqslant -2$. This gives

$$Q(s) = i\int_\tau^{\tau_{2n}(\sigma)} \frac{\Phi(\sigma+it)}{\lambda(\sigma+it)}\,\mathrm{d}t + \int_\sigma^{-2} \frac{\Phi(s_n(v))}{\lambda(s_n(v))} s_n'(v)\,\mathrm{d}v + Q(s_n(-2)). \tag{5.18}$$

Since $\operatorname{Re} s_n(-2) = -2$ and, by (5.7),

$$\operatorname{Im} s_n(-2) = \tau_{2n}(-2) = 2n\pi + O(1) = \tau_{2n}(\sigma) + O(1) = \tau + O(1),$$

the last term in (5.18) may be estimated by (5.14) and is seen to be bounded by the right hand side of (5.15). By (5.8) and our choice for n, we have in the first integral $|\lambda(\sigma+it)| \gg |\lambda(s)|$. Moreover by (5.10), we have for $v \leqslant -2$

$$|\lambda(s_n(v))| \gg |\lambda(s_n(-2))| \gg (|\tau|+1)^{\operatorname{Re} a}.$$

Together with the bounds (5.4) and (5.9), this shows that the integrals in (5.18) are bounded by

$$\ll \|\varphi\|\, e^{(u_0+1)|\sigma|} \left(|\lambda(s)|^{-1} + \int_\sigma^{-2} \frac{e^{(u_0+1)(\sigma-v)}|1+i\tau_{2n}'(v)|}{(|\tau|+1)^{\operatorname{Re} a}}\, \mathrm{d}v \right)$$
$$\ll \|\varphi\|\, e^{(u_0+1)|\sigma|} \big(|\lambda(s)|^{-1} + (|\tau|+1)^{-\operatorname{Re} a} \big)$$

and completes the proof of (5.15).

To prove (5.17), we first note that, by (3.3) and (2.6),

$$\begin{aligned} \alpha_n =< \varphi, G_n >&= u_0\varphi(u_0)G_n(u_0) - b\int_{u_0-1}^{u_0} \varphi(u)G_n(u+1)\, \mathrm{d}u \\ &= u_0\varphi(u_0)\int_{\Delta_n} \frac{e^{-u_0 s}}{\lambda(s)}\, \mathrm{d}s - b\int_{u_0-1}^{u_0} \left(\varphi(u)\int_{\Delta_n} \frac{e^{-(u+1)s}}{\lambda(s)}\, \mathrm{d}s \right) \mathrm{d}u \\ &= \int_{\Delta_n} \frac{\Phi(s)}{\lambda(s)}\, \mathrm{d}s. \end{aligned}$$

Recall that the integration path Δ_n is equal to the line $[-\infty + i\vartheta_n, +\infty + i\vartheta_n]$, where the segment $[-1+i\vartheta_n, 1+i\vartheta_n]$ is deformed in the case $n=0$ in order to avoid the vertical portion of the branchcut $\mathcal{C}$. By Lemma 2 and (5.4) the integrand $\Phi(s)/\lambda(s)$ tends to zero uniformly in τ if $\sigma \to +\infty$, and uniformly in $|\tau-\vartheta_n| \leqslant \frac{1}{2}\pi - \delta$ for any fixed $\delta > 0$ if $\sigma \to -\infty$. Thus we may replace the portion $[1+i\vartheta_n, +\infty+i\vartheta_n]$ of Δ_n by the path $[1+i\vartheta_n, 1, +\infty]$, and the portion to the left of the line $\operatorname{Re} s = -2$ by the curve $s_n(v)$, $-\infty < v \leqslant -2$ together with the straight line segment $[s_n(-2), -2+i\vartheta_n]$ and obtain

$$\alpha_n = \lim_{v\to-\infty} Q(s_n(v)). \tag{5.19}$$

Now let s in the range (5.16) be given. Suppose first that $|\tau-\vartheta_n| \leqslant \frac{2}{3}\pi$. If the constant c_3 in (5.16) is sufficiently large, then (5.7) implies that $|\tau_{2n-1}-(\vartheta_n-\pi)| < \frac{1}{3}\pi$ and $|\tau_{2n+1}-(\vartheta_n+\pi)| < \frac{1}{3}\pi$, so that $\tau \in [\tau_{2n-1}, \tau_{2n+1}]$. We then obtain as before

$$Q(s) = Q(s_n(\sigma)) + O\left(\frac{\|\varphi\| e^{-(u_0+1)\sigma}}{|\lambda(s)|} \right). \tag{5.20}$$

Moreover, for $v \leqslant \sigma$ and in the range (5.16) we have by (5.10)

$$\frac{\mathrm{d}}{\mathrm{d}v} \log |\lambda\big(s_n(v)\big)| \leqslant -(u_0+2)$$

and hence

$$|\lambda(s_n(v))| \geqslant |\lambda(s_n(\sigma))| e^{(u_0+2)(\sigma-v)} \gg |\lambda(s)| e^{(u_0+2)(\sigma-v)},$$

provided the constant c_3 in (5.16) is sufficiently large. Thus (5.19) yields

$$\begin{aligned}|Q(s_n(\sigma)) - \alpha_n| &= \left| \int_{-\infty}^{\sigma} \frac{\Phi(s_n(v))}{\lambda(s_n(v))} s_n'(v) \,\mathrm{d}v \right| \\ &\ll \|\varphi\|\, e^{-(u_0+1)\sigma} \int_{-\infty}^{\sigma} \frac{e^{-(u_0+1)(v-\sigma)}}{|\lambda(s_n(v))|} (1 + |\tau_{2n}'(v)|) \,\mathrm{d}v \\ &\ll \frac{\|\varphi\|\, e^{-(u_0+1)\sigma}}{|\lambda(s)|} \int_{-\infty}^{\sigma} e^{v-\sigma} \,\mathrm{d}v \ll \frac{\|\varphi\| e^{-(u_0+1)\sigma}}{|\lambda(s)|}.\end{aligned}$$

This, with (5.20), proves (5.17) in the case $|\tau - \vartheta_n| \leqslant \frac{2}{3}\pi$. For $\frac{2}{3}\pi < |\tau - \vartheta_n| < \pi$, (5.17) follows from (5.15), since then, by (5.16) and Lemma 2, $|\lambda(s)| \ll (|\tau| + 1)^{\operatorname{Re} a} \ll |\lambda\big(s_n(\sigma)\big)|$ and

$$|\alpha_n| \leqslant |Q(s_n(\sigma)) - \alpha_n| + |Q(s_n(\sigma))| \ll \frac{\|\varphi\| e^{-(u_0+1)\sigma}}{|\lambda\big(s_n(\sigma)\big)|} \ll \frac{\|\varphi\| e^{-(u_0+1)\sigma}}{|\lambda(s)|}$$

by Lemma 2 and (5.15) applied to $Q(s_n(\sigma))$. The proof of Lemma 6 is therefore complete.

Lemma 7. *We have*

$$|\widehat{f}(s)| \ll \frac{\|\varphi\|\, e^{-u_0\sigma}}{|s|} \qquad (\sigma \geqslant -1,\ |s| \geqslant 1), \tag{5.21}$$

(5.22)

$$|\widehat{f}(s)| \ll \frac{\|\varphi\|\, e^{(u_0+1)|\sigma|}}{|s|} \left(1 + \exp\left\{ Re \left(b \frac{e^{-s}}{-s} T(-s) \right) + O\left(\frac{e^{|\sigma|}}{|s|^4} + 1 \right) \right\} \right) \ (\sigma < -1),$$

and in the range (5.16),

$$\widehat{f}(s) = \alpha_n \frac{\lambda(s)}{s} + O\left(\frac{\|\varphi\| e^{(u_0+1)|\sigma|}}{|s|} \right). \tag{5.23}$$

Moreover,

$$\widehat{f}(s)' = -\frac{\Phi(s)}{s} + O\left(\frac{|\widehat{f}(s)| e^{|\sigma|}}{|s|} \right) \qquad (\sigma \leqslant -1). \tag{5.24}$$

Proof. The first three estimates are immediate consequences of Lemmas 2, 4, and 6, and the relation

$$|s^a| (|\tau| + 1)^{-\operatorname{Re} a} \ll \exp\left\{ O\left(e^{|\sigma|} |s|^{-4} \right) \right\} \qquad (\sigma \leqslant -1).$$

The last estimate follows from the relation

$$\widehat{f}(s)' = -\frac{\Phi(s)}{s} - \widehat{f}(s)\Big(\frac{be^{-s}+a-1}{s}\Big),$$

established in the proof of Lemma 4.

Lemma 8. *Suppose $h \notin \mathbb{Z}^+$. Then there is an entire function $E(s)$ such that*

$$\widehat{f}(s) = \frac{\Gamma(1-h)\alpha\lambda(s)}{s} + E(s) \quad (s \in \mathbb{C} \smallsetminus \mathcal{C}). \tag{5.25}$$

Proof. Let $s \in \mathbb{C} \smallsetminus \mathcal{C}$ and consider

$$Q(s) = \lim_{T\to+\infty} \int_s^T \frac{\Phi(z)}{\lambda(z)}\,\mathrm{d}z,$$

where the integral may be taken over any path from s to T $\quad(T \in \mathbb{R}^+)$ within $\mathbb{C} \smallsetminus \mathcal{C}$. Since $z^h\Phi(z)/\lambda(z) = \Phi(z)e^{-bI(-z)}$ is an entire function, we have

$$\frac{\Phi(z)}{\lambda(z)} = z^{-h}\sum_{k\geqslant 0}\gamma_k z^k,$$

where z^{-h} is defined using the same branch of $\log z$ as in the definition of $\lambda(z)$ and the series $\sum \gamma_k z^k$ has an infinite radius of convergence. Integrating termwise, we obtain

$$Q(s) = A + s^{1-h}E_1(s), \tag{5.26}$$

with

$$E_1(s) := -\sum_{k\geqslant 0}\frac{\gamma_k}{k-h+1}s^k, \quad A := Q(1) - E_1(1).$$

Since $h \notin \mathbb{Z}^+$, the coefficients in the last series are $\ll |\gamma_k|$ and $E_1(s)$ is an entire function. In view of the formula $\widehat{f}(s) = \lambda(s)Q(s)/s$, we obtain hence

$$\widehat{f}(s) = A\frac{\lambda(s)}{s} + E(s)$$

with

$$E(s) := \lambda(s)s^{-h}E_1(s) = e^{bI(-s)}E_1(s),$$

which is an entire function.

Thus (5.26) implies (5.25) provided we can show that

$$A = \Gamma(1-h)\alpha.$$

In order to evaluate A, we introduce the function

$$Q_0(s) = \lim_{T\to+\infty} \int_s^{T+i0} \frac{\Phi(z)}{\lambda_0(z)}\,\mathrm{d}z.$$

where $\lambda_0(z)$ is defined as in Section 2, and the integral is taken over an arbitrary path from s to $T+i0$ within $\mathbb{C}\smallsetminus\big(\mathbb{R}^+\cup\{0\}\big)$. An argument similar to that giving (5.26) yields

$$Q_0(s) = B + s^{1-h}E_1(s), \tag{5.28}$$

where s^{1-h} is now defined using the determination of $\arg s$ in $(0,2\pi)$. Since the argument has the same value on $\mathbb{R}^+ + i0$ in (5.26) and in (5.28), we have

$$Q(T) = Q_0(T+i0)$$

for all positive T, and hence $A=B$.

Next, we may write

$$\begin{aligned} Q_0(T-i0) &= A + e^{-2\pi hi}T^{1-h}E_1(T) \\ &= A + e^{-2\pi hi}\big(Q_0(T+i0)-A\big). \end{aligned}$$

This yields

$$A = \frac{Q_0(T-i0) - e^{-2\pi hi}Q_0(T+i0)}{1-e^{-2\pi hi}} \qquad (T>0).$$

Letting $T\to+\infty$, we get

$$A = \frac{L}{1-e^{-2\pi hi}},$$

with

$$\begin{aligned} L := Q(+\infty - i0) &= \int_{\Delta^*} \frac{\Phi(z)}{\lambda_0(z)}\,\mathrm{d}z \\ &= u_0\varphi(u_0)\int_{\Delta^*}\frac{e^{-u_0z}}{\lambda_0(z)}\,\mathrm{d}z - b\int_{u_0-1}^{u_0}\varphi(u)\left(\int_{\Delta^*}\frac{e^{-(u+1)z}}{\lambda_0(z)}\,\mathrm{d}z\right)\mathrm{d}u \\ &= \frac{2\pi i\,e^{-\pi hi}}{\Gamma(h)}\left\{u_0\varphi(u_0)G(u_0) - b\int_{u_0-1}^{u_0}\varphi(u)G(u+1)\,\mathrm{d}u\right\} \\ &= \frac{2\pi i\,e^{-\pi hi}}{\Gamma(h)}\alpha. \end{aligned}$$

Hence

$$A = \frac{\pi\alpha}{\Gamma(h)\sin(\pi h)} = \alpha\Gamma(1-h),$$

as required, by the reflection formula for the Γ-function.

5. Proof of Theorem 2

For ease of exposition we only consider here uniform convergence in the sense of Cauchy's principal value for the series in (3.2). It will be clear to the reader that the proof extends, in a straightforward manner, to usual uniform convergence.

Since $f(u)$ has a continuous derivative for $u > u_0$ and by (5.2) the Laplace integral (5.1) is absolutely convergent for $\sigma > 0$, we have by the Laplace inversion formula

$$f(u) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \widehat{f}(s)e^{us}\,\mathrm{d}s \qquad (u > u_0). \tag{6.1}$$

Let N be any given nonnegative integer. We put

$$T^+ := \vartheta_N + \pi, \qquad T^- := \vartheta_{-N} - \pi,$$

and define the paths

$$W^+ := [-\infty + iT^+, -1 + iT^+, -1 + i\infty],$$

$$W^- := [-1 - i\infty, -1 + iT^-, -\infty + iT^-].$$

Then, the analytic properties of $\widehat{f}(s)$ established in the previous section enable us to derive from (6.1) the formula

$$f(u) = A(u) + \sum_{|n|\leqslant N} A_n(u) + R_N^+(u) + R_N^-(u), \tag{6.2}$$

with

$$A(u) := \frac{1}{2\pi i}\int_{\Gamma^*} \widehat{f}(s)e^{us}\,\mathrm{d}s,$$
$$A_n(u) := \frac{1}{2\pi i}\int_{\Gamma_n} \widehat{f}(s)e^{us}\,\mathrm{d}s,$$
$$R_N^{\pm}(u) := \frac{1}{2\pi i}\int_{W^{\pm}} \widehat{f}(s)e^{us}\,\mathrm{d}s.$$

Indeed, this follows from the residue theorem, observing that, by Lemma 7, $|\widehat{f}(s)| \to 0$, as $|\tau| \to +\infty$, uniformly for $|\sigma| \leqslant 1$, and that, by (5.22), $|\widehat{f}(s)\,e^{us}|$ is exponentially decreasing , for any fixed $u > u_0 + 1$, along the infinite horizontal branches of the paths $\Gamma_n \quad (|n| \leqslant N)$ or $W^{\pm}$.

We are going to prove that we have for $u > u_0 + 1$

$$A(u) = \alpha F(u), \tag{6.3}$$

$$A_n(u) = \alpha_n F_n(u), \tag{6.4}$$

and the error terms $R_N^{\pm}(u)$ satisfy

$$R_N^{\pm}(u) \ll_\delta \frac{\|\varphi\|}{N+1} \qquad (u \geqslant u_0 + 1 + \delta), \tag{6.5}$$

$$|R_N^{\pm}(u)| \ll \|\varphi\| \Big[\exp\{-uZ^+\} + \exp\{-uZ^-\} \Big] \exp\{O(u/\xi^3)\} \quad (u \geqslant u_1(N)), \tag{6.6}$$

with $Z^{\pm} := \mathrm{Re}\,(\zeta_{\mp N \mp 1} - T(\zeta_{\mp N \mp 1}))$.

Applying Theorem 1 and Corollary 1, we see that (6.6) implies the estimate (3.5) for the error term $R_N(u)$ in (3.4) and thus we obtain the assertions of Theorem 2 from (6.2)–(6.6).

To prove (6.3), we first assume that $h \notin \mathbb{Z}$. In this case we can apply Lemma 8 and obtain

$$A(u) = \frac{\alpha\Gamma(1-h)}{2\pi i} \int_{\Gamma^*} \frac{\lambda(s)}{s} e^{us}\, \mathrm{d}s = \alpha F(u),$$

since the integral of the entire function $E(s)$ is trivially zero. The above relation remains valid in the case $h \in \mathbb{Z}$, since, by Remark (ii) of Section 2, both sides are continuous functions in a and b, and hence in h. Thus (6.3) holds in every circumstance.

Next we show (6.4). By the residue theorem we have $A_n(u) = \lim_{\sigma\to-\infty} A_n(u,\sigma)$ with

$$A_n(u,\sigma) := \frac{1}{2\pi i} \int_{\sigma+i(\vartheta_n-\pi)}^{\sigma+i(\vartheta_n+\pi)} \widehat{f}(s) e^{us}\, \mathrm{d}s.$$

Estimating $\widehat{f}(s)$ by (5.23), we obtain for $\sigma \leqslant -c_3(|n|+1)$

$$A_n(u,\sigma) = \alpha_n \frac{1}{2\pi i} \int_{\sigma+i(\vartheta_n-\pi)}^{\sigma+i(\vartheta_n+\pi)} \frac{\lambda(s)}{s} e^{us}\, \mathrm{d}s + O\left(\frac{\|\varphi\| e^{(u-u_0-1)\sigma}}{|\sigma|} \right).$$

Since $u > u_0 + 1$, the error term here tends to zero as $\sigma \to -\infty$. The main term tends to $\alpha_n F_n(u)$, by another application of the residue theorem. This completes the proof of (6.4).

For the proof of (6.5) and (6.6), we restrict ourselves to $R_N^+(u)$, the other case being similar. We begin with (6.5). We select $\sigma = -1$, and obtain by partial summation

$$\begin{aligned} \left| \int_{T^+}^{+\infty} \widehat{f}(s) e^{us}\, \mathrm{d}\tau \right| &\leqslant \frac{1}{u} \left| \widehat{f}(-1+iT^+) e^{-u} \right| + \frac{1}{u} \left| \int_{T^+}^{+\infty} \widehat{f}(s)' e^{us}\, \mathrm{d}\tau \right| \\ &\ll |\widehat{f}(-1+iT^+)| + \left| \int_{T^+}^{+\infty} \frac{\Phi(s) e^{us}}{s}\, \mathrm{d}\tau \right| + \left| \int_{T^+}^{+\infty} \left(\widehat{f}(s)' + \frac{\Phi(s)}{s} \right) e^{us}\, \mathrm{d}\tau \right|. \end{aligned}$$

By (5.21) and (5.24), the first and third terms are

$$\ll \frac{\|\varphi\|}{T^+} \ll \frac{\|\varphi\|}{N+1}. \tag{6.7}$$

Moreover, the second term is plainly

$$\leqslant |u_0\varphi(u_0)|\left|\int_{T^+}^{+\infty}\frac{e^{(u-u_0)s}}{s}\,\mathrm{d}\tau\right| + |b|\int_{u_0-1}^{u_0}\left|\varphi(v)\int_{T^+}^{+\infty}\frac{e^{(u-v-1)s}}{s}\,\mathrm{d}\tau\right|\,\mathrm{d}v,$$

which, by a further integration by parts, is also seen to be of the size (6.7) under the hypothesis $u \geqslant u_0+1+\varepsilon$. This yields the required bound for the contribution to $R_N^+(u)$ of the vertical part of the path W^+. The contribution of the horizontal part can plainly be bounded by

$$\begin{aligned}&\ll \max_{\sigma\leqslant -1}\{|\widehat{f}(\sigma+iT^+)|e^{-|\sigma|(u_0+1)}\}\int_{-\infty}^{-1}e^{(u-u_0-1)\sigma}\,\mathrm{d}\sigma\\ &\ll \max_{\sigma\leqslant -1}\big\{|\widehat{f}(\sigma+iT^+)|e^{-|\sigma|(u_0+1)}\big\}.\end{aligned}$$

By the estimate (5.22) of Lemma 7 this is

$$\ll \frac{\|\varphi\|}{1+|T^+|}\Big(1+e^{M(-1,T^+)}\Big) \tag{6.8}$$

with

$$M(v,T) := \max_{\sigma\leqslant v}\,\mathrm{Re}\,\left\{b\frac{e^{-s}}{-s}T(-s)+c_4\left(\frac{e^{-\sigma}}{|s|^4}+1\right)\right\} \tag{6.9}$$

for a suitable constant c_4. Now, for $s=\sigma+iT^+=\sigma+i(\vartheta_N+\pi)$, $\sigma\leqslant -1$, we have

$$\mathrm{Re}\,\left\{b\frac{e^{-s}}{-s}T(-s)\right\} = -\frac{\kappa e^{|\sigma|}}{|s|}\left\{\cos\varphi-\frac{1}{|s|}\cos(2\varphi)+\frac{2}{|s|^2}\cos(3\varphi)\right\}, \tag{6.10}$$

where $\varphi := \arctan(\tau/\sigma)$. It is easy to check that, for $|\varphi|\leqslant \frac{1}{2}\pi$, the expression in parantheses is $\geqslant c_5/|s|^2$ for a suitable positive constant c_5, provided $|s|$ is sufficiently large. Hence the right-hand side of (6.9) is $\leqslant O(1)$ for $v=-1$, and this completes the proof of (6.5).

Finally, we prove (6.6). Since $|\widehat{f}(s)|\to 0$ as $|\tau|\to+\infty$ uniformly for σ in any fixed vertical strip $\sigma_1\leqslant\sigma\leqslant\sigma_2$, we may replace in the definition of W^+ the number -1 by any real number r. We select $r=-z$, with $z:=\mathrm{Re}\,\zeta_{-N-1}$. The argument used above in (6.8)–(6.10) works equally well in this context and yields the bound

$$\ll \|\varphi\|\,e^{-(u-u_0)z}$$

for the contribution of the horizontal branch of the new path, and this is sufficient for (6.6). It remains to evaluate the contribution of the vertical part of the path. Using the same argument again, as well as Lemmas 2 and 7, we obtain the bound

$$\ll \|\varphi\|e^{-(u-u_0)z}\max_{t\geqslant T^+}|\widehat{f}(-z+it)| \ll \|\varphi\|\exp\{-(u-u_0)z+N(-z,T^+)\}$$

with

$$N(\sigma, T) := \max_{t \geqslant T} \mathrm{Re} \left\{ b \frac{e^{-s}}{-s} T(-s) + c_4 \left(\frac{e^{-\sigma}}{|s|^4} + 1 \right) \right\}.$$

Thus, to prove (6.6), it suffices to show that

$$N(-\mathrm{Re}\,\zeta_{-N-1}, T^+) \leqslant u \left(T(\zeta_{-N-1}) + O\left(\frac{1}{\xi^3}\right) \right),$$

If u is sufficiently large in terms of ε and N this may easily be deduced from the representation (6.10), the definition of ζ_{-N-1}, and Lemma 1. This finishes the proof.

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