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On the normal concentration of divisors, 2

Helmut Maier & Gérald Tenenbaum

Abstract. We improve the current upper and lower bounds for the normal order of the Erdős–Hooley Δ–function

$$\Delta(n) := \sup_{u\in\mathbb{R}} \sum_{\substack{d\mid n\\ 0<\log d-u\leqslant 1}} 1 \qquad (n\in\mathbb{N}^*),$$

obtaining, for almost all integers n, the inequalities

$$(\log_2 n)^{\gamma+o(1)} < \Delta(n) < (\log_2 n)^{\log 2+o(1)}$$

where the exponent $\gamma := (\log 2)/\log\left(\frac{1-1/\log 27}{1-1/\log 3}\right) \approx 0.33827$ is conjectured to be optimal.

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Summary

1 Introduction 1
2 Proofs of Theorem 1.1 and Corollary 1.2 4
 2.1 Proof of Theorem 1.1 4
 2.2 Proof of Corollary 1.2 8
3 Proof of Theorem 1.3 9
4 Proof of Theorem 1.4 10
 4.1 Outline 10
 4.2 Notation and preliminary estimates 11
 4.3 Arithmetic properties of the first pair of divisors 22
 4.4 The second pair of divisors 24
 4.5 The induction step 28

1. Introduction

For positive integer n and real u, we consider

$$\Delta(n,u) := \sum_{\substack{d\mid n\\ \mathrm{e}^u<d\leqslant \mathrm{e}^{u+1}}} 1, \qquad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u).$$

The Δ-function is an interesting example of a concentration function of arithmetical nature. It was introduced by Erdős in [2] more than thirty years ago and Hooley [6] showed that information on its average order

$$s(x) := \frac{1}{x}\sum_{n\leqslant x} \Delta(n)$$

and of a number of its generalizations have applications to a great variety of arithmetical problems. Established in [4] and [11], the best bounds at the time of writing are

$$\log_2 x \ll s(x) \ll \mathrm{e}^{c\sqrt{\log_2 x \log_3 x}} \qquad (x\to\infty) \tag{1·1}$$

where c is a constant and, here and in the sequel, $\log_k$ denotes the k-fold iterated logarithm. See [5] and [12] for further references and descriptions on this question.

The normal order of $\Delta(n)$ is also of crucial interest from the perspective of understanding the fine multiplicative structure of a random integer. It was conjectured by Erdős at the end of the thirties, and referred to in [1], that $\Delta(n)>1$ for almost all n. This was settled, positively, by the authors in [7], and the best known bounds for the normal behaviour of $\Delta(n)$, established in [7] and [8], are

$$(\log_2 n)^{c_0+o(1)} < \Delta(n) < \xi(n)\log_2 n \qquad (n\to\infty) \tag{1·2}$$

where $c_0 := (\log 2)/|\log(1-1/\log 3)| \approx 0.28754$ and $\xi(n)$ is any function tending with infinity with n.

Our aim here is to improve upon both the upper and the lower bound of (1·2). The new lower bound, stated in Theorem 1.4 below, is the most difficult; we believe that it coincides with the actual normal order of the Δ–function, although a line of attack towards such an estimate still eludes us.

As in previous work, we use the notation pp to indicate that a relation holds on a sequence of asymptotic density 1. Furthermore, the notation pp$x$ means that the relation thus designated holds for all but $o(x)$ integers $\leqslant x$ as $x\to\infty$.

We denote by $\{p_j(n)\}_{j=1}^{\omega(n)}$ the increasing sequence of distinct prime factors of an integer n and let $\{d_j(n)\}_{j=1}^{\tau(n)}$ represent the increasing sequence of its divisors.

For $r\geqslant 1$, we define

$$E_r(n) := \min_{1\leqslant j\leqslant \tau(n)-r} \frac{d_{j+r}}{d_j}.$$

Given an integer-valued function $\xi=\xi(x)$ tending to infinity arbitrarily slowly, we put

$$K = K(n,x) := \max\{k : 1\leqslant k\leqslant \omega(n),\, \log_2 p_k(n) < \log_2 x - \xi(x)\}$$

and

$$n_k := \begin{cases} \prod_{\xi<j\leqslant k} p_j(n) & \text{if } k\leqslant K,\\ n_K & \text{if } k>K.\end{cases}$$

It follows from theorems 50 and 51 of [5] (but see also the corollary to lemma 7 of [8]) that, for any given $\varepsilon>0$, we have

$$1+(\mathrm{e}/3)^{(1+\varepsilon)k} < E_1(n_k) < 1+(\mathrm{e}/3)^{(1-\varepsilon)k} \quad (\xi<k\leqslant K) \qquad \mathrm{pp}x \tag{1·3}$$

and the exact pp behaviour of $E_r(n)$ for $r \geqslant 2$ raises an interesting open problem. Using techniques similar to that of the proof of theorem 3 of [3], it can be shown that

$$E_2(n_k) > 1 + \mathrm{e}^{-\alpha_2 k} \quad (\xi < k \leqslant K) \qquad \mathrm{pp}x$$

for some $\alpha_2 < \log 3 - 1$. Moreover, it is a simple consequence of theorem 51 of [5] that, for any $\varepsilon > 0$ and all $r \geqslant 1$, we have

$$(1{\cdot}4) \qquad E_r(n_k) \leqslant 1 + \mathrm{e}^{-(1-\varepsilon)\alpha_r^* k} \quad (\xi < k \leqslant K) \qquad \mathrm{pp}x$$

with

$$\alpha_r^* := \frac{1}{\varrho_0^m - 1}, \quad \varrho_0 := \frac{\log 3}{\log 3 - 1}, \quad 2^{m-1} < r + 1 \leqslant 2^m.$$

The methods and results of the present paper may be used to sharpen these bounds. We can thus replace α_r^* in (1·4) by

$$(1{\cdot}5) \qquad \alpha_r^{**} := \frac{1}{(\varrho_0 - 1)\varrho_1^{m-1}}, \quad \varrho_1 := \tfrac{1}{3}(2\varrho_0 + 1), \quad 2^{m-1} < r + 1 \leqslant 2^m.$$

We do not pursue such goal here and postpone the corresponding study to a future work.

In our first result below, we obtain a new pp-lower bound for $E_r(n_k)$ when r is large. This implies in particular, for a suitable constant $c > 0$, an inequality of the type $E_r(n_k) \geqslant 1 + \{E_1(n_k) - 1\}^{c/r}$ which is non trivial as soon as $r \geqslant 8$. This information will be later exploited through the fact that $\Delta(n, u)$ stays almost equal to its maximum on a fairly long interval, with the consequence that high moments of $\Delta(n, u)$ may be used more efficiently in the process of bounding $\Delta(n)$.

Theorem 1.1. *Let $r \geqslant 1$ be given. Then*

$$(1{\cdot}6) \qquad E_r(n_k) > 1 + 2^{-(k+r+2)/r} \quad (\xi < k \leqslant K) \qquad \mathrm{pp}x.$$

Note that $\tau(n_k) = 2^{k-\xi}$, so it follows from (1·6) that

$$E_r(n_k) > 1 + 2^{-3-\xi/r}\tau(n_k)^{-1/r} \quad (\xi < k \leqslant K) \qquad \mathrm{pp}x.$$

Corollary 1.2. *Let $r = r(n) \to \infty$. Then we have*

$$E_r(n) > 1 + 1/(\log n)^{o(1)} \qquad \mathrm{pp}.$$

As will be seen in Section 3, Theorem 1.1 may be inserted in our previous upper bound iterative method in a fairly standard way. We thus obtain the following estimate.

Theorem 1.3. *We have*

$$\Delta(n) < (\log_2 n)^{\log 2 + o(1)} \qquad \text{pp.} \tag{1·7}$$

In the sequel, we put

$$\varrho_0 := \frac{1}{1 - 1/\log 3} \approx 11.14072, \quad \varrho_1 := \tfrac{1}{3}(2\varrho_0 + 1) = \frac{1 - 1/\log 27}{1 - 1/\log 3} \approx 7.76048. \tag{1·8}$$

Our final result, proved in Section 4, is a lower bound which we believe optimal. The main idea is to show that, in previous lower bounds methods, some prime factors are left over in the involved iterative processes and to develop an extended procedure in order to actually employ these extra primes to manufacture more close divisors. Further details are provided in Section 4.1.

Theorem 1.4. *Let* $\gamma := (\log 2)/\log \varrho_1 \approx 0.33827$. *Then*

$$\Delta(n) > (\log_2 n)^{\gamma + o(1)} \qquad \text{pp.} \tag{1·9}$$

We conjecture that this result is optimal in the strong sense that

$$\Delta(n) = (\log_2 n)^{\gamma + o(1)} \qquad \text{pp.}$$

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2. Proofs of Theorem 1.1 and Corollary 1.2

2·1. Proof of Theorem 1.1

For integers $m \geqslant 1$, $q \geqslant 1$, and real $\varepsilon > 0$, define

$$\begin{aligned}
\Delta_\varepsilon(m; u) &:= \sum_{\substack{d \mid m \\ \mathrm{e}^u < d \leqslant (1+\varepsilon)\mathrm{e}^u}} 1 \quad (u \in \mathbb{R}), \\
D_q(m; \varepsilon) &:= \int_{-\infty}^{+\infty} \binom{\Delta_\varepsilon(m; u)}{q} \mathrm{d}u \\
&= \sum_{\substack{\mathrm{d}_j \mid m \ (1 \leqslant j \leqslant q) \\ d_1 < d_2 < \ldots < d_q \leqslant (1+\varepsilon) d_1}} \log\{(1+\varepsilon) d_1 / d_q\}, \\
D_q^*(m; \varepsilon) &:= \sum_{\substack{\mathrm{d}_j \mid m \ (1 \leqslant j \leqslant q) \\ d_1 < d_2 < \ldots < d_q \leqslant (1+\varepsilon) d_1}} 1.
\end{aligned}$$

We shall prove the following result, from which the required result is a comparatively simple consequence. We use throughout the notation

$$\varepsilon_k := \mathrm{e}^{-\alpha k}.$$

Lemma 2.1. *Let $\alpha > 0$ and $q \geqslant 1$ be fixed. Then we have*

$$(2\cdot1) \qquad D_q(n_k;\varepsilon_k) \leqslant \varepsilon_k^q 2^k \quad (\xi < k \leqslant K) \qquad \text{pp}x.$$

Before embarking on the proof, we check that this implies (1·6). Indeed, let us assume that $r \geqslant 1$ and that $d_0 < d_1 < \ldots < d_r$ are $r+1$ consecutive divisors of n_k in some interval $]\mathrm{e}^u, (1+\varepsilon_k)\mathrm{e}^u]$. Then, we have, trivially,

$$\binom{\Delta_{2\varepsilon_k}(n_k;u)}{r+1} \geqslant 1 \qquad (\log d_0 - \tfrac{1}{2}\varepsilon_k \leqslant u < \log d_0).$$

Therefore, for all non exceptional n, we have $\frac{1}{2}\varepsilon_k \leqslant D_{r+1}(n_k;2\varepsilon_k) \leqslant \varepsilon_k^{r+1}2^{k+r+1}$, from which it readily follows that $\varepsilon_k = \mathrm{e}^{-\alpha k} \geqslant 2^{-(k+r+2)/r}$. This is all needed.

Proof of Lemma 2.1. The result holds trivially for $q = 1$ since we have

$$(2\cdot2) \qquad D_1(n_\ell;\varepsilon) = \tau(n_\ell)\log(1+\varepsilon) \leqslant \varepsilon 2^\ell$$

for all $\ell \geqslant 1$ and $\varepsilon > 0$.

We may hence assume henceforth that $q \geqslant 2$ and also that $0 < \alpha \leqslant \frac{1}{2}$ for, in view of the lower bound in (1·3), we have $D_q(n_k;\varepsilon_k) = 0$ for any fixed $\alpha > \log 3 - 1$.

We argue by induction on q, and set out to establish that

$$(2\cdot3) \qquad D_j(n_\ell;\varepsilon_k) \leqslant \varepsilon_k^j 2^\ell \quad (1 \leqslant j \leqslant q,\ 2\xi/\alpha < \ell \leqslant k \leqslant K) \qquad \text{pp}x.$$

This is sufficient since we may ultimately replace ξ by $\frac{1}{2}\alpha\xi$, for the choice of ξ is arbitrary under the constraint $\xi(x) \to \infty$.

By (2·2), the result holds trivially for $q = 1$. We now consider an integer $q \geqslant 2$ and assume that (2·3) is satisfied when $1 \leqslant j < q$. Put $h(k) := [\alpha k]$. By (1·3), we have

$$E_1\big(n_{h(k)}\big) > 1 + \varepsilon_k \quad (2\xi/\alpha < k \leqslant K) \qquad \text{pp}x,$$

so

$$(2\cdot4) \qquad D_q(n_{h(k)};\varepsilon_k) = 0 \quad (2\xi/\alpha < k \leqslant K) \qquad \text{pp}x.$$

We shall bound $D_q(n_\ell;\varepsilon_k)$ for $h(k) < \ell \leqslant k$ by induction on ℓ, taking (2·4) as initial step.

Before embarking on the proof, we make a technical change, due to the fact that the upper bound induction process is greatly simplified if we have at our disposal an *a priori* lower bound for the quantity under study. So, we introduce

$$D_q^\dagger(m;\varepsilon) := D_q(m;\varepsilon) + \varepsilon^q 2^\xi \tau(m),$$

and note that our induction hypothesis becomes

$$D_j^\dagger(n_\ell;\varepsilon_k) \leqslant \varepsilon_k^j 2^{\ell+1} \quad (1 \leqslant j \leqslant q-1,\ 2\xi/\alpha < \ell \leqslant k \leqslant K) \qquad \text{pp}x. \tag{2·5}$$

Also, we bear in mind that (2·4) implies

$$D_q^\dagger(n_{h(k)};\varepsilon_k) = \varepsilon_k^q 2^{h(k)} \quad (2\xi/\alpha < k \leqslant K) \qquad \text{pp}x. \tag{2·6}$$

For notational simplicity, we put $\varepsilon = \varepsilon_k$ in the sequel of this proof. The basic device is the formula

$$\Delta_\varepsilon(n_{\ell+1},u) = \Delta_\varepsilon(n_\ell,u) + \Delta_\varepsilon(n_\ell, u - \log p_{\ell+1}(n)) \qquad (\xi \leqslant \ell < K(n,x)),$$

from which we readily obtain, for $n \leqslant x$, $\ell < K(n,x)$,

$$D_q(n_{\ell+1};\varepsilon) = 2D_q(n_\ell;\varepsilon) + \sum_{1\leqslant j\leqslant q-1} B_{q,j}\big(n_\ell;\varepsilon,p_{\ell+1}(n)\big), \tag{2·7}$$

where

$$B_{q,j}\big(m;\varepsilon,p\big) := \int_{-\infty}^{+\infty} \binom{\Delta_\varepsilon(m;u)}{j}\binom{\Delta_\varepsilon(m;u-\log p)}{q-j}\,\mathrm{d}u.$$

The main step in our method consists in averaging (2·7) over numbers with fixed n_ℓ and variable $p_{\ell+1}(n)$. This process is only effective if the mean values are taken in a set of integers $n \leqslant x$ whose multiplicative structure is sufficiently close to a statistical one, and we now describe the required properties.

We set $L := [2\log_2 x]$, so that

$$K(n,x) < L \qquad \text{pp}x.$$

Let β denote a sufficiently small positive constant and, for $\xi < \ell \leqslant L$, let A_ℓ denote the set of all integers a satisfying the conditions

$$(A_\ell) \qquad \begin{cases} \mu(a)^2 = 1,\ \omega(a) = \ell - \xi, \\ \log_2 P^+(a) < \log_2 x - \xi,\ \log_2 a < \log_2 x - \tfrac12\xi, \\ (1-\beta)(j+\xi) < \log_2 p_j(a) < (1+\beta)(j+\xi)\ (1 \leqslant j \leqslant \ell-\xi). \end{cases}$$

Set

$$A := \{n \leqslant x : n_\ell \in A_\ell\ (\xi < \ell \leqslant K(n,x))\}.$$

Then, it follows from [8] (corollary to lemma 2 and lemma 4) that

$$n \in A \qquad \text{pp}x.$$

Define $S_\ell(x,a)$ as the number of those $n \leqslant x$ such that $n_\ell = a$. By lemma 5 of [8], we have

$$S_{\ell+1}(x,ap) \ll \mathrm{e}^{(1+\beta)\xi-(1-\beta)\ell}\,\frac{x}{ap} \tag{2·8}$$

uniformly for $\xi < \ell \leqslant L$, $a \in A_\ell$, and $P^+(a) < p < \exp\exp(\log_2 x - \xi)$.

We are now ready for the averaging step mentioned above. For $h(k) \leqslant \ell \leqslant L$, $a \in A_\ell$, $1 \leqslant j \leqslant q-1$, put

$$T_{q,j}(a,x) := \sum_{\substack{n\in A,\, n_\ell=a\\ K(n,x)>\ell}} B_{q,j}\big(a;\varepsilon,p_{\ell+1}(n)\big).$$

Then

$$\begin{aligned} T_{q,j}(a,x) &\leqslant \sum_{\substack{p>P^+(a)\\ ap\in A_{\ell+1}}} S_{\ell+1}(x,ap)B_{q,j}\big(a;\varepsilon,p\big) \\ &\ll \mathrm{e}^{(1+\beta)\xi-(1-\beta)\ell}\,\frac{x}{a}\int_{-\infty}^{+\infty}\binom{\Delta_\varepsilon(a;u)}{j}\sum_{p>P^+(a)}\frac{1}{p}\binom{\Delta_\varepsilon(a;u-\log p)}{q-j}\,\mathrm{d}u, \end{aligned} \tag{2·9}$$

by (2·8). The p-sum equals

$$\sum_{\substack{d_1,\dots,d_{q-j}\mid a\\ d_1<\dots<d_{q-j}<(1+\varepsilon)d_1}}\ \sum_{\substack{\mathrm{e}^u/d_1<p\leqslant(1+\varepsilon)\mathrm{e}^u/d_{q-j}\\ p>P^+(a)}}\frac{1}{p}. \tag{2·10}$$

In the inner sum, p covers an interval of bounds e^v, e^w, say. By the prime number theorem, this is

$$\int_v^w \frac{\mathrm{d}t}{t} + O\big(\mathrm{e}^{-\sqrt{v}}\big).$$

We rearrange the main terms and add the remainders, noticing that

$$w > \log P^+(a) > \mathrm{e}^{(1-\beta)\ell}.$$

We obtain that the double sum (2·10) does not exceed

$$\int_{\log P^+(a)}^{\infty}\binom{\Delta_\varepsilon(a;u-t)}{q-j}\frac{\mathrm{d}t}{t} + O\Big(D^*_{q-j}(a;\varepsilon)\exp\big(-\mathrm{e}^{(1-\beta)\ell/2}\big)\Big). \tag{2·11}$$

Using the crude upper bound $D^*_{q-j}(a;\varepsilon) \leqslant \tau(a)^{q-j} \leqslant 2^{\ell(q-j)}$, we plainly obtain

$$D^*_{q-j}(a;\varepsilon)\exp\big(-\mathrm{e}^{(1-\beta)\ell/2}\big) \leqslant \exp\big(q\ell-\mathrm{e}^{(1-\beta)\ell/2}\big) \ll \mathrm{e}^{-\ell}D^\dagger_{q-j}(a;\varepsilon)$$

since $\varepsilon > \mathrm{e}^{-\ell-1}$ for $\ell \geqslant h(k)$. Appealing to the lower bound $\log P^+(a) > \mathrm{e}^{(1-\beta)\ell}$, we thus derive from (2·9) and (2·11)

$$T_{q,j}(a,x) \ll \mathrm{e}^{(1+\beta)\xi-2(1-\beta)\ell}\,\frac{x}{a}D_j(a;\varepsilon)D^\dagger_{q-j}(a;\varepsilon). \tag{2·12}$$

Put $G_q(m) := \sum_{1\leqslant j\leqslant q-1} D_j^\dagger(m;\varepsilon)D_{q-j}^\dagger(m;\varepsilon)$ $(q \geqslant 2,\ m \geqslant 1)$. Since

$$D_q^\dagger(n_{\ell+1};\varepsilon) - 2D_q^\dagger(n_\ell;\varepsilon) = D_q(n_{\ell+1};\varepsilon) - 2D_q(n_\ell;\varepsilon)$$

when $\xi \leqslant \ell < K(n,x)$, we deduce from (2·7) and (2·12) that

$$\sum_{\substack{n\in A\\ n_\ell=a}} \frac{\big\{D_q^\dagger(n_{\ell+1};\varepsilon) - 2D_q^\dagger(n_\ell;\varepsilon)\big\}^+}{G_q(n_\ell)} \ll \frac{x}{a}\mathrm{e}^{(1+\beta)\xi-2(1-\beta)\ell}.$$

We now sum this over $a \in A_\ell$ using the bound

$$\sum_{a\in A_\ell} \frac{1}{a} \leqslant \prod_{(1-\beta)\xi<\log_2 p\leqslant(1+\beta)\ell} \Big(1+\frac{1}{p}\Big) \ll \mathrm{e}^{(1+\beta)\ell-(1-\beta)\xi}.$$

We obtain

$$\sum_{n\in A} \frac{\big\{D_q^\dagger(n_{\ell+1};\varepsilon) - 2D_q^\dagger(n_\ell;\varepsilon)\big\}^+}{G_q(n_\ell)} \ll x\mathrm{e}^{-(1-5\beta)\ell}.$$

This implies, by a standard argument, that the inequalities

$$D_q^\dagger(n_{\ell+1};\varepsilon) \leqslant 2D_q^\dagger(n_\ell;\varepsilon) + \mathrm{e}^{-(1-6\beta)\ell}G_q(n_\ell) \qquad (2\xi/\alpha < \ell \leqslant k \leqslant K) \tag{2·13}$$

hold simultaneously for all but at most $o(x)$ integers $n \leqslant x$.[1] We now divide (2·13) by $2^{\ell+1}$ and sum for $h(k) \leqslant \ell < s \leqslant k$, taking into account (2·6) and the induction hypothesis (2·5)—in the form $G_q(n_\ell) \leqslant q\varepsilon^q 4^{\ell+1}$. We obtain, for a suitable choice of the parameter β,

$$\frac{D_q^\dagger(n_s;\varepsilon)}{2^s} \leqslant \varepsilon^q + 2q\varepsilon^q \sum_{h(k)\leqslant\ell<s} \Big(\frac{2}{\mathrm{e}^{1-6\beta}}\Big)^\ell \leqslant 2\varepsilon^q$$

provided x, and therefore ξ, is large enough. This establishes the induction hypothesis for $j = q$ and therefore completes the proof. □

2·2. Proof of Corollary 1.2

For $\varepsilon > 0$, let $\Delta_\varepsilon(n) := \sup_{u\in\mathbb{R}} \Delta_\varepsilon(n,u)$. Then, we have

$$\Delta_\varepsilon(ab) \leqslant \Delta_\varepsilon(a)\tau(b) \qquad (a \geqslant 1,\ b \geqslant 1). \tag{2·14}$$

This is proved in lemma 61.1 of [5] for $\varepsilon = \mathrm{e}-1$, but the proof immediately extends to the general case. Let $\alpha > 0$. Selecting, $\varepsilon := (\log x)^{-\alpha}$, $a = n_K$ and $b = n/n_K$ and noticing that $\Omega(b) \leqslant \{1+o(1)\}\xi$, pp$x$, we infer from (2·14) and (1·6) that

$$\Delta_\varepsilon(n) \ll_\alpha 4^\xi \qquad \mathrm{pp}x.$$

Since the growth of ξ is arbitrarily slow, this implies the stated result in the form $\Delta_\varepsilon(n) \leqslant r(n)$, pp$x$. □

1. We remark that crucial use is made here of the fact that we only need to consider (2·13) when $\ell \geqslant h(k)$.

3. Proof of Theorem 1.3

This is a simple reappraisal of the upper bound proof in [8], where we take advantage of the supplementary information provided by Theorem 1.1.

For integers $q \geqslant 1$, $n \geqslant 1$, define

$$M_q(n) := \int_{-\infty}^{+\infty} \Delta(n,u)^q \,\mathrm{d}u, \quad R_q(n) := \sum_{1\leqslant j\leqslant q-1} \binom{q}{j} M_j(n) M_{q-j}(n).$$

Let $\mathrm{e}_1 \in]1, \mathrm{e}[$, $\alpha > 0$, $r > 1/\alpha$. We have

$$\Delta(n) \leqslant 2^{1-1/q} M_q(n)^{1/q} \qquad (n \geqslant 1, q \geqslant 1), \tag{3·1}$$

$$M_q(n_{k+1}) \leqslant 2M_q(n_k) + \mathrm{e}_1^{-k} R_q(n_k) \quad (\xi < k \leqslant K,\, 1 \leqslant q \leqslant k) \quad \mathrm{pp}x, \tag{3·2}$$

$$\Delta(n_k) \leqslant r + \mathrm{e}^{\alpha k/q} M_q(n_k)^{1/q} \qquad (\xi < k \leqslant K,\, q \geqslant 1) \quad \mathrm{pp}x. \tag{3·3}$$

The first inequality is stated and proved in theorem 72 of [5], the second stems from equation (13) of [8]. To prove the third, we first check that, for large k, we have

$$\log\big(1 + 2^{-1-k/r}\big) > \mathrm{e}^{-k\alpha}, \tag{3·4}$$

we consider u_0 such that $\Delta(n_k) = \Delta(n_k, u_0)$, and observe that, if $\Delta(n_k) \geqslant r$ and $d_1 < \ldots < d_r$ are the r smallest divisors of n_k in $]\mathrm{e}^{u_0}, \mathrm{e}^{u_0+1}]$, then, by Theorem 1.1, we have, in view of (3·4),

$$\log d_r > \log d_1 + \mathrm{e}^{-\alpha k} \quad (\xi < k \leqslant K) \qquad \mathrm{pp}x. \tag{3·5}$$

Thus $\Delta(n_k, u) \geqslant \Delta(n_k) - r$ for $\log d_1 \leqslant u < \log d_r$, that is on an interval of length $\geqslant \mathrm{e}^{-\alpha k}$. This implies

$$M_q(n_k) \geqslant \big(\Delta(n_k) - r\big)^q \mathrm{e}^{-\alpha k} \quad (\xi < k \leqslant K,\, q \geqslant 1) \qquad \mathrm{pp}x,$$

from which (3·3) follows.

We are now ready for the main step of our proof. Let c, δ be fixed with

$$c > \alpha + \log 2, \quad 1 < \delta < c + 1/\log 2 - 1.$$

We show by induction over k that

$$M_q(n_k) \leqslant 2^{\delta k} (q!)^c \qquad (1 \leqslant q \leqslant k) \qquad \mathrm{pp}x. \tag{3·6}$$

For $k = \xi + 1$, we have $M_q(n_k) = 2$ for all $q \geqslant 1$ provided ξ, and therefore k, is large enough. Hence (3·6) holds.

We assume (3·6) holds for k, $\xi < k \leqslant K$, and establish it still holds for $k + 1$.

When $q \leqslant k$, we may appeal to the induction hypothesis (3·6) to bound the right-hand side of (3·2). We select e_1 sufficiently close to e and obtain

$$\begin{aligned} M_q(n_{k+1}) &\leqslant 2^{\delta(k+1)} (q!)^c \bigg\{ 2^{1-\delta} + \Big(\frac{2^\delta}{\mathrm{e}_1}\Big)^k \sum_{1\leqslant j\leqslant q-1} \binom{q}{j}^{1-c} \bigg\} \\ &\leqslant 2^{\delta(k+1)} (q!)^c \big\{ 2^{1-\delta} + \big(2^\delta/\mathrm{e}_1\big)^k 2^{(1-c)q} q^c \big)\big\} \\ &\leqslant 2^{\delta(k+1)} (q!)^c, \end{aligned}$$

for large ξ, by the choice of δ.

When $q = k+1$, we can still invoke (3·6) to estimate $R_q(n_k)$ which is expressed in terms of $M_j(n_k)$ with $j \leqslant q-1 = k$. To bound $M_{k+1}(n_k)$, we utilise (3·3):

$$\begin{aligned} M_{k+1}(n_k) &\leqslant \Delta(n_k)M_k(n_k) \leqslant \big\{r + \mathrm{e}^{\alpha}M_k(n_k)^{1/k}\big\}M_k(n_k) \\ &\leqslant 2^{\delta(k+1)}\{(k+1)!\}^c\Big\{\frac{2^{-\delta}r}{(k+1)^c} + \frac{\mathrm{e}^{\alpha}(k!)^{c/k}}{(k+1)^c}\Big\} \\ &\leqslant 2^{\delta(k+1)}\{(k+1)!\}^c\Big\{\mathrm{e}^{\alpha-c} + o(1)\Big\}, \end{aligned}$$

by Stirling's formula. Inserting this in (3·2), we obtain

$$\begin{aligned} M_{k+1}(n_{k+1}) &\leqslant 2^{\delta(k+1)}\{(k+1)!\}^c\Big\{2\mathrm{e}^{\alpha-c} + o(1) + k^c\Big(\frac{2^{\delta+1-c}}{\mathrm{e}_1}\Big)^k\Big\} \\ &\leqslant 2^{\delta(k+1)}\{(k+1)!\}^c, \end{aligned}$$

for large enough ξ.

This establishes (3·6). In particular, by (3·1),

$$\Delta(n_K) \ll M_K(n_K)^{1/K} \ll K^c \qquad \text{pp}x.$$

Now, by (2·14) for $\varepsilon = \mathrm{e}-1$, we have

$$\Delta(n) \leqslant \Delta(n_K)2^{\Omega(n/n_K)} \ll \Delta(n_K)4^{\xi} \qquad \text{pp}x.$$

Since α may be chosen arbitrarily small and we may take ξ tending to ∞ arbitrarily slowly, this finishes the proof. □

4. Proof of Theorem 1.4

4·1. Outline

Let ϱ_0 be as defined in (1·8) and recall that $\varrho_1 := \frac{1}{3}(2\varrho_0 + 1)$. We fix $\varrho_0^* > \varrho_0$, $\varrho_1^* := \frac{1}{3}(2\varrho_0^* + 1) > \varrho_1$, $\varepsilon_0 > 0$, put $u_0 := (\log_2 x)^{\varepsilon_0}$ and define

$$u_k := {\varrho_1^*}^{k-1}\varrho_0^* u_0 \qquad (k \geqslant 1). \tag{4·1}$$

Let $K = K_x := (1-2\varepsilon_0)(\log_3 x)/\log \varrho_1^*$. We shall show by induction upon k that there is a positive constant c such that, for any integer k, $1 \leqslant k \leqslant K$, all integers n but at most $\ll xk/u_0^c$ have $2k$ divisors d_{0n}, d'_{0n}, d_{1n}, d'_{1n}, …, $d_{k-1,n}$, $d'_{k-1,n}$ satisfying:

(i) $\prod_{0\leqslant j<k} d_{jn}d'_{jn} \,\big|\, n$;

(ii) $0 < |\log(d'_{jn}/d_{jn})| \leqslant 1 \quad (0 \leqslant j < k)$;

(iii) $p \,\big|\, \prod_{0\leqslant j<k} d_{jn}d'_{jn} \Rightarrow u_0 < \log_2 p \leqslant u_k$.

Forming all possible products of K factors by selecting for each $j < K$ one of the two divisors d_{jn}, d'_{jn}, we obtain that all but $\ll xKu_0^{-c} = o(x)$ integers $n \leqslant x$

have 2^K divisors lying in a interval of logarithmic length $2K$. By Dirichlet's box principle, it follows that

$$\Delta(n) > 2^{K-1}/K.$$

Since ε_0 may be chosen arbitrarily small and ϱ_1^* may be chosen arbitrarily close to ϱ_1, we obtain the desired lower bound.

A heuristic explanation for this construction is as follows. Given u and v with $u < v < (1-\varepsilon)\log_2 x$, a normal integer has roughly $w := v - u$ prime divisors p in $\mathcal{J}_n(u,v) := \{p : p|n,\, u < \log_2 p \leqslant v\}$ from which about 3^w irreducible ratios of divisors d'/d may be formed. From a general device, based on a uniform distribution hypothesis and made effective in [7] and [5], we expect to find two distinct divisors d, d' such that $|\log(d'/d)| \leqslant 1$ as soon as (essentially) $3^w > \mathrm{e}^v$. However, the distribution of $\omega(dd')$ among the 3^w ratios d'/d such that $dd' | \prod_{p\in\mathcal{J}_n(u,v)} p$ is binomial and has a peak when $\omega(dd') \approx 2w/3$. Therefore, we expect, and actually show, that about $w/3$ of the prime factors in $\mathcal{I}_n(u,v)$ have been left over in the construction of the pair of close divisors $\{d,\, d'\}$. If the multiplicative structure of dd' is sufficiently regular (and to actually establish this will turn out to be the most difficult part of the proof) we can construct a new pair of divisors using these prime factors and, for suitable z, the primes from $\mathcal{J}_n(v,z)$. Assuming good behaviour of this set of prime factors, we now only require

$$3^{z-v+w/3} > \mathrm{e}^z,$$

which is satisfied for the choice $u = u_0$, $v = u_1$, $z = u_2$. An iterative procedure is then implemented on this basis. At stage k, we may use approximately $u_k - u_{k-1}$ prime divisors from the last interval, $\frac{1}{3}(u_{k-1} - u_{k-2})$ from the previous one, and so on. The basic condition on the u_j may thus be written

$$\sum_{1\leqslant j\leqslant k} \frac{u_j - u_{j-1}}{3^{k-j}} > \frac{u_k}{\log 3}.$$

A simple computation shows that this is indeed the case for the choice (4·1) provided ϱ_0^* is sufficiently close to ϱ_0.

4·2. Notation and preliminary estimates

In the sequel, we define $\mathbb{N}(u,v)$ as the subset of the positive integers all of whose prime factors lie in the interval $]\exp\exp u, \exp\exp v]$ and we note that $1 \in \mathbb{N}(u,v)$ for all u, v. It shall be convenient to use the notation

$$\sum_{n\in A}^{u,v}$$

to denote a sum restricted to integers those n belonging to $A \cap \mathbb{N}(u,v)$ for some given set A.

For integer $n \geqslant 1$, we use the notation

$$n_{u,v} := \prod_{\substack{p^\nu \| n \\ u < \log_2 p \leqslant v}} p^\nu, \qquad \Omega(n;u,v) := \Omega(n_{u,v}) = \sum_{\substack{p^\nu \| n \\ u < \log_2 p \leqslant v}} \nu.$$

We also set

$$\nabla(m;t;z) := \sum_{\substack{dd'|m, d\neq d' \\ |\log(d'/d)-z|\leqslant t}} \mu(dd')^2, \qquad \nabla(m;t) := \nabla(m;t;0),$$

and write

$$Q(y) := y\log y - y + 1 \qquad (y>0). \tag{4·2}$$

We first quote, with a slight change of notation, theorem 51 of [5].

Theorem 4.1 ([5]). *Let $\beta := Q(1/\log 3) \approx 0.00415$. Uniformly for*

$$x \geqslant 3, \quad -\tfrac12 \leqslant u \leqslant v \leqslant \log_2 x, \quad w := v-u \geqslant 2, \quad 1 \leqslant \xi \leqslant \sqrt{w}, \quad 0 \leqslant t \leqslant \mathrm{e}^v,$$

we have

$$\nabla(n_{u,v};t) > t3^w \mathrm{e}^{-v-\xi\sqrt{w}} - 1 \tag{4·3}$$

for all integers $n \leqslant x$ but at most $\ll x\big\{\mathrm{e}^{-\xi^2/50} + \xi^{-\beta} w^{-\beta/2}(\log w)^{4\beta}\big\}$.

The following device will be also be of crucial use.

Lemma 4.2. *Uniformly for $x \geqslant 3$, $v \leqslant \log_2 x$, $T \geqslant 1$ we have*

$$\sum_{\substack{n\leqslant x \\ n_{1,v} > \exp(T\mathrm{e}^v)}} 1 \ll x\,\mathrm{e}^{-T/2}. \tag{4·4}$$

Proof. This is Exercise III.5.6 of [13], with solution in [15]. □

Lemma 4.2 has the following useful corollary.

Corollary 4.3. *Let F denote a non-negative arithmetic function. Assume the real numbers ε, ε_1, ε_2, u, v, x satisfy $\varepsilon \in [0,1[$, $x \geqslant 3$, $-\frac12 \leqslant u \leqslant v \leqslant (1-\varepsilon)\log_2 x$ and also that there exists a subset E_x of $\mathbb{N}\cap[1,x]$ such that*

$$|E_x| \leqslant \varepsilon_1 x, \qquad \sum_{n\in[1,x]\smallsetminus E_x} F(n_{u,v}) \leqslant \varepsilon_2 x. \tag{4·5}$$

Then, there exists a subset $\mathcal{E}$ of $\mathbb{N}(u,v)$ such that

$$\mathrm{e}^{-w}\sum_{m\in\mathcal{E}} \frac1m \ll \varepsilon_1 + \exp\big\{-\tfrac12 \mathrm{e}^{\varepsilon v}\big\}, \quad \mathrm{e}^{-w}\sum_{m\in\mathbb{N}(u,v)\smallsetminus\mathcal{E}} \frac{F(m)}{m} \ll \varepsilon_2. \tag{4·6}$$

The implicit constants depend only on ε.

Proof. We may plainly assume $\varepsilon_1 \gg \exp\{-\frac{1}{2}\mathrm{e}^{\varepsilon v}\}$ and therefore, in view of (4·4), that E_x contains all integers $n \leqslant x$ such that $n_{u,v} > V := \exp\{\mathrm{e}^{(1+\varepsilon)v}\}$. We then set $\mathcal{E} := \{m \in \mathbb{N}(u,v) : \exists n \in E_x,\, n_{u,v} = m\}$. Since $(1+\varepsilon)v \leqslant (1-\varepsilon^2)\log_2 x$, we have by the sieve

$$\sum_{n\in[1,x]\smallsetminus E_x} F(n_{u,v}) \gg x\mathrm{e}^{-w} \sum_{m\in\mathbb{N}(u,v)\smallsetminus\mathcal{E}} \frac{F(m)}{m},$$

so the second bound in (4·6) is satisfied. Similarly

$$|E_x| \geqslant \sum_{\substack{n\leqslant x\\ n_{u,v}\in\mathcal{E}\\ n_{u,v}\leqslant V}} 1 \gg x\mathrm{e}^{-w} \sum_{\substack{m\in\mathcal{E}\\ m\leqslant V}} \frac{1}{m}.$$

Hence the required result follows from the Rankin type bound

$$\sum_{\substack{m\in\mathcal{E}\\ m>V}} \frac{1}{m} \leqslant \sum_{\substack{m\in\mathbb{N}(u,v)\\ m>V}} \frac{1}{m^{1-\alpha}V^{\alpha}} \leqslant \frac{1}{V^{\alpha}} \prod_{u<\log_2 p\leqslant v} \Big(1+\frac{1}{p^{1-\alpha}}\Big) \ll \exp\big\{w - \tfrac{1}{2}\mathrm{e}^{\varepsilon v}\big\},$$

with $\alpha := \frac{1}{2}\mathrm{e}^{-v}$. □

We also need a variant of theorem 50 of [5].

Theorem 4.4. *Uniformly for*

$$x \geqslant 3,\ -\tfrac{1}{2} \leqslant u \leqslant v \leqslant \log_2 x,\ w := v-u \geqslant 2,\ 1 \leqslant \xi \leqslant \sqrt{w},\ 0 \leqslant t \leqslant \mathrm{e}^v,\ |z| \leqslant \tfrac{1}{2}\mathrm{e}^u,$$

we have

$$\nabla(n_{u,v};t;z) \leqslant t3^w\mathrm{e}^{-v+\xi\sqrt{w}} \tag{4·7}$$

for all integers $n \leqslant x$ but at most $\ll xw\mathrm{e}^{-\xi^2/11}$.

Moreover, under the above assumptions, there exists a subset $\mathcal{E} = \mathcal{E}(u,v)$ of $\mathbb{N}(u,v)$ such that

$$\mathrm{e}^{-w}\sum_{m\in\mathcal{E}} \frac{1}{m} \ll w\mathrm{e}^{-\xi^2/11} \tag{4·8}$$

and

$$\mathrm{e}^{-w} \sum_{m\in\mathbb{N}(u,v)\smallsetminus\mathcal{E}} \frac{\nabla(m;t;z)}{m} \ll t3^w\mathrm{e}^{-v+\xi\sqrt{w}}. \tag{4·9}$$

The implicit constants in (4·8) *and* (4·9) *are absolute.*

We omit the details since this may be established exactly as in [5]: the introduction of the parameter z is innocuous inasmuch we have, for all fixed $y\in]0,1]$,

$$\sum_{d\leqslant \mathrm{e}^z D}^{u,v} y^{\Omega(d)} \ll D\mathrm{e}^z(\log D)^{y-1}\mathrm{e}^{-yu} \qquad (D>\exp\exp u).$$

The second statement immediately follows from the first by Corollary 4.3.

Theorem 4.1 ensures, for suitable values of u and v and all but at most $o(x)$ integers $n\leqslant x$, the existence of the first pair of divisors, $\{d_{0n},d'_{0n}\}$. In the remaining part of this section, we prove a number of estimates in order to show that arithmetic properties of these divisors are such that the prime factors of $n_{uv}/d_{0n}d'_{0n}$ can be used to produce the second pair at comparatively low cost. These same estimates will later be used to tackle $n_{u,v}/D_{kn}$ with

$$D_{kn}:=d_{0n}d'_{0n}\cdots d_{k-1,n}d'_{k-1,n}. \tag{4·10}$$

We start with a familiar device on the distribution of prime factors of $n_{u,v}$. We write

$$G(n;u,v;\varepsilon):=\max_{0\leqslant j\leqslant w}\{\Omega(n;v-j,v)-(1+\varepsilon)j\}, \tag{4·11}$$

where $w:=v-u$.

Lemma 4.5. *(i) Uniformly for $-\frac12\leqslant u\leqslant v\leqslant \log_2 x$, $w:=v-u$, $0\leqslant \xi_1\leqslant\sqrt{w}$, $0\leqslant\xi_2\leqslant\frac{9}{10}\sqrt{w}$, we have*

$$-\xi_1\sqrt{w}\leqslant \omega(n_{u,v})-w\leqslant\Omega(n_{u,v})-w\leqslant\xi_2\sqrt{w}$$

for all integers $n\leqslant x$ but at most $\ll x\big\{\mathrm{e}^{-\xi_1^2/2}+\mathrm{e}^{-\xi_2^2/3}\big\}$.

(ii) For any fixed $\varepsilon\in]0,\frac12]$ and uniformly for $-\frac12\leqslant v\leqslant\log_2 x$, $T\geqslant 1$, we have

$$G(n;0,v,\varepsilon)=\max_{j\leqslant v}\Big\{\Omega(n;v-j,v)-(1+\varepsilon)j\Big\}\leqslant T$$

for all integers $n\leqslant x$ except at most $\ll x\varepsilon^{-2}(1+\varepsilon)^{-T}$.

Moreover, for all u, v such that $-\frac12\leqslant u\leqslant v$, we have

$$\mathrm{e}^{-w}\sum_{G(m;u,v;\varepsilon)>T}^{u,v}\frac1m\ll(1+\varepsilon)^{-T}\varepsilon^{-2}, \tag{4·12}$$

(iii) For any fixed $\varepsilon\in]0,\frac12]$ and uniformly for $-\frac12\leqslant u\leqslant v\leqslant\log_2 x$, $0\leqslant T\leqslant w$, we have

$$\min_{T\leqslant j\leqslant w}\omega(n_{v-j,v})/j\geqslant 1-\varepsilon$$

for all integers $n\leqslant x$ except at most $\ll x\varepsilon^{-2}\mathrm{e}^{-TQ(1-\varepsilon)}$.

Proof. Property (i) is a straightforward generalization of theorem III.3.7 from [13]. It can be established by inserting, into the proof given in [13], the bounds

$$Q(1-y) > y^2/2, \qquad Q(1+y) > y^2/3 \qquad (0 < y \leqslant 1).$$

We omit further details.

Property (ii) is similar to lemma 50.1 of [5] and may be proved by the same technique. The second assertion follows by Corollary 4.3.

Property (iii) is identical with the statement of Lemma 51.2 of [5]. □

We put, for $m \in \mathbb{N}^*$, $\vartheta \in \mathbb{R}$,

$$\varrho(m;\vartheta) := \sum_{dd'|m} \mu(dd')^2 (d'/d)^{i\vartheta} = \prod_{p|m} \{1 + 2\cos(\vartheta \log p)\} \tag{4·13}$$

and also introduce the function

$$\omega_\vartheta(n) := \sum_{p|n,\, p\leqslant \exp(1/|\vartheta|)} 1, \tag{4·14}$$

with the convention that $\omega_0(n) = \omega(n)$.

We now establish a result stating that, in a certain average sense upon n, the quantity $\varrho\big(n/d_{0n}d'_{0n};\vartheta\big)^2$ has the same upper bound than $\varrho(n;\vartheta)^2$ itself. We define, with the notations (4·13) and (4·14),

$$R(m;\vartheta) := \frac{\varrho(m;\vartheta)^2}{3^{\omega(m)+\omega_\vartheta(m)}} \qquad (m \in \mathbb{N}^*,\ \vartheta \in \mathbb{R}) \tag{4·15}$$

and

$$\nabla^\dagger(m;t;\vartheta) := \sum_{\substack{dd'|m \\ dd'\neq 1 \\ |\log(d'/d)|\leqslant t}} R(m/dd';\vartheta) \qquad (m \in \mathbb{N}^*,\ t \in \mathbb{R}^+,\ \vartheta \in \mathbb{R}).$$

Theorem 4.6. *Assume*

$$-\tfrac{1}{2} \leqslant u \leqslant v, \quad w = v - u, \quad 0 \leqslant \xi \leqslant \sqrt{w}.$$

There exists a subset $\mathcal{E} = \mathcal{E}(u,v)$ *of* $\mathbb{N}(u,v)$ *such that*

$$\mathrm{e}^{-w} \sum_{m\in\mathcal{E}} \frac{1}{m} \ll w\mathrm{e}^{-\xi^2/11} \tag{4·16}$$

and, for all $t \geqslant 0$,

$$\mathrm{e}^{-w} \sum_{m\in\mathbb{N}(u,v)\smallsetminus\mathcal{E}} \frac{\nabla^\dagger\big(m;t;\vartheta\big)}{m} \ll t3^w \mathrm{e}^{-v+\xi\sqrt{w}} \{\log(3+|\vartheta|)\}^4. \tag{4·17}$$

The implicit constant in (4·17) *depends at most on* ε.

Proof. When $\vartheta = 0$, the result readily follows from theorem 50 of [5], and Corollary 4.3 with $F(m) := \nabla^\dagger(m;t,0)$. Note that, in [5], the sum is restricted to pairs $\{d,d'\}$ such that $(d,d')=1$, but this is actually irrelevant.

The same method applies when $\vartheta \neq 0$. The corresponding extension of theorem 50 of [5] may be established by a straightforward reappraisal of the proof given in [5]: the details are essentially identical, simply noting that all innermost sums may be estimated by appealing to the estimate

$$\sum_{m\leqslant x} y^{\Omega(m_{u,z})} R(m;\vartheta) \ll x\{\log(3+|\vartheta|)\}^4 \mathrm{e}^{(y-1)(z-u)}, \tag{4·18}$$

valid uniformly for $x \geqslant 1$, $0 \leqslant u \leqslant z \leqslant \log_2(3x)$, $\vartheta \in \mathbb{R}$, $0 < y \leqslant 1$. This is established in [5] (lemma 51.3) when $y=1$ for the subsum corresponding to squarefree integers. The general case is derived similarly. □

The next result provides the main argument for the construction of the second pair (and actually all subsequent pairs) of divisors in our proof of Theorem 1.4.

For $1 \leqslant h \leqslant H$, we define

$$g(m;h,H,\sigma) := \max_{h\leqslant j\leqslant H} |\Omega(m;\sigma-j,\sigma)/j - 2/3|,$$

and we put for $n \geqslant 1$, $-\frac12 \leqslant u \leqslant v$, $t \geqslant 0$, $\alpha \geqslant 0$, $h \geqslant 1$,

$$\nabla(n_{u,v};t;\alpha,h) := \sum_{\substack{dd'|n_{u,v}\\ 0<|\log(d'/d)|\leqslant t\\ g(dd';h,w,v)\geqslant\alpha}} 1.$$

Other variants of the ∇-function have been considered in [9] and [14] with the purpose of counting close divisors with prescribed conditions on the distribution of prime factors.

We are now in a position to state and prove our main lemma. Like Theorem 4.6, it generalises theorem 50 of [5], which essentially corresponds to the case $\alpha = 0$. However, and although the same method is still applicable, there are now some extra complications, due to the nature of the new condition on dd'.

Theorem 4.7. *Assume*

$$-\tfrac12 \leqslant u \leqslant v, \quad w = v-u, \quad 0 \leqslant \xi \leqslant \tfrac{9}{10}\sqrt{w}, \quad 0 < \alpha \leqslant \tfrac{1}{10}, \quad 1 \leqslant h \leqslant w,$$

and set $\beta_1 := \frac13 Q(1+\frac32\alpha)$, *where* Q *is the function defined in* (4·2). *There exists a subset* E_x *of of* $[1,x]$ *such that*

$$|E_x| \ll x\big\{w\mathrm{e}^{-\xi^2/11} + \mathrm{e}^{-\beta_1 h}\alpha^{-2}\big\} \tag{4·19}$$

and, for all $t \geqslant 0$,

$$\nabla(n_{u,v};t;\alpha,h) \leqslant t\mathrm{e}^{-v}3^w\mathrm{e}^{\xi\sqrt{w}-\beta_1 h} \qquad \big(n\in[1,x]\smallsetminus E_x\big). \tag{4·20}$$

Proof. We start by noting that we may assume henceforth that $\xi \geqslant \xi_0$ where ξ_0 is any absolute constant because the result is trivial otherwise.

We may also, without loss of generality, restrict the parameter t to the range

$$t_0 \leqslant t \leqslant t_1, \quad \text{with} \quad t_0 := \tfrac{1}{2}\mathrm{e}^v 3^{-w}\mathrm{e}^{-\xi\sqrt{v}}, \quad t_1 := \tfrac{1}{10}\mathrm{e}^v.$$

For, when $t < t_0$, (4·20) implies $\nabla(n_{u,v}; t; \alpha, h) = 0$. If $t > t_1$, we drop the condition on t to obtain, for all z_0, z_1, with $0 < z_0 \leqslant 1 \leqslant z_1$,

$$\nabla(n_{u,v}; t; \alpha, h) \leqslant \sum_{h\leqslant j\leqslant w} \sum_{m|n_{u,v}} \tau(m)\big\{z_1^{\Omega(m;v-j,v)-(2/3+\alpha)j} + z_0^{\Omega(m;v-j,v)-(2/3-\alpha)j}\big\}.$$

Therefore, we deduce from Lemma 4.5(i) that, for all $n \leqslant x$ but at most $\ll x\mathrm{e}^{-4\xi^2/27}$ exceptions, we have

$$\begin{aligned}\nabla(n_{u,v}; t; \alpha, h)&\\ &\leqslant 3^{w+2\xi\sqrt{w}/3} \sum_{h\leqslant j\leqslant w} \big\{z_0^{(\alpha-2/3)j} f_j(n; z_0)\} + z_1^{-(2/3+\alpha)j} f_j(n; z_1)\big\},\end{aligned} \tag{4·21}$$

where $f_j(n; z)$ is for each z the multiplicative function of n defined by

$$f_j(n; z) := \big(\tfrac{1}{3}\big)^{\Omega(n;u,v)} \sum_{m|n_{u,v}} \tau(m) z^{\Omega(m;v-j,v)}.$$

Standard bounds on non-negative multiplicative functions (see, e.g., [13], corollary III.3.5.1) apply to yield

$$\sum_{n\leqslant x} f_j(n; z) \ll_z x\mathrm{e}^{2(z-1)j/3}.$$

Inserting this back into (4·21), selecting $z_i := 1 + \frac{3}{2}(-1)^{i-1}\alpha$, and summing upon j furnishes that the number of integers $n \leqslant x$ that satisfy (4·21) and

$$\nabla(n_{u,v}; t; \alpha, h) > \tfrac{1}{10} 3^{w+2\xi\sqrt{w}/3}\mathrm{e}^{-\beta_1 h} = t_1\mathrm{e}^{-v}3^w\mathrm{e}^{2\xi\sqrt{w}/3-\beta_1 h}$$

is $\ll x\alpha^{-2}\mathrm{e}^{-\beta_1 h}$.

We therefore assume henceforth that

$$t_0 \leqslant t \leqslant t_1. \tag{4·22}$$

Let us set

$$\nabla_1(n) := \sum_{\substack{d|n\\ 1<d\leqslant \mathrm{e}^t\\ g(d;h,w,v)\geqslant\alpha}} 1, \qquad \nabla_2(n) := \sum_{\substack{dd'|n\\ 1<d<d'\leqslant d\mathrm{e}^t\\ g(dd';h,w,v)\geqslant\alpha}} 1,$$

so that we have, for all n,

$$\nabla(n_{u,v};t;\alpha,h) \leqslant 2\nabla_1(n_{u,v}) + 2\nabla_2(n_{u,v}). \tag{4·23}$$

Write $d_1' = 1$, $d_2' = d'$, and, for $i = 1, 2$, let ∇_i^+ designate the subsum of ∇_i corresponding to the condition

$$\max_{H\leqslant j\leqslant w} \Omega(dd_i'; v-j, v)/j \geqslant \tfrac{2}{3} + \alpha,$$

and ∇_i^- the similar quantity corresponding to the condition

$$\min_{H\leqslant j\leqslant w} \Omega(dd_i'; v-j, v)/j \leqslant \tfrac{2}{3} - \alpha.$$

We shall only describe the bounds for, say, ∇_i^+, since the other case is similar.

We begin by establishing an upper bound for $\nabla_1^+(n)$. We may assume $t > \mathrm{e}^u$ since otherwise $\nabla_1^+(n_{u,v}) = 0$. We then have, for all $z \in]1, 2[$,

$$\begin{aligned}\nabla_1^+(n_{u,v}) &\leqslant \sum_{h\leqslant j\leqslant w}\sum_{1<d\leqslant \mathrm{e}^t} z^{\Omega(d;v-j,v)-(2/3+\alpha)j} \leqslant \sum_{h\leqslant j\leqslant w} z^{-(2/3+\alpha)j} \sum_{d\mid n}^{u,\log t} z^{\Omega(d;v-j,v)} \\ &\leqslant (1+z)^{\Omega(n;u,\log t)} \sum_{h\leqslant j\leqslant w} z^{-(2/3+\alpha)j} \\ &\leqslant (1+z)^{\Omega(n;u,\log t)} \frac{z^{-(2/3+\alpha)h}}{1-z^{-(2/3+\alpha)}}.\end{aligned}$$

By Lemma 4.5(i), this yields that, for large enough ξ_0,

$$\nabla_1^+(n_{u,v}) \leqslant \tfrac{1}{4} t^{\log(1+z)} (1+z)^{-u+\xi\sqrt{w}} z^{-(2/3+\alpha)h}$$

for all $n \leqslant x$ with at most $\ll_z x\mathrm{e}^{-\xi^2/3}$ exceptions. We select $z := \mathrm{e} - 1$ and observe that $\mathrm{e}^{-u} \leqslant 3^w \mathrm{e}^{-v}$. Since $Q(1 + 3\alpha/2) < (1 + 3\alpha/2)\log(\mathrm{e} - 1)$ for $0 < \alpha \leqslant \frac{1}{10}$, we obtain that the inequality

$$\nabla_1^+(n_{u,v}) \leqslant \tfrac{1}{4} t\mathrm{e}^{-v} 3^w \mathrm{e}^{\xi\sqrt{w}-\beta_1 h} \tag{4·24}$$

holds with the required upper bound for the size of the exceptional set.

We now turn our attention to $\nabla_2^+(n_{u,v})$. Any counted pair $\{d, d'\}$ must verify

$$\mathrm{e}^{-t}/d < 1/d' < \log(d'/(d'-1)) \leqslant \log(d'/d) \leqslant t,$$

moreover $d > 1$, $\log_2 P^-(d) > u$. Hence

$$d > d_0(t) := \max\big(\exp \mathrm{e}^u, \mathrm{e}^{-t}/t\big). \tag{4·25}$$

We define

$$I_1 := \big]d_0(t), \max(1, 1/t^2)\big], \qquad I_2 := \big]\max(1, 1/t^2), x^{1/4}\big], \qquad I_3 := \big]x^{1/4}, \sqrt{x}\,\big],$$

and for $k = 1, 2, 3$, denote by $\nabla_{2k}^+(n_{u,v})$ the contribution to $\nabla_2^+(n_{u,v})$ arising from pairs $\{d, d'\}$ such that $d \in I_k$. The interval I_1 is empty unless $t^2 < \exp(-\mathrm{e}^u)$. In this case,

$$\sum_{n \leqslant x} \nabla_{21}^+(n_{u,v}) \leqslant x \sum_{d \leqslant 1/t^2}^{u,v} \frac{1}{d} \sum_{d < d' \leqslant d\mathrm{e}^t} \frac{1}{d'}.$$

The inner sum is $\ll t + 1/d \ll t$ by (4·25) since $t \ll 1$. Therefore

$$\begin{aligned}\sum_{n \leqslant x} \nabla_{21}^+(n_{u,v}) &\ll tx \sum_{d \leqslant 1/t^2}^{u,v} \frac{1}{d} \\ &\ll tx\mathrm{e}^{-u} \log(1/t) \ll txw\mathrm{e}^{-u} = tx\mathrm{e}^{-v} 3^w w (\mathrm{e}/3)^w.\end{aligned}$$

Since $\log 3 - 1 - \frac{1}{3}Q(1 + \frac{3}{2}\alpha) > \frac{1}{11}$ for $0 \leqslant \alpha \leqslant \frac{1}{10}$, we infer that

$$\nabla_{21}^+(n_{u,v}) \leqslant \tfrac{1}{12} t\mathrm{e}^{-v} 3^w \mathrm{e}^{-\beta_1 h} \tag{4·26}$$

holds for all but an acceptable number of exceptional $n \leqslant x$.

Next, we apply Lemma 4.5(ii) with $\varepsilon := \xi/(3\sqrt{w})$, $T := \varepsilon w$, to obtain that, with $\ll xw\mathrm{e}^{-\xi^2/11}$ exceptional $n \leqslant x$, we have

$$\Omega(n; u, \log_2 d') \leqslant (1 + \varepsilon)(\log_2 d' - u) + \tfrac{1}{3}\xi\sqrt{w} \tag{4·27}$$

simultaneously for all $d' \mid n_{u,v}$. For those integers n such that (4·27) holds, we may write, for $k = 2$ or 3 and any $y \in]0, 1]$, $z \geqslant 1$,

$$\nabla_{2k}^+(n_{u,v}) \leqslant y^{-\xi\sqrt{w}/3} \sum_{h \leqslant j \leqslant w} z^{-(2/3+\alpha)j} \nabla_{2k}^*(n_{u,v}; j) \tag{4·28}$$

with

$$\nabla_{2k}^*(n_{u,v}; j) := \sum_{\substack{d \mid n \\ d \in I_k}}^{u,v} \sum_{\substack{d' \mid n/d \\ d < d' \leqslant d\mathrm{e}^t}}^{u,v} y^{\Omega(n; u, \log_2 d')} z^{\Omega(dd'; v-j, v)} \Big(\frac{\log d'}{\mathrm{e}^u}\Big)^{-(1+\varepsilon)\log y}.$$

We shall choose later the values of the parameters y and z.

Let us first consider the case $k = 2$. We have, for $h \leqslant j \leqslant w$,

$$S_{22}^*(j) := \sum_{n \leqslant x} \nabla_{22}^*(n_{u,v}, j) \\ \leqslant \sum_{d \in I_2}^{u,v} \sum_{d < d' \leqslant d\mathrm{e}^t}^{u,v} y^{\Omega(dd')} z^{\Omega(dd';v-j,v)} \Big(\frac{\log d'}{\mathrm{e}^u}\Big)^{-(1+\varepsilon)\log y} \sum_{m \leqslant x/dd'} y^{\Omega(m;u,\log_2 d')}.$$

Now $dd' \leqslant \mathrm{e}^t d^2 \leqslant x^{1/10} x^{1/2} = x^{3/5}$ by (4·22). The inner sum is hence

$$\ll \frac{x}{dd'} \Big(\frac{\log d'}{\mathrm{e}^u}\Big)^{y-1}.$$

Thus

$$S_{22}^*(j) \ll x \sum_{d \in I_2}^{u,v} \frac{y^{\Omega(d)} z^{\Omega(d;v-j,v)}}{d} \sum_{d < d' \leqslant d\mathrm{e}^t}^{u,v} \frac{y^{\Omega(d')} z^{\Omega(d';v-j,v)}}{d'} \Big(\frac{\log d'}{\mathrm{e}^u}\Big)^A, \tag{4·29}$$

with $A := y - 1 - (1+\varepsilon)\log y$. The inner sum may be evaluated by partial summation using the estimate

$$\sum_{n \leqslant x}^{u,v} y^{\Omega(n)} z^{\Omega(n;v-j,v)} \ll x^{1-c/\exp(v)} (\log x)^{y-1} \mathrm{e}^{y(z-1)j - yu}, \tag{4·30}$$

valid uniformly for $x \geqslant 2$, $0 < y \leqslant 1$, $1 \leqslant z \leqslant z_0 < 2$, with an absolute constant $c > 0$. This can be shown in much the same way as lemma 50.2 of [5] and we leave the details to the reader.

When $t \geqslant 1$, and assuming that

$$B := A + y - 1 = 2y - 2 - (1+\varepsilon)\log y < 0, \tag{4·31}$$

we thus obtain that the innermost sum in (4·29) is

$$\ll t\mathrm{e}^{-u + y(z-1)j} \Big(\frac{\log d}{\mathrm{e}^u}\Big)^B.$$

When $t < 1$, we check, by applying Shiu's theorem [10] on short sums of non-negative multiplicative functions, that the same estimate persists: indeed we have $d > 1/t^2$ whenever $d \in I_2$, so the length of the interval involved is always as large as the square root of the size on its elements. Taking our estimates back into (4·29), we obtain

$$S_{22}^*(j) \ll xt\mathrm{e}^{-u+y(z-1)j} \sum_{d \in I_2}^{u,v} \frac{y^{\Omega(d)} z^{\Omega(d;v-j,v)}}{d} \Big(\frac{\log d}{\mathrm{e}^u}\Big)^B.$$

Using (4·30) and partial summation again, we arrive at

$$S_{22}^*(j) \ll xt\mathrm{e}^{-u+2y(z-1)j+w(B+y)}, \tag{4·32}$$

assuming now that $B+y \gg 1$. We now check that the optimal choice for y and z is compatible with this last condition and (4·31). This optimal choice is determined by noting that, if we set

$$C(y) := 3y - 2 - (1+2\varepsilon)\log y, \quad D(y,z) := 2y(1-z) + (\tfrac{2}{3}+\alpha)\log z,$$

the quantity

$$t\mathrm{e}^{-u+wC(y)} \sum_{h \leqslant j \leqslant w} \mathrm{e}^{-jD(y,z)}$$

is, up to constant factor, an upper bound for $\nabla_{22}^+(n_{u,v})$ on average over integers n satisfying (4·28). Now $C(y)$ is minimal when $y = (1+2\varepsilon)/3$, and, with this choice, $D(y,z)$ is maximal when $z = (1+\frac{3}{2}\alpha)/(1+2\varepsilon)$. However, these choices are not always admissible regarding the conditions upon B and the fact that we need $z \geqslant 1$, so we select $y = (1+\varepsilon)/3$ and $z = 1+\frac{3}{2}\alpha$ instead. This yields

$$\begin{aligned} B &= \tfrac{2}{3}(1+\varepsilon) - 2 + (1+\varepsilon)\log\{3/(1+\varepsilon)\} \leqslant \tfrac{4}{3}\log\tfrac{9}{4} - \tfrac{10}{9} < 0, \\ B+y &= (1+\varepsilon)\log 3 - 1 - Q(1+\varepsilon) \geqslant \log 3 - 1 > 0. \end{aligned}$$

We easily check that

$$C\Big(\frac{1+\varepsilon}{3}\Big) \leqslant \log 3 - 1 + 2\varepsilon\log 3, \quad D\Big(\frac{1+\varepsilon}{3}, 1+\tfrac{3}{2}\alpha\Big) \geqslant \tfrac{2}{3}Q(1+\tfrac{3}{2}\alpha) - \alpha\varepsilon.$$

We have therefore shown that

$$\nabla_{22}^+(n_{u,v}) \leqslant \tfrac{1}{12}t\mathrm{e}^{-v}3^{w+2\xi\sqrt{w}/3}\mathrm{e}^{-\beta_1 h+\xi\sqrt{w}/30} \tag{4·33}$$

except for at most $\ll x\{w\mathrm{e}^{-\xi^2/11} + \alpha^{-2}\mathrm{e}^{-\beta_1 h}\}$ integers $n \leqslant x$.

We proceed similarly to bound $S_{23}^*(j) := \sum_{n\leqslant x} \nabla_{23}^*(n_{u,v};j)$. We first obtain

$$S_{23}^*(j) \ll \frac{x\mathrm{e}^{u(1+\varepsilon)\log y}}{(\log x)^{(1+\varepsilon)\log y}} \sum_{d\in I_3}^{u,v} \frac{y^{\Omega(d)}z^{\Omega(d;v-j,v)}}{d} T(x,d;j;y,z) \tag{4·34}$$

with

$$T(x,d;j;y,z) := \sum_{d<d'\leqslant \min(\mathrm{e}^t d, x/d)}^{u,v} \frac{y^{\Omega(d')}z^{\Omega(d';v-j,v)}}{d'} \bigg(\frac{\log(2x/dd')}{\mathrm{e}^u}\bigg)^{y-1}.$$

Arguing as in [5], pp. 103–104 with the help of (4·30), we show that

$$T(x,d;j;y,z) \ll (t/y)\mathrm{e}^{y(z-1)j-(2y-1)u}(\log d)^{y-1}\Big(\log\frac{2x}{d^2}\Big)^{y-1}.$$

Inserting back in (4·34) and appealing to (4·30) again to perform partial integration, we finally arrive at

$$\begin{aligned}S_{23}^*(j) &\ll (t/y)x^{1-c/(4\exp v)}\mathrm{e}^{2y(z-1)j-u}\Big(\frac{\log x}{\mathrm{e}^u}\Big)^{3y-2-(1+\varepsilon)\log y}\\ &\ll (t/y)x\mathrm{e}^{2y(z-1)j-u+w(3y-2-(1+\varepsilon)\log y)}.\end{aligned}$$

The upper bound above is identical to that of (4·32), so we conclude as previously that

$$\nabla_{23}^+(n_{u,v}) \leqslant \tfrac{1}{12}t\mathrm{e}^{-v}3^{w+2\xi\sqrt{w}/3}\mathrm{e}^{-\beta_1 h+\xi\sqrt{w}/30} \tag{4·35}$$

holds for integer $n\leqslant x$ except at most $\ll x\{w\mathrm{e}^{-\xi^2/11}+\alpha^{-2}\mathrm{e}^{-\beta_1 h}\}$ exceptions.

The required conclusion now follows from (4·23), (4·24), (4·26), (4·33) and (4·35). □

4·3. Arithmetic properties of the first pair of divisors

Recall the definitions of ϱ_0 in (1·8), $Q(y)$ in (4·2), $\omega_\vartheta(n)$ in (4·14) and $u_1 := \varrho_0^* u_0$ where $\varrho_0^*>\varrho_0$. We also introduce a parameter $\delta>0$ so small that

$$(1+\delta)\varrho_0 \leqslant \varrho_0^*/(1+\delta)$$

and set $v_1 := u_1/(1+\delta)$.

Proposition 4.8. *Let ϱ_0 denote the constant defined in (1·8), let $\sigma\in]0,\frac{1}{20}[$, $\varepsilon_0\in]0,1[$ and assume that $\eta:=\varrho_0^*-\varrho_0$ is sufficiently small. There exists a constant $c>0$ such that the following assertion holds for all sufficiently large real numbers x, with $u_0 := (\log_2 x)^{\varepsilon_0}$: all integers $n\leqslant x$ but at most $\ll x/u_0^c$ exceptions possess two distinct divisors d_{0n}, d_{0n}' such that*

(i) $d_{0n}d_{0n}' | n_{u_0,v_1}$;

(ii) $|\log(d_{0n}'/d_{0n})|\leqslant 1$;

(iii) the inequality $\omega\big(n_{u_0,u_1}/d_{0n}d_{0n}'\big)-\omega_\vartheta\big(n_{u_0,u_1}/d_{0n}d_{0n}'\big) > \frac{1}{3}(1-\sigma)(u_1+\log\vartheta)$ *holds simultaneously for all ϑ such that* $\mathrm{e}^{-(1-\sigma)u_1}<\vartheta\leqslant \mathrm{e}^{-u_0}$;

(iv) there exists a subset $\mathcal{E}_0$ of $\mathbb{N}^*\cap[1,x]$ such that $|\mathcal{E}_0|\ll x/u_0^c$ and, for all $z\in[u_1,(1-\varepsilon)\log_2 x]$,

$$\sum_{\substack{n\leqslant x\\ n\notin\mathcal{E}_0}} R\big(n_{u_0,z}/d_{0n}d_{0n}';\vartheta\big) \ll x\big\{\log(3+|\vartheta|)\big\}^8\mathrm{e}^{\sigma u_0}, \tag{4·36}$$

where the implicit constant depends at most upon ε_0.

Proof. We apply Theorem 4.1 with $u := u_0$, $v := v_1$, $w := v_1 - u_1$, $\xi := \frac{1}{20}\eta\sqrt{w}$, $t := 1$, and check that, for large x, we have

$$3^w > 2\mathrm{e}^{v+\xi\sqrt{w}}.$$

Indeed, since $(\varrho_0 - 1)\log 3 = \varrho_0$, we obtain after a small computation that

$$w\log 3 - v - \xi\sqrt{w} = \big\{\log 3 - 1 - \tfrac{1}{20}(1 + \eta_1\log 3)\big\}\eta_1 u_0$$

with $\eta_1 := \varrho_0^*/(1+\delta) - \varrho_0 > \delta\varrho_0$. This yields that (i) and (ii) hold for all $n \leqslant x$ except at most $\ll xu_0^{-1/250}$ exceptions.

To establish (iii) with the required number of exceptions, we first apply Lemma 4.5(iii) with $u := u_0$, $v := u_1$, $\varepsilon := \sigma/7$ and $T := \sigma u_1$. This yields that the inequality

$$\omega(n_{u_0,u_1}) - \omega_\vartheta(n_{u_0,u_1}) > (1 - \tfrac{1}{7}\sigma)[u_1 + \log\vartheta] > (1 - \tfrac{1}{6}\sigma)(u_1 + \log\vartheta) \tag{4·37}$$

holds uniformly for $\mathrm{e}^{-(1-\sigma)u_1} \leqslant \vartheta \leqslant \mathrm{e}^{-u_0}$ for all integers $n \leqslant x$ but at most

$$\ll_\sigma x\mathrm{e}^{-Q(1-\sigma/7)\sigma u_1} \ll_\sigma x/u_0$$

exceptions.

Next, we apply Theorem 4.7 with $u := u_0$, $v := u_1$, $w := w_1 = u_1 - u_0$, $\xi := \eta\sqrt{w_1}$, $\alpha := \sigma/6 > 0$, $h := \sigma u_1$, so that $\beta_1 := \frac{1}{3}Q(1 + \frac{1}{4}\sigma)$, and note that, since

$$w_1\log 3 + \xi\sqrt{w_1} = w_1(\log 3 + \eta) \leqslant u_1(1 + 12\eta),$$

we have, if, as we may assume, η is small enough in terms of σ,

$$\mathrm{e}^{-u_1}3^{w_1}\mathrm{e}^{\xi\sqrt{w_1} - \beta_1 h} \leqslant \mathrm{e}^{-\beta_1\sigma u_1/2}.$$

It follows that there is a subset E_x of $[1, x]$ such that

$$|E_x| \ll x\{w_1\mathrm{e}^{-\xi^2/11} + \mathrm{e}^{-\beta_1 h}\} \ll x/u_0$$

and

$$\sum_{\substack{n\leqslant x\\ n\notin E_x}} \nabla(n_{u_0,u_1}; 1, 1 + \tfrac{1}{6}\sigma, h) \ll x\mathrm{e}^{-\beta_1\sigma u_1/2}.$$

Thus, except perhaps for $\ll x/u_0$ exceptional n, the sum $\nabla(n_{u_0,u_1}; 1; 1 + \frac{1}{6}\sigma, h)$ is empty. For the non exceptional integers n, we have, whenever $\mathrm{e}^{-(1-\sigma)u_1} \leqslant \vartheta \leqslant \mathrm{e}^{-u_0}$,

$$\begin{aligned}\omega(d_{0n}d'_{0n}) - \omega_\vartheta(d_{0n}d'_{0n}) &\leqslant \Omega(d_{0n}d'_{0n}; \log(1/\vartheta), u_1)\\ &\leqslant (\tfrac{2}{3} + \tfrac{1}{6}\sigma)(u_1 + \log\vartheta)\end{aligned}$$

Taking (4·37) into account, we infer that (iii) certainly holds with at most $\ll x/u_0$ exceptions.

Finally, we observe that, in the case $z = u_1$, (iv) is an immediate consequence of Theorem 4.6 with, say, $\xi := \eta w_1$, since the summand in (4·36) does not exceed $\nabla^\dagger(n_{u_0,u_1}; 1; \vartheta)$: we select $\mathcal{E}_0 := \{n \leqslant x : n_{u_0,u_1} \notin \mathcal{E}\}$, where $\mathcal{E}$ is as in the statement of Theorem 4.6. When $u_1 < z \leqslant (1-\varepsilon_0)\log_2 x$, we first apply the case $z = u_1$ and note that, by the sieve,

$$\sum_{\substack{n\leqslant x\\ n\notin\mathcal{E}_0}} R\big(n_{u_0,u_1}/d_{0n}d'_{0n};\vartheta\big) \asymp xe^{-w_1}\sum_{m\notin\mathcal{E}}^{u_0,u_1} \frac{R(m/d_{0m}d'_{0m};\vartheta)}{m}$$

because the subsum on the left corresponding to those n such that, say, $n_{u_0,u_1} > \sqrt{x}$ may be neglected. Hence it follows from (4·17) that

$$e^{-w_1}\sum_{m\notin\mathcal{E}}^{u_0,u_1} \frac{R(m/d_{0m}d'_{0m};\vartheta)}{m} \ll t\mathrm{e}^{-u_1+\xi\sqrt{w_1}}3^{w_1}\{\log(3+|\vartheta|)\}^4. \tag{4·38}$$

Therefore, we have, still by the sieve,

$$\sum_{\substack{n\leqslant x\\ n\notin\mathcal{E}_0}} R\big(n_{u,z}/d_{0n}d'_{0n};\vartheta\big) \ll x\mathrm{e}^{-(z-u)}\sum_{m\notin\mathcal{E}}^{u_0,u_1} \frac{R(m/d_{0m}d'_{0m};\vartheta)}{m}\sum_{\ell}^{u_1,z}\frac{R(\ell;\vartheta)}{\ell}.$$

Using (4·18) with $y = 1$ and a standard sieve argument, we see that inner sum is $\ll \mathrm{e}^{z-u_1}\{\log(3+|\vartheta|)\}^4$. Thus, the required bound follows from (4·38). □

4·4. The second pair of divisors

Recall the definition of ϱ_0, ϱ_1 in (1·8), and let $\varrho_0^* > \varrho_0$, $\varrho_1^* := \frac{1}{3}(2\varrho_0^*+1) > \varrho_1$. For large x, we define $u_k := {\varrho_1^*}^{k-1}\varrho_0^* u_0$ $(k \geqslant 1)$ as in (4·1). For $u_1 \leqslant j \leqslant u_2$ and non-exceptional integer $n \leqslant x$ in the sense of Proposition 4.8, we put

$$n_j := \prod_{p\mid n_{u_0,j}/(d_{0n}d'_{0n})} p.$$

For simplicity, we define $n_j := 1$ when n is exceptional in the above sense. We shall show that, for all integers $n \leqslant x$ but at most $\ll x/u_0^c$ exceptions, n_{u_2} has two divisors d_{1n}, d'_{1n} such that $0 < |\log(d'_{1n}/d_{1n})| \leqslant 1$. This will complete the second of the K inductive constructions described in section 4.1.

We introduce a parameter $\delta > 0$ so small that

$$(1+\delta)^3\varrho_1 \leqslant \varrho_1^*$$

and put

$$z_1 := (1+\delta)\varrho_1 u_1 \leqslant u_2/(1+\delta)^2. \tag{4·39}$$

For integer $m \geqslant 1$, we define

$$\mathcal{L}(m) := \bigcup_{dd'|m} \Big(\log(d'/d) + [-\tfrac{1}{2}, \tfrac{1}{2}] \Big)$$

and denote by $\lambda(m)$ the Lebesgue measure of this set. The following statement constitutes a fundamental lemma.

We retain from Theorem 4.1 the notation $\beta := Q(1/\log 3) \approx 0.00415$.

Lemma 4.9. *Let δ and z_1 be as above. If $\eta := \varrho_0^* - \varrho_0 > 0$ is sufficiently small, there exists a constant $c = c(\delta, \eta) > 0$ such that, for any $j \in [z_1, u_2]$ and any $\varepsilon > 0$, we have*

$$\lambda(n_j) > \varepsilon \mathrm{e}^j$$

for all $n \leqslant x$ but at most $\ll x\{\varepsilon^\beta + u_1^{-c}\}$ exceptions.

Proof. We may plainly assume that ε is sufficiently small. By Lemma 3 of [7], we have, for all m,

$$\lambda(m) \int_{-2}^{2} \varrho(m;\vartheta)^2 \,\mathrm{d}\vartheta \geqslant \frac{3^{2\omega(m)}}{2\pi},$$

so we need an upper bound for

$$I(n_j) := \int_{-2}^{2} \frac{\varrho(n_j;\vartheta)^2}{3^{2\omega(n_j)}} \,\mathrm{d}\vartheta. \tag{4·40}$$

We consider the contributions $I_s(n_j)$ $(1 \leqslant s \leqslant 4)$ from several ranges $\mathcal{D}_s$ for the integration variable ϑ, with the aim to show that, except perhaps for an acceptable number of exceptional integers n, we have

$$I_s(n_j) \leqslant \frac{\mathrm{e}^{-j}}{8\pi\varepsilon} \qquad (1 \leqslant s \leqslant 4). \tag{4·41}$$

We start with $\mathcal{D}_1 := \big[-\mathrm{e}^{-j}/(16\pi\varepsilon), \mathrm{e}^{-j}/(16\pi\varepsilon)\big]$, and observe that (4·41) then follows trivially because the integrand in (4·40) has absolute value at most 1.

For the other ranges, namely

$$\begin{aligned} \mathcal{D}_2 &:= \{\vartheta : \mathrm{e}^{-j}/(16\pi\varepsilon) < |\vartheta| \leqslant \mathrm{e}^{-u_1}\}, \\ \mathcal{D}_3 &:= \{\vartheta : \mathrm{e}^{-u_1} < |\vartheta| \leqslant \mathrm{e}^{-u_0}\}, \\ \mathcal{D}_4 &:= \{\vartheta : \mathrm{e}^{-u_0} < |\vartheta| \leqslant 2\}, \end{aligned}$$

we seek a uniform lower bound for the quantity

$$F(n_j;\vartheta) := 3^{\omega(n_j) - \omega_\vartheta(n_j)}$$

as a function of ϑ, and write, with the notation (4·15),

$$\frac{\varrho(n_j;\vartheta)^2}{3^{2\omega(n_j)}} = \frac{R(n_j;\vartheta)}{F(n_j;\vartheta)}.$$

Let $\kappa := (1+\sqrt{\varepsilon})/\log 3$. Since $d_{0n}d'_{0n} \mid n_{u_0,u_1}$, we have

$$\omega(n_j) - \omega_\vartheta(n_j) = \Omega(n; \log(1/|\vartheta|), j) \qquad (\vartheta \in \mathcal{D}_2)$$

and so, we deduce from lemma 51.2 of [5] that

$$F(n_j; \vartheta) > |\vartheta \mathrm{e}^j|^{\kappa \log 3}$$

holds simultaneously for all $\vartheta \in \mathcal{D}_2$ and all $n \leqslant x$ but at most $\ll \varepsilon^{Q(\kappa)} x \ll x\varepsilon^\beta$. We thus obtain, with the same number of exceptions,

$$I_2(n_j) \leqslant \int_{\mathcal{D}_2} \frac{R(n_j; \vartheta)}{|\vartheta \mathrm{e}^j|^{1+\sqrt{\varepsilon}}} \,\mathrm{d}\vartheta.$$

By (4·18) with $y = 1$, it follows that, for a suitable subset E_2 of $[1, x]$ such that

$$|E_2| \ll x\varepsilon^\beta,$$

we have

$$\sum_{\substack{n \leqslant x \\ n \notin E_2}} I_2(n_j) \ll x \int_{\mathrm{e}^{-j}}^\infty \frac{\mathrm{d}\vartheta}{(\vartheta \mathrm{e}^j)^{1+\sqrt{\varepsilon}}} \ll \varepsilon^{-1/2} x \mathrm{e}^{-j}.$$

Thus, we have (4·41) for $s = 2$ and all $n \leqslant x$ but at most $\ll \varepsilon^\beta x$.

We now turn our attention to $\mathcal{D}_3$. In this range, we have

$$\omega(n_j) - \omega_\vartheta(n_j) = \omega(n_{u_1,j}) + \omega\big(n_{u_0,u_1}/d_{0n}d'_{0n}\big) - \omega_\vartheta\big(n_{u_0,u_1}/d_{0n}d'_{0n}\big),$$

so we deduce from Lemma 4.5(i) and property (iii) of Proposition 4.8 that, given any $\sigma_1 \in]0, \frac{1}{40}[$, we have

$$\omega(n_j) - \omega_\vartheta(n_j) > (1-\sigma_1)\{j - u_1 + \tfrac{1}{3}(u_1 + \log \vartheta)\}$$

uniformly for $\mathrm{e}^{-(1-\sigma_1)u_1} < |\vartheta| \leqslant \mathrm{e}^{-u_0}$ and $n \in [1, x] \smallsetminus E_3$, with $|E_3| \ll x/u_0^c$. However, by Lemma 4.5(i), we have

$$\Omega(n_j; (1-\sigma_1)u_1, u_1) \leqslant \Omega(n; (1-\sigma_1)u_1, u_1) \leqslant 2\sigma_1 u_1$$

for all but at most $\ll x/u_0$ integers $n \leqslant x$. Observing that

$$j - u_1 \geqslant \{(1+\delta)\varrho_1 - 1\}u_1 > 5u_1 \qquad (z_1 \leqslant j \leqslant u_2),$$

we infer that, for $\mathrm{e}^{-u_1} < |\vartheta| \leqslant \mathrm{e}^{-(1-\sigma_1)u_1}$, we have

$$\begin{aligned}\omega(n_j) - \omega_\vartheta(n_j) \geqslant \omega(n_{u_1,j}) &> (1-\sigma_1)\{j - u_1 + \tfrac{1}{3}(u_1 + \log\vartheta) - \tfrac{2}{3}\sigma_1 u_1\} \\ &\geqslant (1-\sigma_1)(1 - \tfrac{2}{15}\sigma_1)\{j - u_1 + \tfrac{1}{3}(u_1 + \log \vartheta)\}\end{aligned}.$$

Thus, for any $\sigma \in]2\sigma_1, \frac{1}{20}[$, all $n \in [1,x] \smallsetminus E_3$ and uniformly for all $\vartheta \in \mathcal{D}_3$, we have

$$\omega(n_j) - \omega_\vartheta(n_j) \geqslant (1-\sigma)\{j - u_1 + \tfrac{1}{3}(u_1 + \log\vartheta)\}. \tag{4·42}$$

Consequently,

$$F(n_j, \vartheta) > 3^{(1-\sigma)(j-2u_1/3)}|\vartheta|^{(1-\sigma)(\log 3)/3}$$

provided $n \notin E_3$. At the cost of modifying the value of the exponent c in our bound for the size of the exceptional set E_3, we therefore deduce from the above and property (iv) of Proposition 4.8 that

$$\begin{aligned}\sum_{\substack{n\leqslant x\\ n\notin E_3}} I_3(n_j) &\ll x\mathrm{e}^{\sigma u_0}\int_{\mathrm{e}^{-u_1}}^{\mathrm{e}^{-u_0}} \frac{\mathrm{d}\vartheta}{3^{(1-\sigma)(j-2u_1/3)}\vartheta^{(1-\sigma)(\log 3)/3}}\\ &\ll x\exp\Big\{\sigma u_0 - (1-\sigma)(\log 3)(j - \tfrac{2}{3}u_1) - u_0\big(1 - \tfrac{1}{3}(1-\sigma)\log 3\big)\Big\}\\ &= x\exp\Big\{-(1-\sigma)(\log 3)j + (1-\sigma)(\varrho_1^*\log 3 - 1)u_0\Big\},\end{aligned} \tag{4·43}$$

with $\varrho_1^* := \frac{1}{3}(2\varrho_0^* + 1)$. Now observe that, since $\varrho_0^* > \varrho_0$ and $(\log 3 - 1)\varrho_0 = \log 3$, we have

$$\begin{aligned}\{(1-\sigma)(\log 3) - 1\}j &\geqslant (1+\delta)\{(1-\sigma)(\log 3) - 1\}\varrho_1\varrho_0^* u_0\\ &> (1+\delta)\{(\log 3 - 1)\varrho_0 - \sigma\log 3\}\varrho_1 u_0\\ &= (1+\delta)(1-\sigma)(\log 3)\varrho_1 u_0.\end{aligned} \tag{4·44}$$

Therefore, the upper bound in (4·43) is

$$\leqslant x\mathrm{e}^{-j-(1-\sigma)u_0}3^{(1-\sigma)(\varrho_1^*-(1+\delta)\varrho_1)u_0} \leqslant \mathrm{e}^{-j-u_0/2}$$

provided $\eta = \varrho_0^* - \varrho_0$ is sufficiently small. We deduce that (4·41) holds for $s = 3$ and all $n \leqslant x$ but an acceptable number of exceptions.

It remains to deal with $\mathcal{D}_4$. Applying (4·42) with $\vartheta = \mathrm{e}^{-u_0}$, we see that, with an acceptable number of exceptional n, we have

$$F(n_j, \vartheta) > 3^{(1-\sigma)(j-2u_1/3-u_0/3)}.$$

Appealing to property (iv) of Proposition 4.8 again, we obtain that, for a suitably bounded exceptional set E_4, we have

$$\sum_{\substack{n\leqslant x\\ n\notin E_4}} I_4(n_j) \ll x3^{-(1-\sigma_2)(j-2u_1/3-u_0/3)} \qquad (z_1 \leqslant j \leqslant u_2)$$

for all $\sigma_2 \in]2\sigma, \frac{1}{20}[$. Using (4·44) for some $\sigma_3 > \sigma_2$ in place of σ, we deduce as previously that the above bound is $\ll x\mathrm{e}^{-(1+c_1)j}$ for some fixed $c_1 = c_1(\delta,\eta) > 0$. This yields (4·41) for $s = 4$ and finishes the proof of our lemma. □

Having at hand the necessary arithmetic information on $n_{u_0,u_1}/d_{0n}d'_{0n}$ stated in Proposition 4.8, it is now a simple matter to complete the proof of the existence of the pair $\{d_{1n}, d'_{1n}\}$. The details being very similar to those of [7] (theorem 1) or [5] (theorem 51), we only provide brief indications.

Let z_1 be defined as in (4·39) and, for $z_1 \leqslant j \leqslant v_2 := u_2/(1+\delta)$, let us consider the number N_j of those integers $n \leqslant x$ such that

$$\min_{\substack{dd'|n_j\\ d'\neq d}} |\log(d'/d)| > 1.$$

This is clearly a non-increasing function of j. Using Lemma 4.9 and a sieve argument as in [7] or [5], we obtain that, for $r \asymp \log u_1$ and on the assumption that $N_{v_2} \gg x\{\varepsilon^\beta + u_0^{-c}\}$, we have

$$N_{j+r} - N_j \gg \varepsilon N_j/(\log u_1)^2 \qquad (z_1 \leqslant j \leqslant v_2).$$

A simple iteration then shows that, in any event,

$$N_{v_2} \ll x\Big\{\varepsilon^\beta + u_0^{-c} + \exp\big(-c_2\varepsilon u_1^{2/3}/(\log u_1)^3\big)\Big\}$$

where c_2 is an absolute positive constant. Selecting $\varepsilon = 1/\sqrt{u_1}$ yields

$$N_{v_2} \ll x u_1^{-\beta/2},$$

as required.

4·5. The induction step

In what follows, we extend the definitions of the arithmetic functions ω and ω_ϑ to positive rational numbers by setting

$$\omega(a/b) := \omega(a/(a,b)) \qquad (a, b \in \mathbb{N}^*).$$

We let $\delta > 0$ be sufficiently small and put

$$v_k := u_k/(1+\delta), \quad w_k := u_k - u_{k-1} \qquad (k \geqslant 1).$$

In this section, we assume that d_{sn}, d'_{sn} have been constructed for $s < k$ and we show how to construct the next pair of divisors $\{d_{kn}, d'_{kn}\}$. Our induction hypothesis is that all integers $n \leqslant x$ but at most $\ll kx/u_0^c$ have $2k$ divisors d_{0n}, d'_{0n}, d_{1n}, d'_{1n}, …, $d_{k-1,n}$, $d'_{k-1,n}$ satisfying the following conditions where $\sigma \in]0, \frac{1}{20}]$ is chosen sufficiently small:

(i) $D_{kn} := \prod_{0\leqslant s<k} d_{sn}d'_{sn} \,\big|\, n_{u_0,v_k}$ with

(ii) $0 < |\log(d'_{sn}/d_{sn})| \leqslant 1 \quad (0 \leqslant s < k)$;

(iii) for $0 \leqslant m < k$, the inequality

$$\omega\big(n_{u_m,u_{m+1}}/D_{kn}\big) - \omega_\vartheta\big(n_{u_m,u_{m+1}}/D_{kn}\big) > \frac{(1-\sigma)(u_{m+1}+\log\vartheta)}{3^{k-m}} \tag{4·45}$$

holds simultaneously for all ϑ in the interval $\mathrm{e}^{-(1-\sigma)u_{m+1}} < \vartheta \leqslant \mathrm{e}^{-u_m}$;

(iv) there exists a subset $\mathcal{E}_0$ of $\mathbb{N}^* \cap [1, x]$ such that $|\mathcal{E}_0| \ll x/u_0^c$ and, for all $z \in [u_k, (1-\varepsilon)\log_2 x]$,

$$\sum_{\substack{n \leqslant x \\ n \notin \mathcal{E}_0}} R\big(n_{u_0,z}/D_{kn}; \vartheta\big) \ll x\big\{\log(3+|\vartheta|)\big\}^8 \mathrm{e}^{\sigma u_{k-1}}, \tag{4·46}$$

where the implicit constant is absolute.

We have to show that the above properties are still satisfied at rank k. We establish the existence of $\{d_{k,n},\, d'_{k,n}\}$ exactly as in section 4.4, with now $k+2$ integration domains in the analogue of Lemma 4.9, and appealing to the induction hypothesis instead of Proposition 4.8.

The main difficulty consists in establishing property (iii). To ease exposition, we restrict to the case $k = 2$, $m = 0$, which is fully representative of the general case.

We first note that it is sufficient to consider integers n satisfying

$$n_{u_0,u_1} = d_0 d'_0 t_0 t'_0 s_0, \quad n_{u_1,u_2} = t_1 t'_1 s_1$$

with the conditions

$$\begin{cases} (a) \quad 0 < |\log(d'_0/d_0)| \leqslant 1,\ d_1 = t_0 t_1,\ d'_1 = t'_0 t'_1,\ 0 < |\log(d'_1/d_1)| \leqslant 1, \\ (b) \quad d_0 d'_0 t_0 t'_0 s_0 \in \mathbb{N}(u_0, u_1),\ t_1 t'_1 s_1 \in \mathbb{N}(u_1, u_2),\ \mu(n_{u_0,u_2})^2 = 1, \\ (c) \quad |\Omega(d_0 d'_0 t_0 t'_0 s_0) - w_1| \leqslant w_1^{2/3}, \\ (d) \quad \min\limits_{\sigma^4 u_1 \leqslant j \leqslant w_1} \Omega(d_0 d'_0 t_0 t'_0 s_0; u_1 - j, u_1)/j \geqslant 1 - \sigma^4, \\ (e) \quad |\log(t'_0/t_0)| \leqslant (\log u_0)\mathrm{e}^{v_1} \leqslant \tfrac{1}{2}\mathrm{e}^{u_1}, \\ (f) \quad |\Omega(d_0 d'_0; u_1 - j, u_1)/j - \tfrac{2}{3}| \leqslant \sigma^4 \qquad (\sigma^4 u_1 \leqslant j \leqslant w_1). \end{cases} \tag{4·47}$$

Indeed, the last condition in (b) follows from the fact that all integers $n \leqslant x$ but at most $\ll x\exp\{-\mathrm{e}^{u_0}\}$ are such that $p^2 \mid n \Rightarrow \log_2 p \leqslant u_0$; condition (c) follows from Lemma 4.5(i); condition (d) is a consequence of Lemma 4.5(iii) with $\varepsilon := \sigma^4$, $T := \sigma^4 u_1$; condition (e) follows from Lemma 4.2, since this statement guarantees that the exceptional n are at most $\ll x/\sqrt{u_0}$; and condition (f) follows from Theorem 4.7 with $u = u_0$, $v = u_1$, $\alpha := \sigma^4$, $t = 1$, $\xi := \eta\sqrt{w_1}$, $h := \sigma^4 u_1$: we check as in the proof of Proposition 4.8 that, if η is sufficiently small in terms of σ, then the number of contravening integers is $\ll x\mathrm{e}^{-\beta_1 \sigma u_1/2}$.

We set out to show that the inequalities

$$\Omega(d_0 d'_0 t_0 t'_0; u_1 - j, u_1) \leqslant (\tfrac{8}{9} + \tfrac{1}{10}\sigma)j \qquad (\sigma u_1 \leqslant j \leqslant w_1) \tag{4·48}$$

hold for all integers $n \leqslant x$ except at most $\ll x/u_0^c$. Indeed, combined with condition (d), this immediately yields (iii) of the induction hypothesis.

The number of exceptional integers is

$$\ll x\mathrm{e}^{-w_1 - w_2} \sum \frac{1}{d_0 d'_0 t_0 t'_0 s_0 t_1 t'_1 s_1} \tag{4·49}$$

where the sum runs over all eight-tuples satisfying (4·47) and contravening (4·48) for at least one j.

Condition (e) of (4·47) enables us to apply Theorem 4.4 with

$$u = u_1, \quad v = u_2, \quad \xi := \sigma^4\sqrt{w}, \quad z := \log(t_0'/t_0).$$

As a consequence, we obtain that, for each fixed d_0, d_0', t_0, t_0', s_0 the sum over the three other variables is

$$\ll 3^{w_2}\mathrm{e}^{-u_2+w_2(1+\sigma^4)} \tag{4·50}$$

except for a set of integers n of cardinality $\ll xw_1\mathrm{e}^{-\sigma^8w_1/11}$, which is an acceptable bound.

Next, we need an estimate for the contribution S_0 to the sum in (4·49) of the five remaining variables. To this end, we deduce from the last condition in (4·47) that if (4·48) is not fulfilled then there is a $m \in [\sigma u_1, w_1]$ such that

$$\Omega(t_0t_0'; u_1 - m, u_1) > (\tfrac{2}{9} + \tfrac{1}{11}\sigma)m. \tag{4·51}$$

We split each of the variables t_0, t_0', s_0 into two factors, respectively belonging to $\mathbb{N}(u_0, u_1 - m)$ and $\mathbb{N}(u_1 - m, u_1)$, and write accordingly

$$t_0 = ab, \quad t_0' = a'b', \quad s_0 = fg.$$

We let $S_0(m; A, B, D, E, F, G)$ denote the subsum of S_0 corresponding to conditions (4·51) and

$$\begin{cases} \Omega(aa') = A, \quad \Omega(bb') = B, \\ \Omega(f) = F, \quad \Omega(g) = G, \\ \Omega(d_0) = D, \quad \Omega(d_0') = E. \end{cases}$$

The contribution of a, a', f, is

$$\ll \frac{2^A(w_1 - m + 1)^{A+F}}{A!F!},$$

that of b, b', g, is

$$\ll \frac{2^B(m+1)^{B+G}}{B!G!}\cdot$$

By conditions (f) and (a), we must have $\log_2 d_0 > u_1(1 - \sigma^4) - 1$. So, for each given d_0, we have

$$\sum_{\substack{d_0/\mathrm{e} < d_0' \leqslant d_0 \\ \Omega(d_0') = E}} \frac{1}{d_0'} \ll \frac{(w_1+1)^E}{E!\log d_0} \ll \mathrm{e}^{-u_1(1-\sigma^4)}\frac{(w_1+1)^E}{E!}$$

and similarly

$$\sum_{\Omega(d_0) = D} \frac{1}{d_0} \ll \frac{(w_1+1)^D}{D!}\cdot$$

We thus obtain

$$S_0(m;A,B,D,E,F,G)\ll\frac{2^{A+B}(w_1-m+1)^{A+F}(m+1)^{B+G}w_1^{D+E}}{\mathrm{e}^{u_1(1-\sigma^4)}A!B!D!E!F!G!}.$$

We sum this quantity over the range

$$\begin{cases}(\alpha)\quad \sigma u_1\leqslant m\leqslant w_1,\\ (\beta)\quad |A+B+D+E+F+G-w_1|\leqslant w_1^{2/3},\\ (\gamma)\quad B>(\frac29+\frac1{11}\sigma)m\\ (\delta)\quad D+E>\frac23(1-\frac19\sigma^4)w_1.\end{cases}$$

Indeed, condition (β) corresponds to (c), condition (γ) to (4·51), and condition (δ) to (f) with $j:=[w_1]+1$.

We introduce two parameters ϑ, ψ, with $\vartheta\geqslant1$, $\psi\geqslant1$, and take (γ) and (δ) into account by inserting a factor

$$\vartheta^{B-(\frac29+\frac1{11}\sigma)m}\psi^{D+E-\frac23(1-\frac19\sigma^4)w_1},$$

and extending the sum to all admissible values of the parameters A,B,D,E,F,G. This yields

$$S_0\ll\frac{\mathrm{e}^{-u_1(1-\sigma^4)}}{\psi^{\frac23(1-\frac19\sigma^4)w_1}}\sum_{|N-w_1|\leqslant w_1^{2/3}}\ \sum_{\sigma u_1\leqslant m\leqslant w_1}\frac{\{(2\psi+3)w_1+(2\vartheta-2)m\}^N}{\vartheta^{(\frac29+\frac1{11}\sigma)m}N!}.$$

The ratio of two consecutive terms in the m-sum is

$$\leqslant\vartheta^{-(\frac29+\frac1{11}\sigma)}\Big(1+\frac{2\vartheta-2}{(2\psi+3)w_1}\Big)^N.$$

This is exceeded by a constant <1 provided the parameters ϑ and ψ are chosen in such a way that

$$r:=\tfrac29(1+\tfrac9{22}\sigma)\log\vartheta-\frac{2\vartheta-2}{2\psi+3}>0. \tag{4·52}$$

Under this new hypothesis, we obtain

$$\begin{aligned}S_0&\ll\frac{\mathrm{e}^{-\{1-\sigma^4+r\sigma(1-\sigma)\}u_1}}{\psi^{\frac23(1-\frac19\sigma^4)w_1}}\sum_{|N-w_1|\leqslant w_1^{2/3}}\frac{\{(2\psi+3)w_1\}^N}{N!}\\ &\ll\frac{\mathrm{e}^{-\{1-\sigma^4+r\sigma(1-\sigma)\}u_1}\{(2\psi+3)\mathrm{e}\}^{w_1+w_1^{2/3}}}{\psi^{\frac23(1-\frac19\sigma^4)w_1}}.\end{aligned}$$

We select $\psi:=3$, $\vartheta:=1+\frac9{22}\sigma$, and check that (4·52) is satisfied: indeed, $r=\frac29Q(1+\frac9{22}\sigma)>\frac3{242}\sigma^2$. This yields, for small enough, positive σ,

$$S_0\ll\mathrm{e}^{w_1-u_1}3^{(4/3)w_1}\mathrm{e}^{-\sigma^3u_1/81}. \tag{4·53}$$

Taking (4·50) into account, we obtain that the upper bound (4·49) is

$$\ll x3^{(4/3)w_1+w_2}\mathrm{e}^{-u_1-u_2-\sigma^3u_1/82}. \tag{4·54}$$

We have

$$u_1=\varrho_0^*u_0,\quad u_2=\varrho_1^*\varrho_0^*u_0,\quad w_1=(\varrho_0^*-1)u_0,\quad w_2=\varrho_0^*(\varrho_1^*-1)u_0.$$

Now we observe that

$$\begin{aligned}\tfrac{4}{3}(\log 3)(\varrho_0-1)+(\log 3)\varrho_0(\varrho_1-1)-\varrho_0-\varrho_1\varrho_0 \\ &= \tfrac{4}{3}(\log 3)(\varrho_0-1)+\tfrac{2}{3}(\log 3)\varrho_0(\varrho_0-1)-\tfrac{2}{3}\varrho_0(\varrho_0+2)\\ &= \tfrac{2}{3}(\log 3)(\varrho_0-1)(\varrho_0+2)-\tfrac{2}{3}\varrho_0(\varrho_0+2)=0.\end{aligned}$$

Therefore, the upper bound (4·54) is certainly $\ll 1/u_0$ provided $\varrho_0^*-\varrho_0$ is chosen sufficiently small in terms of σ.

This achieves the proof of property (iii) of the induction hypothesis.

Finally, property (iv) follows from (4·17), with now $t:=k$, as explained in the corresponding part of the proof of Proposition 4.8.

This concludes the proof of the induction step and hence of Theorem 1.4.

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Helmut Maier
Universität Ulm
Abt. Mathematik III
Helmholtzstrasse 18
D-89069 Ulm
Germany

Gérald Tenenbaum
Institut Élie Cartan
Nancy-Université, CNRS, INRIA,
Boulevard des Aiguillettes
BP 239
54506 Vandœuvre Cedex
France