On the friable mean-value of the Erdős-Hooley Delta function*

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Abstract. For integer n and real u, define $\Delta(n,u) := |\{d : d \mid n,\ \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|$. Then, the Erdős-Hooley Delta function is defined as $\Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u)$. We provide uniform upper and lower bounds for the mean-value of $\Delta(n)$ over friable integers, i.e. integers free of large prime factors.

1. Introduction and statement of results

For integer $n \geqslant 1$ and real u, put

$$\Delta(n,u) := |\{d : d \mid n,\ \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \qquad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u).$$

The Δ-function was introduced by Erdős in 1974 and was highlighted in 1979 by Hooley [13]. It turned out to be a key-concept in many branches of analytic number theory such as Waring type problems, circle method, Diophantine approximation, distribution of prime factors in polynomial sequences, etc.

However, the behaviour of $\Delta(n)$ remains rather mysterious. For instance, the average order is still not known with desirable precision. Hall and Tenenbaum [8] obtained in 1982 the lower bound

$$D(x) := \sum_{n\leqslant x} \Delta(n) \gg x \log_2 x \qquad (x \geqslant 3), \tag{1.1}$$

whereas Tenenbaum [16] showed in 1985 that for suitable $c > 0$ we have

$$D(x) \ll x\mathrm{e}^{c\sqrt{\log_2 x \log_3 x}} \qquad (x \geqslant 16). \tag{1.2}$$

Here and in the sequel, we let $\log_k$ denote the k-fold iterated logarithm. Recently, La Bretèche and Tenenbaum [3, th. 1.1] obtained a slight improvement to (1.2) by removing the triple logarithm in the exponent and, even more recently, Koukoulopoulos and Tao [14] obtained the remarkable bound

$$D(x) \ll x(\log_2 x)^{11/4} \qquad (x \geqslant 3).$$

A few months later, Ford, Koukouloulos and Tao [7] improved (1.1) by showing

$$D(x) \gg x(\log_2 x)^{1+\eta+o(1)} \qquad (x \geqslant 3),$$

where the exponent $\eta \approx 0.3533227$ appears in the work of Ford, Green and Koukoulopoulos [6] on the normal order of $\Delta(n)$. Both bounds have been recently improved by La Bretèche and Tenenbaum [4]: we have

$$x(\log_2 x)^{3/2} \ll D(x) \ll x(\log_2 x)^{5/2} \qquad (x \geqslant 3), \tag{1.3}$$

which constitutes the current state of the art.

Let $P^+(n)$ denote the largest prime factor of an integer $n > 1$ and let us agree that $P^+(1) = 1$. Following usual notation, we define $S(x,y)$ as the set of y-friable integers not exceeding x, and denote by $\Psi(x,y)$ its cardinality, viz.

$$S(x,y) := \{n \leqslant x : P^+(n) \leqslant y\}, \qquad \Psi(x,y) = |S(x,y)| \quad (x \geqslant 1, y \geqslant 1).$$

Structural properties of the set $S(x,y)$ motivated a vast array of the literature in the last fourty years. The applications are indeed numerous and significant: circle method, Waring-type problems, cryptology, sieve theory, probabilistic models in number theory.

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Given an arithmetical function f, let us use the notation $\Psi(x,y;f) := \sum_{n\in S(x,y)} f(n)$. In this work we investigate bounds for the friable mean-value

$$\mathfrak{S}(x,y) := \frac{\Psi(x,y;\Delta)}{\Psi(x,y)} \quad (x \geqslant y \geqslant 2). \tag{1.4}$$

We now define some quantities arising in our statements. Given $\kappa > 0$, denote by ϱ_κ the continuous solution on $]0,\infty[$ of the delay differential system

$$\begin{cases} \varrho_\kappa(v) = v^{\kappa-1}/\Gamma(\kappa) & (0 < v \leqslant 1),\\ v\varrho_\kappa'(v) + (1-\kappa)\varrho_\kappa(v) + \kappa\varrho_\kappa(v-1) = 0 & (v > 1),\end{cases}$$

and set $\varrho_\kappa(v) := 0$ for $v < 0$.

Thus (see, e.g., [12]) ϱ_κ is the order κ fractional convolution power of $\varrho := \varrho_1$, the Dickman function, which provides a continuous approximation to $\Psi(x,y)$ in

$$H_\varepsilon := \Big\{(x,y) : \ x \geqslant 3, \ \mathrm{e}^{(\log_2 x)^{5/3+\varepsilon}} \leqslant y \leqslant x\Big\} \qquad (\varepsilon > 0). \tag{1.5}$$

Indeed, improving on results by Dickman and de Bruijn, Hildebrand [10] proved the asymptotic formula

$$\Psi(x,y) = x\varrho(u)\bigg\{1 + O\bigg(\frac{\log(2u)}{\log y}\bigg)\bigg\} \quad ((x,y)\in H_\varepsilon), \tag{1.6}$$

with the standard notation

$$u = \frac{\log x}{\log y}.$$

The asymptotic behaviour of the functions ϱ_κ (and in fact of more general delay differential equations, as displayed in [12]) may be described in terms of the function $\xi(t)$ defined as the unique positive solution to $\mathrm{e}^\xi = 1 + t\xi$ for $t \neq 1$ and by $\xi(1) = 0$. From [17, lemma III.5.11] and the remark following [17, th. III.5.13], we quote the estimates

$$\xi(t) = \log t + \log_2 t + O\Big(\frac{\log_2 t}{\log t}\Big), \quad \xi'(t) = \frac{1}{t} + \frac{1}{t\log t} + O\Big(\frac{\log_2 t}{t(\log t)^2}\Big) \qquad (t\to\infty). \tag{1.7}$$

Applying [12, cor. 2] in the case $(a,b) = (1-\kappa,\kappa)$, we have

$$\varrho_\kappa(v) = \sqrt{\frac{\xi'(v/\kappa)}{2\pi\kappa}}\exp\Big\{\kappa\gamma - \kappa\int_1^{v/\kappa}\xi(t)\,\mathrm{d}t\Big\}\Big\{1 + O\Big(\frac{1}{v}\Big)\Big\} \qquad (v \geqslant 1+\kappa), \tag{1.8}$$

where γ denotes Euler's constant. We put

$$\mathfrak{r}(v) := \frac{\varrho_2(v)}{\sqrt{v}\varrho(v)} \asymp \frac{1}{\sqrt{v}}\exp\bigg(\int_1^v \{\xi(t) - \xi(t/2)\}\,\mathrm{d}t\bigg) \asymp 2^{v+O(v/\log 2v)} \qquad (v \geqslant 1), \tag{1.9}$$

while a genuine asymptotic formula follows from (1.8).

Let $\tau(n)$ denote the total number of divisors of an integer n. We trivially have

$$\tau(n)/\log 2n \ll \Delta(n) \leqslant \tau(n) \qquad (n \geqslant 1), \tag{1.10}$$

where the lower bounds follows from the pigeon-hole principle. Since, by [19, cor. 2.3], we have

$$\Psi(x,y;\tau) = \bigg\{1 + O\bigg(\frac{\log(2u)}{\log y}\bigg)\bigg\}x\varrho_2(u)\log y \quad ((x,y)\in H_\varepsilon),$$

we may state as a benchmark that

$$\frac{\mathfrak{r}(u)}{\sqrt{u}} \ll \mathfrak{S}(x,y) \ll 2^{u+O(u/\log 2u)}\log y \qquad ((x,y)\in H_\varepsilon).$$

We obtain the following results, where the following notation is used:

$$\overline{u} := \min\Big(\frac{y}{\log y}, u\Big) \qquad (x \geqslant y \geqslant 2), \tag{1.11}$$

$$g(t) := \log\bigg(\frac{(1+2t)^{1+2t}}{(1+t)^{1+t}(4t)^t}\bigg) \qquad (t > 0), \tag{1.12}$$

$$\varepsilon_y := \frac{1}{\sqrt{\log y}} \quad (y \geqslant 2). \tag{1.13}$$

Theorem 1.1. (i) *Let $\varepsilon > 0$. For a suitable absolute constant $c > 0$ and uniformly for $(x, y) \in H_\varepsilon$, we have*

$$\log_2 y + \mathfrak{r}(u) \ll \mathfrak{S}(x,y) \ll 2^{u+O(u/\log 2u)} \mathrm{e}^{c\sqrt{\log_2 y \log_3 y}}. \tag{1.14}$$

(ii) *For $2 \leqslant y \leqslant x^{1/(2\log_2 x \log_3 x)}$, and with $\lambda := y/\log x$, we have*

$$\mathfrak{S}(x,y) \asymp \mathrm{e}^{\{1+O(\varepsilon_y + 1/\log 2u)\}g(\lambda)u}. \tag{1.15}$$

Note that g is positive and strictly increasing on $(0, +\infty)$. The asymptotic behaviour of this function is given by

$$g(\lambda) = \begin{cases} \log 2 - 1/(4\lambda) + O(1/\lambda^2) & \text{as } \lambda \to \infty, \\ \lambda \log(1/\lambda) - \lambda(\log 4 - 1) + O(\lambda^2) & \text{as } \lambda \to 0. \end{cases} \tag{1.16}$$

Morever, the lower bound $g(\lambda)u \gg \overline{u}$ holds on the whole range $x \geqslant y \geqslant 2$.

The error term in (1.15) may be simplified to $1/\log 2u$ if $\log y > (\log_2 x)^2$ and to ε_y otherwise.

Note that (1.10) implies

$$\mathfrak{S}(x,y) \asymp \frac{\Psi(x,y;\tau)}{u^K \Psi(x,y)} \qquad \Big(\log y \leqslant \sqrt{\log x}\Big),$$

with $K = K(x,y) \in [0,2]$, so that, to the stated accuracy, the evaluation of $\mathfrak{S}(x,y)$ reduces in this range to that of $\Psi(x,y;\tau)/\Psi(x,y)$. This is consistent with the Gaussian tendency of the distribution of the divisors of friable integers: as the friability parameter y decreases, the divisors of friable n concentrate around the mean-value $\sqrt{n}$ and $\Delta(n)$ resembles more and more to $\tau(n)$, the total number of divisors. Another description of this phenomenon appears in [5].

Considering available methods, Theorem 1.1 essentially agrees with standard expectations regarding methodology. We leave to a further project the task of adapting the method of [14] or [4] in the upper bound of (1.14). We note right away that, in the present context, such an improvement would only be relevant for very large values of y since the exponent $\sqrt{\log_2 y \log_3 y}$ is absorbed by the remainder $O(u/\log 2u)$ as soon as $y \leqslant x^{1/(\log_2 x)^c}$ with $c > 1/2$.

2. Preliminary estimates

Here and throughout, the letter p denotes a prime number. In [11], Hildebrand and Tenenbaum provided a universal estimate for $\Psi(x,y)$ by the saddle-point method. Define

$$\zeta(s,y) := \prod_{p \leqslant y} \Big(1 - \frac{1}{p^s}\Big)^{-1}, \quad \varphi_y(s) := -\frac{\zeta'(s,y)}{\zeta(s,y)} \quad (\Re s > 0, y \geqslant 2),$$

and, for $2 \leqslant y \leqslant x$, let $\alpha = \alpha(x,y)$ denote the unique positive solution to the equation $\varphi_y(\alpha) = \log x$. According to [11, th. 1], we have

$$\Psi(x,y) = \frac{x^\alpha \zeta(\alpha,y)}{\alpha\sqrt{2\pi|\varphi_y'(\alpha)|}} \Big\{1 + O\Big(\frac{1}{u} + \frac{\log y}{y}\Big)\Big\} \quad (x \geqslant y \geqslant 2). \tag{2.1}$$

By [11, (2.4)]), we have

$$\alpha = \frac{\log(1 + y/\log x)}{\log y} \Big\{1 + O\Big(\frac{\log_2 y}{\log y}\Big)\Big\} \quad (x \geqslant y \geqslant 2). \tag{2.2}$$

Moreover, by [11, (7.8)], we have, for any given $\varepsilon > 0$,

$$\alpha = 1 - \frac{\xi(u)}{\log y} + O\Big(\mathrm{e}^{-(\log y)^{(3/5)-\varepsilon}} + \frac{1}{u(\log y)^2}\Big) \quad (x \geqslant x_0(\varepsilon),\ (\log x)^{1+\varepsilon} \leqslant y \leqslant x). \tag{2.3}$$

Finally, by [11, (2.5)], we have

$$|\varphi_y'(\alpha)| = \Big(1 + \frac{\log x}{y}\Big) \log x \, \log y \Big\{1 + O\Big(\frac{1}{\log(u+1)} + \frac{1}{\log y}\Big)\Big\} \quad (x \geqslant y \geqslant 2). \tag{2.4}$$

3. Proof of Theorem 1.1(i): lower bound

Let $\tau(n)$ denote the total number of divisors of a natural integer n. The following inequality is established in [9, lemma 60.1]

$$\Delta(n)\tau(n) \geqslant \sum_{\substack{d,d'\mid n\\ 0<\log(d'/d)\leqslant 1}} 1 = \sum_{\substack{dd'\mid n\\ (d,d')=1\\ 0<\log(d'/d)\leqslant 1}} \tau\Big(\frac{n}{dd'}\Big) \qquad (n\geqslant 1),$$

the equality above being obtained by representing the ratios d'/d in reduced form.

Put

$$u_t := \frac{\log t}{\log y} \quad (t\geqslant 1, y\geqslant 2), \quad \Omega(n) := \sum_{p^\nu\| n} \nu \quad (n\geqslant 1).$$

Since $\tau(ab)\leqslant \tau(a)2^{\Omega(b)}$ $(a,b\geqslant 1)$, we have, for $(x,y)\in H_\varepsilon$,

$$\mathfrak{S}(x,y) \geqslant \frac{1}{\Psi(x,y)} \sum_{\substack{dd'\in S(x,y)\\ (d,d')=1\\ 0<\log(d'/d)\leqslant 1}} \frac{1}{2^{\Omega(dd')}}\Psi\Big(\frac{x}{dd'},y\Big) \gg \sum_{\substack{dd'\in S(x,y)\\ (d,d')=1\\ 0<\log(d'/d)\leqslant 1}} \frac{\varrho(u-u_{dd'})}{\varrho(u)dd'2^{\Omega(dd')}}, \tag{3.1}$$

where the last inequality follows from (1.6). To evaluate the double sum in (3.1), we establish an asymptotic formula for

$$T_d(x,y) := \sum_{\substack{m\in S(x,y)\\ (m,d)=1}} \frac{1}{2^{\Omega(m)}}\cdot$$

We shall make use of the following notation

$$C := \prod_p \frac{\sqrt{1-1/p}}{1-1/2p}, \quad \kappa_y := \frac{1}{(\log y)^{2/5}},$$

$$\varphi_y(d) := \prod_{p\mid d}\Big(1+\frac{1}{2p^{1-\kappa_y}}\Big), \quad \vartheta_y(d) := \sum_{p\mid d}\frac{\log p}{p^{1-\kappa_y}}, \quad \mathfrak{q}(d) := \prod_{p\mid d}\Big(1-\frac{1}{2p}\Big) \quad (d\geqslant 1).$$

Lemma 3.1. *Let $\varepsilon>0$. For $x\geqslant 1$, $y>\exp\{(\log_2 3x)^{5/3+\varepsilon}\}$, $d\in S(x,y)$, we have*

$$T_d(x,y) = \frac{Cx\varrho_{1/2}(u)}{\sqrt{\log y}}\Big\{\mathfrak{q}(d) + O\Big(\kappa_y\varphi_y(d)\{1+\vartheta_y(d)\}\Big)\Big\}. \tag{3.2}$$

Proof. We have

$$T_d(x,y) = \sum_{m\in S(x,y)} \frac{1}{2^{\Omega(m)}}\sum_{t\mid (m,d)}\mu(t) = \sum_{t\mid d}\frac{\mu(t)}{2^{\Omega(t)}}T_1\Big(\frac{x}{t},y\Big).$$

An estimate for the inner T_1-term follows from [19, cor. 2.3], which, in the domain

$$x\geqslant 1, \quad y>\exp\{(\log_2 3x)^{5/3+\varepsilon}\},$$

we rewrite as

$$T_1(x,y) = \frac{Cx\varrho_{1/2}(u)}{\sqrt{\log y}}\Big\{1+O\Big(\frac{\log(u+1)}{\log y}+\frac{1}{\sqrt{\log y}}+\frac{1}{\log(2x)}\Big)\Big\}. \tag{3.3}$$

Here the error term $1/\log(2x)$ enables to include the case $1\leqslant x<y$: the corresponding estimate follows from [17, th. II.6.2]. Since $\log(u+1)\ll(\log y)^{3/5}$ in H_ε, we get

$$T_d(x,y) = \frac{Cx}{\sqrt{\log y}} \sum_{\substack{t\mid d\\ t\leqslant x/\sqrt{y}}} \frac{\mu(t)\varrho_{1/2}(u-u_t)}{t2^{\Omega(t)}} + R_1 + R_2, \tag{3.4}$$

with

$$R_1 \ll \frac{x}{(\log y)^{9/10}} \sum_{\substack{t\mid d\\ t\leqslant x/\sqrt{y}}} \frac{\mu(t)^2\varrho_{1/2}(u-u_t)}{t2^{\Omega(t)}} \ll \frac{x\varrho_{1/2}(u)}{(\log y)^{9/10}}\sum_{t\mid d}\frac{\mu(t)^2}{2^{\Omega(t)}t^{1-\xi(2u)/\log y}},$$

$$R_2 \ll \sum_{\substack{t\mid d\\ x/\sqrt{y}<t\leqslant x}} \frac{x}{t\sqrt{\log 2x/t}},$$

where the bound for R_1 follows from

$$\varrho_{1/2}(u-v) \ll \varrho_{1/2}(u)\mathrm{e}^{v\xi(2u)} \qquad (u \geqslant 1,\ 0 \leqslant v \leqslant u - \tfrac{1}{2}) \tag{3.5}$$

proved in [15][1]. By multiplicativity, we thus get

$$R_1 \ll \frac{x\varrho_{1/2}(u)\varphi_y(d)}{(\log y)^{9/10}}\cdot \tag{3.6}$$

Since $d \leqslant x$, we have $p_{\omega(d)} \ll \log x$, where $p_{\omega(d)}$ denotes the $\omega(d)$th prime number. Hence, using de Bruijn's estimate for $\log \Psi(x,y)$ as refined in [17, th. III.5.2], we plainly obtain, for a suitable absolute constant $c > 0$,

$$\sum_{t|d,\, t\leqslant z} 1 \leqslant \Psi(z, p_{\omega(d)}) \leqslant z^{c/\log_2 x} \qquad \big(\sqrt{x} \leqslant z \leqslant x\big). \tag{3.7}$$

As a consequence

$$R_2 \ll x \int_{\sqrt{x}}^{x} \frac{1}{z}\,\mathrm{d}O\big(z^{c/\log_2 x}\big) \ll \sqrt{x}\mathrm{e}^{c\log x/\log_2 x},$$

and we conclude that

$$R_1 + R_2 \ll \frac{x\varrho_{1/2}(u)\varphi_y(d)}{(\log y)^{9/10}}\cdot \tag{3.8}$$

To estimate the main term of (3.4), we approximate $\varrho_{1/2}(u-u_t)$ by $\varrho_{1/2}(u)$, using the bound

$$\varrho_{1/2}'(w) \ll \varrho_{1/2}(w)\log(1+w) \qquad (w \geqslant \tfrac{1}{2})$$

which, with an appropriate modification of the range of validity, is also proved in [15, lemma 6.2]. In view of (3.5), this implies that

$$\varrho_{1/2}(u-u_t) - \varrho_{1/2}(u) \ll u_t\varrho_{1/2}(u)t^{\kappa_y}\log(u+1).$$

Thus,

$$\begin{aligned}\sum_{\substack{t|d\\ t\leqslant x/\sqrt{y}}} \frac{\mu(t)\varrho_{1/2}(u-u_t)}{t2^{\Omega(t)}\varrho_{1/2}(u)} &= \sum_{\substack{t|d\\ t\leqslant x/\sqrt{y}}} \frac{\mu(t)}{t2^{\Omega(t)}} + O\Bigg(\sum_{\substack{t|d\\ t\leqslant x/\sqrt{y}}} \frac{\mu(t)^2(\log t)\log(u+1)}{t^{1-\kappa_y}2^{\Omega(t)}\log y}\Bigg)\\ &= \mathfrak{q}(d) + O\Bigg(\sum_{\substack{t|d\\ x/\sqrt{y}<t\leqslant x}} \frac{1}{t} + \kappa_y\sum_{t|d}\frac{\mu(t)^2\log t}{t^{1-\kappa_y}2^{\Omega(t)}}\Bigg).\end{aligned}$$

By (3.7), the first error term is $\ll \sqrt{y}x^{-1+c/\log_2 x}$, which is compatible with (3.2). To estimate the second, we write $\log t = \sum_{p|t}\log p$ since $\mu^2(t) = 1$ and invert summations. This yields the required estimate (3.2). □

By (3.1), we have

$$\mathfrak{S}(x,y) \gg \sum_{d\in S(\sqrt{x}/\mathrm{e},y)} \frac{\varrho(u-2u_d)\{T_d(\mathrm{e}d,y) - T_d(d,y)\}}{\varrho(u)d^2 2^{\Omega(d)}}\cdot$$

We insert (3.2) to evaluate the difference between curly brackets and sum separately the resulting main term and the remainder terms. This can be done by partial summation, using a variant of (3.3) in which the inclusion of the factors $\mathfrak{q}(d)$ or $\varphi_y(d)\{1+\vartheta_y(d)\}$ has as sole effects to alter the value of the constant C. This yields

$$\begin{aligned}\mathfrak{S}(x,y) &\gg \sum_{d\in S(\sqrt{x}/\mathrm{e},y)} \frac{\mathfrak{q}(d)\varrho(u-2u_d)\varrho_{1/2}(u_d)}{\varrho(u)2^{\Omega(d)}d\sqrt{\log y}}\\ &\gg \frac{1}{\varrho(u)}\int_{1/\log y}^{u/2} \varrho(u-2v)\varrho_{1/2}(v)^2\,\mathrm{d}v = \frac{1}{2\varrho(u)}\int_{2/\log y}^{u} \varrho(u-v)\varrho_{1/2}(\tfrac{1}{2}v)^2\,\mathrm{d}v.\end{aligned} \tag{3.9}$$

The contribution of the interval $[2/\log y, 2]$ to the last integral is

$$\geqslant 2\varrho(u)\int_{1/\log y}^{1} \varrho_{1/2}(v)^2\,\mathrm{d}v = \frac{2\varrho(u)}{\pi}\int_{1/\log y}^{1}\frac{\mathrm{d}v}{v} = \frac{2\varrho(u)}{\pi}\log_2 y. \tag{3.10}$$

[1] In [15, lemma 6.1], this bound is claimed for $0 \leqslant v \leqslant u$, but it is necessary to exclude the case when $u - v$ is small.

Now observe that (1.8) implies

$$\varrho_{1/2}(\tfrac{1}{2}v)^2 \asymp \frac{\varrho(v)}{\sqrt{v}} \qquad (v \geqslant 1).$$

Since ϱ_2 is the convolution square of ϱ, it follows that

$$\frac{1}{\varrho(u)}\int_2^u \varrho(u-v)\varrho_{1/2}(\tfrac{1}{2}v)^2\,\mathrm{d}v \gg \frac{\varrho_2(u)}{\sqrt{u}\varrho(u)} = \mathfrak{r}(u). \tag{3.11}$$

Carrying back into (3.9) and taking (3.10) into account, we obtain the required estimate.

4. Proof of Theorem 1.1(i): upper bound

We adapt to the friable case the iterative method developed by Tenenbaum in [16] (see also [9, §7.4]) for bounding the mean-value of the Δ-function. Throughout this proof the letters c and C, with or without index, stand for absolute positive constants.

Given an integer $n \geqslant 2$, let us denote by $\{p_j(n)\}_{1\leqslant j\leqslant \omega(n)}$ the increasing sequence of its distinct prime factors. Following [16] (see also [9]), define

$$M_q(n) = \int_{\mathbb{R}} \Delta(n,u)^q\,\mathrm{d}u,$$

and, for squarefree n, put

$$n_k := \begin{cases} \prod_{j\leqslant k} p_j(n) & \text{if } k \leqslant \omega(n),\\ n & \text{otherwise.}\end{cases}$$

Now, let

$$L_{k,q} = L_{k,q}(x,y) := \sum_{P^+(n)\leqslant y} \frac{\mu(n)^2 M_q(n_k)^{1/q}}{n^\beta},$$

where $\beta := \alpha\big(\sqrt{x},y\big)$ is the saddle-point related to the friable mean-value of $\tau(n)$, the divisor function.

We aim at bounding $L_{k,q}$ from above for large k and q. The starting point is the identity

$$\Delta(mp,u) = \Delta(m,u) + \Delta(m,u-\log p) \quad (u\in\mathbb{R},\ p\nmid m).$$

Apply this to $m = n_k$, $p = p_{k+1}(n)$. Raising to the power q and expanding out, we obtain

$$M_q(n_{k+1}) = 2M_q(n_k) + E_q(n_k,p_{k+1}) \qquad (\omega(n) > k),$$

with

$$E_q(m,p) := \sum_{1\leqslant j<q}\binom{q}{j}\int_{\mathbb{R}} \Delta(m;v)^j\Delta(m;v-\log p)^{q-j}\,\mathrm{d}v.$$

It follows that

$$L_{k+1,q} \leqslant 2^{1/q}L_{k,q} + \sum_{\substack{P^+(m)\leqslant y\\ \omega(m)=k}} \mu(m)^2 \sum_{P^+(m)<p\leqslant y} E_q(m,p)^{1/q} \sum_{\substack{P^+(n)\leqslant y\\ \omega(n)\geqslant k+1\\ n_{k+1}=mp}} \frac{\mu(n)^2}{n^\beta}.$$

The latter sum is

$$\ll \frac{\zeta_1(\beta,y)}{p^\beta m^\beta}\prod_{\ell\leqslant p}\frac{1}{1+\ell^{-\beta}} =: \frac{\zeta_1(\beta,y)g_\beta(p)}{p^\beta m^\beta},$$

where, here and in the remainder of this proof, ℓ denotes a prime number, and

$$\zeta_1(\sigma,y) := \prod_{\ell\leqslant y}(1+\ell^{-\sigma}).$$

Hölder's inequality yields

$$\sum_{z<p\leqslant y}\frac{E_q(m,p)^{1/q}}{p^\beta} \leqslant \bigg\{\sum_{p\geqslant 2}\frac{E_q(m,p)\log p}{p}\bigg\}^{1/q}\bigg\{\sum_{z<p\leqslant y}\frac{1}{p^{(q\beta-1)/(q-1)}(\log p)^{1/(q-1)}}\bigg\}^{(q-1)/q}.$$

and the prime number theorem enables to bound the last sum over p by

$$\ll \frac{qy^{q(1-\beta)/(q-1)}}{(\log z)^{1/(q-1)}}.$$

Now, we have (see, e.g., [9, th. 73])

$$\sum_p \frac{E_q(m,p)\log p}{p} \leqslant C4^q\tau(m)^{q/(q-1)}M_q(m)^{(q-2)/(q-1)}.$$

It follows that

$$L_{k+1,q} \leqslant 2^{1/q}L_{k,q} + C_1 q\mathrm{e}^{\xi(u/2)}G_k \leqslant 2^{1/q}L_{k,q} + C_2qu^2G_k, \tag{4.1}$$

with

$$G_k := \zeta_1(\beta,y)\sum_{\substack{P^+(m)\leqslant y\\ \omega(m)=k}} \frac{\mu(m)^2\tau(m)^{1/(q-1)}M_q(m)^{(q-2)/q(q-1)}g_\beta(P^+(m))}{m^\beta(\log P^+(m))^{1/q}}\cdot$$

Since

$$\frac{\mu(m)^2\zeta_1(\beta,y)g_\beta(P^+(m))}{m^\beta} = \sum_{\substack{P^+(n)\leqslant y\\ n_k=m}}\frac{\mu(n)^2}{n^\beta},$$

we infer that

$$G_k \leqslant \sum_{\substack{P^+(n)\leqslant y\\ \omega(n)\geqslant k}}\frac{\mu(n)^2\tau(n_k)^{1/(q-1)}M_q(n_k)^{(q-2)/q(q-1)}}{n^\beta(\log p_k(n))^{1/q}}\cdot$$

A new application of Hölder's inequality yields

$$G_k \leqslant L_{k,q}^{(q-2)/(q-1)}S_k^{1/(q-1)},$$

where

$$\begin{aligned}
S_k &:= \sum_{\substack{P^+(n)\leqslant y\\ \omega(n)\geqslant k}} \frac{\mu(n)^2\tau(n_k)}{n^\beta\{\log p_k(n)\}^{(q-1)/q}}\\
&\leqslant 2\sum_{\substack{P^+(m)\leqslant y\\ \omega(m)=k-1}}\frac{\mu(m)^2\tau(m)}{m^\beta}\sum_{P^+(m)<p\leqslant y}\frac{1}{p^\beta(\log p)^{1-1/q}}\prod_{p<\ell\leqslant y}\Big(1+\frac{1}{\ell^\beta}\Big)\\
&\leqslant \frac{\zeta_1(\beta,y)}{(k-1)!}\sum_{p\leqslant y}\frac{g_\beta(p)}{p^\beta(\log p)^{1-1/q}}\Big(\sum_{\ell\leqslant p}\frac{2}{\ell^\beta}\Big)^{k-1} \ll \frac{\zeta_1(\beta,y)y^{1-\beta}}{(k-1)!}\sum_{p\leqslant y}\frac{\mathrm{e}^{-T(p)}\{2T(p)\}^{k-1}}{p(\log p)^{1-1/q}},
\end{aligned}$$

where we set

$$T(p) := \sum_{\ell\leqslant p}\frac{1}{\ell^\beta}\cdot$$

(Recall that the letter ℓ denotes generically a prime number.)

We evaluate $T(p)$ by [2, lemma 3.6]. Writing

$$\mathcal{L}(z) := \mathrm{e}^{(\log z)^{3/5}/(\log_2 z)^{1/5}}, \quad w(t) := \frac{t^{1-\beta}-1}{(1-\beta)\log t},$$

we have

$$T(p) = \log_2 p + \int_1^{w(p)} t\xi'(t)\,\mathrm{d}t + b + O\Big(\frac{w(p)}{\mathcal{L}(p)^c} + \frac{\log(u+1)}{\log y}\Big).$$

where b is a suitable constant. Note that $w(y) = u/2 + O(u/\log y)$. Defining

$$h(v) := \int_1^{w(\exp \mathrm{e}^v)} t\xi'(t)\,\mathrm{d}t + b_1,$$

with b_1 sufficiently large so that $T(p) \leqslant \log_2 p + h(\log_2 p)$, and writing $z_v := v + h(v)$, we have, by the prime number theorem,

$$W_k(y) := \sum_{p\leqslant y}\frac{\mathrm{e}^{-T(p)}\{T(p)\}^{k-1}}{p(\log p)^{1-1/q}} \ll \int_0^{\log_2 y}\mathrm{e}^{-(2-1/q)z_v+(1-1/q)h(\log_2 y)}z_v^{k-1}\,\mathrm{d}v.$$

Since $h(\log_2 y) \leqslant u/2 + O(u/\log 2u)$ and since $h'(v)\geqslant 0$, the change of variables $z = z_v$ yields

$$W_k(y) \ll \mathrm{e}^{u/2+O(u/\log 2u)}\int_0^\infty \mathrm{e}^{-(2-1/q)z}z^{k-1}\,\mathrm{d}z \ll \frac{\mathrm{e}^{u/2+O(u/\log 2u)}(k-1)!}{(2-1/q)^{k-1}}\cdot$$

Thus,

$$S_k \ll \frac{\zeta_1(\beta,y)\mathrm{e}^{u/2+O(u/\log 2u)}}{(1-1/2q)^k} \ll \frac{\zeta_1(\alpha,y)\mathrm{e}^{O(u/\log 2u)}}{(1-1/2q)^k},$$

since $\zeta(\beta,y)=\zeta(\alpha,y)\mathrm{e}^{-u/2+O(u/\log 2u)}$ — see [18, (4.2)].

Finally, for q sufficiently large and $\frac{1}{2}<\lambda<\log 2$, we obtain

$$G_k \leqslant C_3 L_{k,q}^{(q-2)/(q-1)}\zeta_1(\alpha,y)^{1/(q-1)}\mathrm{e}^{c_0u/(q\log 2u)+\lambda k/q(q-1)}. \tag{4.2}$$

At this stage, we introduce

$$L_{k,q}^* = L_{k,q} + 2^{k/q}u^{2q}\mathrm{e}^{c_0u/\log 2u}\zeta_1(\alpha,y),$$

so that (4.1) still holds for $L_{k,q}^*$ in place of $L_{k,q}$. Setting $q(k):=\lfloor c_1\sqrt{k/\log k}\rfloor$ with sufficiently small, absolute c_1, we thus have, for large k,

$$L_{k+1,q}^* \leqslant \Big\{2^{1/q}+\frac{1}{k}\Big\}L_{k,q}^* \qquad \big(q\leqslant q(k)\big),$$

whence

$$L_{k+1,q}^* \leqslant 3^{1/q}L_{k,q}^* \qquad \big(q\leqslant q(k)\big). \tag{4.3}$$

To carry out a double induction on k and q, we also need a bound on $L_{k,q+1}^*$ in terms of $L_{k,q}^*$. This is achieved by the inequality $M_{q+1}(n)^{1/(q+1)} \leqslant 2M_q(n)^{1/q}$ proved in [9, th. 72], which yields

$$L_{k,q+1}^* \leqslant 2u^2 L_{k,q}^*. \tag{4.4}$$

With the aim of bounding $L_{k,q(k)}^*$ in terms of $L_{2,q(2)}^*$, we use (4.3) to reduce the parameter k and (4.4) to secure the condition $q\leqslant q(k)$. The first handling provides an overall factor

$$\leqslant \prod_{1\leqslant q\leqslant q(k)} q^{c_2} \leqslant \mathrm{e}^{c_3\sqrt{k\log k}}$$

whereas the second induces a global factor $\ll u^{c_4q(k)}$.

Finally, we obtain

$$L_{k,q}^* \ll L_{2,q(2)}^* u^{c_5q(k)}\mathrm{e}^{c_5\sqrt{k\log k}}.$$

Let $K:=\log_2 y+u$. It can be shown (see [1] and use a bound similar to [9, (7.44)]) that the contribution to $L_{k,q}$ of those integers n such that $\omega(n)>CK$ is negligible, and we omit the details. Eventually, we arrive at

$$L_{k,q} \ll \mathrm{e}^{c_5\sqrt{K\log K}}u^{c_6\sqrt{K/\log K}}\zeta(\alpha,y)\mathrm{e}^{c_0u/\log 2u} \ll \zeta(\alpha,y)\mathrm{e}^{c_7\sqrt{\log_2 y\log_3 y}+O(u/\log 2u)},$$

and so

$$\sum_{n\in S(x,y)}\frac{\mu(n)^2\Delta(n)}{n^\beta} \ll \zeta(\alpha,y)\mathrm{e}^{c\sqrt{(\log_2 y)\log_3 y}+O(u/\log 2u)}.$$

Employing the representation $n=mr^2$, $\mu(m)^2=1$, we obtain that the same bound holds for

$$\sum_{n\in S(x,y)}\frac{\Delta(n)}{n^\beta}.$$

This is the key to our upper bound for $D(x,y):=\sum_{n\in S(x,y)}\Delta(n)$. We have

$$\begin{aligned} D(x,y)\log x-\int_1^x\frac{D(t,y)}{t}\,\mathrm{d}t &= \sum_{n\in S(x,y)}\Delta(n)\log n \leqslant \sum_{\substack{mp^\nu\leqslant x\\ P^+(mp)\leqslant y}}\Delta(m)(\nu+1)\log p^\nu \\ &\ll yD\Big(\frac{x}{y},y\Big)+\sum_{\substack{x/y<n\leqslant x\\ P^+(n)\leqslant y}}\frac{x\Delta(n)}{n}+\sum_{\substack{n\leqslant x\\ P^+(n)\leqslant y}}\Delta(n)\sqrt{\frac{x}{n}}.\end{aligned}$$

The trivial bound

$$D(x,y)\leqslant\sum_{n\in S(x,y)}\tau(n)\ll x\varrho_2(u)\log y,$$

that holds in H_ε (see [19, Cor. 2.3]), furnishes

$$\int_1^x\frac{D(t,y)}{t}\,\mathrm{d}t \ll x\varrho_2(u)\log y, \quad yD(x/y,y)\ll x\varrho_2(u-1)\log y.$$

Moreover, in the same region, for y sufficiently large, $\beta > 1/2$ and

$$\sum_{\substack{n\leqslant x\\ P^+(n)\leqslant y}} \Delta(n)\sqrt{\frac{x}{n}} + \sum_{\substack{x/y<n\leqslant x\\ P^+(n)\leqslant y}} \frac{x\Delta(n)}{n} \ll x^\beta \mathrm{e}^{\xi(u/2)} \sum_{n\in S(x,y)} \frac{\Delta(n)}{n^\beta}\cdot$$

Collecting these estimates, we obtain

$$\begin{aligned} D(x,y) &\ll x\frac{\varrho_2(u)}{u} + x\varrho_2(u)\log 2u + \frac{x^\beta\zeta(\alpha,y)\mathrm{e}^{c\sqrt{(\log_2 y)\log_3 y}+O(u/\log 2u)}}{\log x} \\ &\ll \Psi(x,y)2^{u+O(u/\log 2u)}\mathrm{e}^{c\sqrt{\log_2 y\log_3 y}}, \end{aligned}$$

where we used (1.9), (1.6), the estimate

$$\frac{x^\beta\zeta(\alpha,y)}{\log x} \asymp \Psi(x,y)2^{u+O(u/\log u)},$$

which follows from (2.1), (2.4) and

$$(\beta-\alpha)\log x = -u\int_{u/2}^u \xi'(t)dt + O(1) = u\log 2 + O\Big(\frac{u}{\log u}\Big).$$

This concludes the proof of the upper bound included in (1.14).

5. Proof of Theorem 1.1(ii)

We retain notation $g(t)$ from (1.12), ε_y from (1.13), define $\eta_y := (\log_2 y)/\log y$. Since $\max(1,\lfloor\tau(n)/\log n\rfloor) \leqslant \Delta(n) \leqslant \tau(n)$ holds for all $n \geqslant 1$ (see e.g. [9, th. 60, (6.7)]), we have

$$\frac{\Psi(x,y;\tau)}{2\Psi(x,y)\log x} \leqslant \mathfrak{S}(x,y) \leqslant \frac{\Psi(x,y;\tau)}{\Psi(x,y)} \qquad (x\geqslant y\geqslant 2)\cdot \tag{5.1}$$

Now, by [18, th. 1.2] and [18, (1.6)], we have, with $\lambda := y/\log x$,

$$\frac{\Psi(x,y;\tau)}{\Psi(x,y)} \asymp \zeta(\alpha,y)\mathrm{e}^{-uh(\lambda)\{1+O(\varepsilon_y)\}} \quad (x\geqslant y\geqslant 2), \tag{5.2}$$

where we have put

$$h(t) := t\log 4 - (1+2t)\log\Big(\frac{1+2t}{1+t}\Big) = t\log\Big(1+\frac{1}{t}\Big) - g(t) \quad (t\geqslant 0),$$

and, for the purpose of further reference, note that

$$uh(\lambda) \sim (1-\log 2)u \quad (u\to\infty,\ \lambda\to\infty), \quad uh(\lambda)\asymp \overline{u} \qquad (x\geqslant y\geqslant 2). \tag{5.3}$$

We shall show that

$$\zeta(\alpha,y) = \mathrm{e}^{\lambda u\log(1+1/\lambda)\{1+O(\varepsilon_y+1/\log 2u)\}} \qquad \Big(2\leqslant y\leqslant x^{1/(2\log_2 x\log_3 x)}\Big). \tag{5.4}$$

Since $g(\lambda)u \gg \overline{u}$ for $x\geqslant y\geqslant 2$, we see that (1.15) follows from (5.1) and (5.4) in any subregion where $\overline{u}(\varepsilon_y + 1/\log 2u) \gg \log_2 x$: the condition above corresponds to this requirement when y is large. However, for bounded y, we have $\Psi(x,y;\tau)/\Psi(x,y) \asymp (\log x)^{\pi(y)}$, and so (1.15) holds trivially. Therefore, we may assume in the sequel that y is sufficiently large.

Let us now embark on the proof of (5.4).

Observe that

$$\zeta(\alpha,y) = \zeta(1,y)\exp\Big\{\int_\alpha^1 \varphi_y(\sigma)\,\mathrm{d}\sigma\Big\}. \tag{5.5}$$

Using the estimate for $\varphi_y(\sigma)$ given in [11, lemma 13], we may write

$$\int_\alpha^1 \varphi_y(\sigma)\,\mathrm{d}\sigma = \Big\{1+O\Big(\frac{1}{\log y}\Big)\Big\}\int_\alpha^1 \frac{y^{1-\sigma}-1}{(1-\sigma)(1-y^{-\sigma})}\,\mathrm{d}\sigma. \tag{5.6}$$

By inspection of the proof of (2.2) in [11, pp. 285-7], we see that, for a suitable constant C, we have

$$\alpha(x,y) = 1 - \frac{\xi(u)}{\log y} + O\Big(\frac{1}{(\log y)^2}\Big) \qquad \Big(C(\log x)(\log_2 x)^3 < y\leqslant x\Big). \tag{5.7}$$

This implies $y^\alpha \gg y\mathrm{e}^{-\xi(u)} \gg \log y$ in the same domain, so the contribution of the term $1 - y^{-\sigma}$ in (5.6) is absorbed by the error term. The change of variables defined by $(1-\sigma)\log y = \xi(t)$ then provides, in view of (5.7) and (1.7),

$$\begin{aligned}\int_\alpha^1 \varphi_y(\sigma)\,\mathrm{d}\sigma &= \Big\{1 + O\Big(\frac{1}{\log y}\Big)\Big\}\int_1^u t\xi'(t)\,\mathrm{d}t\\ &= u + \frac{u}{\log u} + O\Big(\frac{u}{\log y}\Big) = u + O\Big(\frac{u}{\log 2u} + \varepsilon_y u\Big).\end{aligned}$$

Since, in the domain of (5.7),

$$u\lambda \log\Big(1 + \frac{1}{\lambda}\Big) = u + O\Big(\frac{u}{\log 2u}\Big),$$

we obtain (5.4) in the range $C(\log x)(\log_2 x)^3 < y \leqslant x^{1/(2\log_2 x \log_3 x)}$. Indeed the factor $\zeta(1,y) \asymp \log y$ appearing in (5.5) is absorbed by the error term.

When $2 \leqslant y \leqslant C(\log x)(\log_2 x)^3$, we put $t = y^\sigma$ in (5.6) to get

$$\int_\alpha^1 \varphi_y(\sigma)\,\mathrm{d}\sigma = \Big\{1 + O\Big(\frac{1}{\log y}\Big)\Big\}\int_{y^\alpha}^y \frac{y/t - 1}{(t-1)\log(y/t)}\,\mathrm{d}t.$$

Note that (2.2) now implies $\alpha \log y \ll \log_2 2y$. Put $T := (\log y)^K$, where K is so large so that $T > y^\alpha$. The contribution of the interval $[T, y]$ to the above integral is

$$\ll \int_T^\infty \frac{y}{t^2}\,\mathrm{d}t \ll \frac{y}{(\log y)^K},$$

where we used the bound $\mathrm{e}^v - 1 \ll v\mathrm{e}^v$ $(v \geqslant 0)$. Then,

$$\begin{aligned}\int_{y^\alpha}^T \frac{y/t - 1}{(t-1)\log(y/t)}\,\mathrm{d}t &= \Big\{1 + O\Big(\frac{\log_2 y}{\log y}\Big)\Big\}\frac{y}{\log y}\int_{y^\alpha}^T \frac{1}{t(t-1)}\,\mathrm{d}t\\ &= \big\{1 + O(\eta_y)\big\}\frac{y}{\log y}\log\Big(\frac{1 - 1/T}{1 - 1/y^\alpha}\Big).\end{aligned}$$

>From [11, (7.18)], it follows that

$$\log\Big(\frac{1}{1 - y^{-\alpha}}\Big) = \log\Big(1 + \frac{1}{\lambda}\Big)\Big\{1 + O\Big(\frac{\log_2 y}{\log y}\Big)\Big\} \quad \Big(2 \leqslant y \leqslant C(\log x)(\log_2 x)^3\Big).$$

Therefore

$$\begin{aligned}\int_\alpha^1 \varphi_y(\sigma)\,\mathrm{d}\sigma &= \big\{1 + O(\eta_y)\big\}\frac{y}{\log y}\log\Big(1 + \frac{1}{\lambda}\Big) + O\Big(\frac{y}{(\log y)^K}\Big)\\ &= \big\{1 + O(\eta_y)\big\}u\lambda\log\Big(1 + \frac{1}{\lambda}\Big).\end{aligned}$$

This establishes (5.4) in the complementary range $2 \leqslant y \leqslant C(\log x)(\log_2 x)^3$.

This completes the proof of theorem 1.1(ii).

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