ON THE NORMAL CONCENTRATION OF DIVISORS

H. MAIER AND G. TENENBAUM

1. *Introduction*

The concept of concentration function was introduced in 1937 by Paul Lévy as a tool for the study of sums of random variables. It is defined, for any random variable X with distribution function F, as

$$Q(l) = \sup_{x \in \mathbb{R}} (F(x+l) - F(x)), \quad l > 0.$$

Since then, concentration functions have been used by probabilists mainly to investigate convergence problems, but they have also been applied to several other questions. A full account of the subject may be found in [**9**].

In number theory, the study of concentration functions has been initiated by Erdös who considered the case of additive arithmetical functions [**1**]. More recent work in this direction is due to Halász [**5**] and Ruzsa [**12**].

An instance of the occurrence of a concentration function in arithmetic is related to the old conjecture of Erdös according to which almost all integers possess at least two divisors d, d', with the property that $d < d' \leqq 2d'$. Let n be a positive integer and let $Q_n(l)$ denote the concentration function of the random variable D_n taking the values $\log d$, as d runs through all divisors of n, with equal probability $1/\tau(n)$. In Erdös's conjecture, if one replaces the constant 2 by e (which has no important consequence), an alternative statement is

$$\Delta(n) := \tau(n)\, Q_n(1) = \max_x \operatorname{card}\{d : d \mid n, e^x < d \leqq e^{x+1}\} > 1, \quad \text{p.p.}$$

Here and throughout the paper the notation p.p. indicates that a relation holds in a sequence of asymptotic density 1.

The function $\Delta(n)$ was studied in several recent papers [**3, 7, 8, 10, 11**], and in particular it was shown by Hooley that its average order has many applications in different branches of number theory. In [**11**], we showed that

$$\Delta(n) > (\log\log n)^\gamma, \quad \text{p.p.},$$

for any $\gamma < -\log 2/\log(1-1/\log 3) = 0.28754\ldots$. This settled Erdös's conjecture, and it seemed desirable to obtain a satisfactory upper estimate for the normal order of the Δ-function.

Sperner's theorem readily implies (see [**10**]) that for square-free n

$$\Delta(n) \leqq 2\binom{\omega(n)}{\frac{1}{2}\omega(n)} \leqq 2\tau(n)\,\omega(n)^{-\frac{1}{2}},$$

where $\omega(n)$ stands for the number of distinct prime factors of n. The same inequality (possibly with another constant) follows for all n from the Kolmogorov–Rogozin

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inequality on concentration functions (see [9]) on noticing that D_n can be written canonically as a sum of independent random variables:

$$D_n = \sum_{p^\nu \| n} D_{p^\nu}.$$

The above upper bound is optimal, but it can be expected that it is only attained for scarce exceptional integers and that $\Delta(n)$ is usually much smaller. Indeed, estimates of the type

$$\Delta(n) \ll_\varepsilon (\log n)^{\alpha+\varepsilon}, \quad \text{p.p.},$$

have been obtained in turn with fairly small values of α. Hooley's average bound [10] implies that $\alpha \leqq (4/\pi)-1 = 0.27323\ldots$, and Hall and Tenenbaum prove in [7] that $\alpha \leqq (\log 2)\,(1-1/\log 3) = 0.06221\ldots$.

In this paper, our aim is to establish the following result, showing that a power of log log n is the right order of magnitude for the normal behaviour of $\Delta(n)$.

THEOREM. *Let $\psi(n) \to \infty$ as slowly as we wish. Then*

$$\Delta(n) < \psi(n) \log\log n, \quad \text{p.p.}$$

It is shown in [7] that the average order of $\Delta(n)$ is at least $C \log\log n$. It could be that this mean value is dominated by those integers n such that $\omega(n) = (1+o(1)) \log\log n$ and that $\Delta(n)/\log\log n$ has a distribution function. Our methods do not seem delicate enough, at present, to yield such a result.

2. *Notation and conventions*

In the sequel $\xi = \xi(x)$ is a function tending to infinity with x, arbitrarily slowly. It will be convenient to suppose that it takes integer values.

The letter p denotes exclusively a prime number; $\omega(n)$ (respectively $\Omega(n)$) stands for the number of prime factors of n counted without (respectively with) multiplicity. We designate by $p_1(n) < \ldots < p_\omega(n)$ the ordered sequence of the distinct prime factors of n and set $P^-(n) = p_1(n)$, $P^+(n) = p_\omega(n)$. By convention, $P^-(1) = +\infty$, $P^+(1) = 1$.

For $n \leqq x$, we define

$$K = K(n,x) = \max\{k, 1 \leqq k \leqq \omega(n) \colon p_k(n) < \exp\exp(\log\log x - \xi(x))\}$$

and we put

$$n_k = \begin{cases} \prod_{\xi < j \leqq k} p_j(n), & \xi < k \leqq K, \\ n_K, & k > K. \end{cases}$$

We use the notation p.p. to indicate that a relation holds in a sequence of asymptotic density 1; the notation p.p.x means for at least $x+o(x)$ integers $\leqq x$. We put $L = L(x) = [2\log\log x]$. Finally, we write $(u)^+ = \max(u,0)$, for $u \in \mathbb{R}$.

3. *Preliminary results*

We shall need the following lemmas.

LEMMA 1. *Let f be a real multiplicative function such that, for all p, $0 \leqq f(p^\nu) \leqq \lambda_1 \lambda_2^\nu$, for $\nu = 0, 1, 2, \ldots$, with $0 < \lambda_1$, $0 < \lambda_2 < 2$. Then, for all $x \geqq 1$, we have*

$$\sum_{n \leqq x} f(n) \ll_{\lambda_1, \lambda_2} x \prod_{p \leqq x} (1-p^{-1}) \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu}.$$

This is a weak form of a theorem of Halberstam and Richert **[6]** generalizing a result of Hall; it has an elementary proof.

The next lemma is included in [**4**, Lemme 11]. A stronger version, which we shall not need here, appears in **[13]**.

LEMMA 2. *For $2 \leqq u \leqq v \leqq x$, we have*

$$\operatorname{card}\{n \leqq x: \prod_{p^\nu \| n,\, p \leqq u} p^\nu \geqq v\} \ll x \exp\left(-c\frac{\log v}{\log u}\right),$$

where c is a positive absolute constant.

COROLLARY. *We have*

$$\log\log n_K < \log\log x - \tfrac{1}{2}\xi(x), \quad \text{p.p.}x. \tag{1}$$

LEMMA 3. *We have*

$$\Omega(n/n_K) < (2+o(1))\,\xi(x), \quad \text{p.p.}x. \tag{2}$$

$$K < L, \quad \text{p.p.}x. \tag{3}$$

This follows immediately from the Turán–Kubilius inequality.

LEMMA 4. *Let $\varepsilon > 0$ be fixed. We have*

$$(1-\varepsilon)\,k < \log\log p_k(n) < (1+\varepsilon)\,k, \quad \xi < k \leqq K, \quad \text{p.p.}x. \tag{4}$$

This is a classical result of Erdös **[1]**.

Henceforth, we fix $\varepsilon > 0$ sufficiently small. For $\xi < k \leqq L$, we define A_k as the set of all integers a satisfying the following conditions:

$$\left.\begin{aligned} &\mu(a)^2 = 1,\ \omega(a) = k-\xi,\ \log\log P^+(a) < \log\log x - \xi(x),\\ &\log\log a < \log\log x - \tfrac{1}{2}\xi(x),\\ &(1-\varepsilon)(j+\xi) < \log\log p_j(a) < (1+\varepsilon)(j+\xi),\ 1 \leqq j \leqq k-\xi. \end{aligned}\right\} \tag{A_k}$$

Set

$$A := \{n \leqq x : n_k \in A_k\ (\xi < k \leqq K)\}.$$

By the corollary to Lemma 2, and Lemmas 3, 4, we see that

$$n \in A, \quad \text{p.p.}x.$$

The next lemma concerns the quantity

$$S_k(x, a) := \operatorname{card}\{n \in A : n_k = a\}.$$

LEMMA 5. *For $\xi < k \leqq L$, $a \in A_k$, and $P^+(a) < p < \exp\exp(\log\log x - \xi(x))$, we have*

$$S_{k+1}(x, ap) \ll \exp((1+\varepsilon)\,\xi - (1-\varepsilon)\,k)\,x/ap.$$

Proof. Let b denote a generic interger such that $\omega(b) = \xi$, $\mu(b)^2 = 1$, and $P^+(b) < P^-(a)$. Then

$$S_{k+1}(x, ap) \ll \sum_b \sum_{m \leqq x/abp} f(m),$$

where f is the strongly multiplicative function defined by

$$f(p') = \begin{cases} 0 & \text{if } p' \leqq p \text{ and } p' \nmid abp, \\ 1 & \text{otherwise.} \end{cases}$$

Since $b < \exp\{\xi e^{(1+\varepsilon)(\xi+1)}\} = x^{o(1)}$ and $ap = x^{o(1)}$, we have $p < x/abp$ for every b and we may estimate the inner sum by Lemma 1. It comes to

$$S_{k+1}(x, ap) \ll \sum_b \frac{x}{\phi(abp) \log p} \ll \frac{x}{ap \log p} \prod_{p' < \exp\exp((1+\varepsilon)(\xi+1))} (1 + 1/(p'-1)).$$

This easily implies the desired result.

We now define for positive integer n and real u

$$\Delta(n, u) := \operatorname{card}\{d : d \mid n, u < \log d \leqq u+1\}.$$

Thus $\Delta(n) = \max_u \Delta(n, u)$. For integer $q \geqq 1$, we put

$$M_q(n) := \int_{-\infty}^{+\infty} \Delta(n, u)^q \, du,$$

$$M_q^*(n) := \sum \left\{1 : d_1, \ldots, d_q \mid n, \log\left(\frac{\max d_i}{\min d_i}\right) \leqq 1\right\}.$$

LEMMA 6. *For $n, q \geqq 1$, we have*

$$\Delta(n) \leqq 2^{1+1/q} M_q(n)^{1/q}, \tag{5}$$

$$M_q^*(n) \leqq 2^q M_q(n). \tag{6}$$

Proof. Let u_0 be such that $\Delta(n, u_0) = \Delta(n)$. Then one of the two intervals $(e^{u_0}, e^{u_0+\frac{1}{2}}]$, $(e^{u_0+\frac{1}{2}}, e^{u_0+1}]$ contains at least $\frac{1}{2}\Delta(n)$ divisors of n. Suppose for instance it is the first. Then $\Delta(n, u) \geqq \frac{1}{2}\Delta(n)$ for $u_0 - \frac{1}{2} < u \leqq u_0$. This implies (5).

Set $\bar{\Delta}(n, u) = \Delta(n, u) + \Delta(n, u+1)$, for $u \in \mathbb{R}$. By a classical inequality

$$\bar{\Delta}(n, u)^q \leqq 2^{q-1}(\Delta(n, u)^q + \Delta(n, u+1)^q),$$

whence

$$\int_{-\infty}^{+\infty} \bar{\Delta}(n, u)^q \, du \leqq 2^q M_q(n).$$

The left-hand side of this inequality is equal to

$$\prod_{d_1, \ldots, d_q \mid n} \left(2 - \log\left(\frac{\max d_i}{\min d_i}\right)\right)^+ \geqq M_q^*(n).$$

This completes the proof.

We shall need an upper bound for $\Delta(n_k)$ in terms of $M_q(n_k)$ sharper than (5) for q close to k. This is the content of Lemma 8 below. The next two results are preparatory estimates for the proof of this lemma. For $v \geqq 0$, we denote by $n(v)$ the largest divisor m of n such that $P^+(m) < \exp\exp v$.

LEMMA 7. *Put* $e_1 = e^{1-2\varepsilon}$. *There exists a positive constant* $\lambda(\varepsilon)$, *depending only on* ε, *such that for* $v \leqq \log\log x - \frac{1}{2}\xi(x)$,

$$\operatorname{card}\{n \leqq x : \exists d, d' \mid n(v), d \neq d', |\log(d'/d)| \leqq (3/e_1)^{-v}\} \ll x v^{-\lambda(\varepsilon)}.$$

Proof. Set $\Omega(n,t) := \sum_{p^\nu \| n,\, p \leqq t} \nu$. By Lemma 1, we have for $0 < y < 2$, $2 \leqq t \leqq x$, that

$$\sum_{n \leqq x} y^{\Omega(n,t) - y \log\log t} \ll_y x (\log t)^{-Q(y)}$$

with $Q(y) = y\log y - y + 1 \geqq 0$. Choosing $y = 1 + \frac{1}{2}\varepsilon$ and taking successively $t = t_k = \exp(e^k v \log(3/e_1))$, $k = 0, 1, 2, \ldots$, we obtain that

$$\Omega(n,t) < (1+\varepsilon)\log\log t, \quad t \geqq (3/e_1)^v, \tag{7}$$

except for at most $O(x v^{-Q(1+\frac{1}{2}\varepsilon)})$ integers $n \leqq x$.

Next, we use the method developed in [**2**]. Disregarding the exceptions above, the integers having the required property contribute at least 1 in the following sum:

$$\sum_{n \leqq x} \sum_{dd' \mid n(v)}{}' z^{\Omega(n,d)} (\log d)^{-(1+\varepsilon)\log z}, \quad 0 < z \leqq 1, \tag{8}$$

where the dash indicates that d, d' satisfy $0 < \log(d/d') \leqq (3/e_1)^{-v}$. Indeed, this last condition ensures that $d \geqq (3/e_1)^v$ (since $\log(d/d') \geqq \log(d/(d-1)) \geqq 1/d$), and we deduce that the inner sum is $\geqq 1$ if it is not empty and n satisfies (7).

The computations are similar to those in [**2**]. We write $n = mdd'$, permute summations and estimate the m-sum by Lemma 1. To deal with the d'-sum, we ignore the condition $P^+(d') < \exp\exp v$ and we match the trivial bound obtained in replacing $z^{\Omega(d')}$ by 1 against the bound yielded by partial summation using the classical formula

$$\sum_{d' \leqq w} z^{\Omega(d')} = C(z)\, w (\log w)^{z-1} (1 + O_z(1/\log w)), \quad w \geqq 2,$$

proved by contour integration. We obtain that the d'-sum is

$$\ll_z (3/e_1)^{-v} d \min(1, (\log d)^{z-1}(1 + (3/e_1)^v/\log d)).$$

We then complete the calculation with the help of the following estimate proved in [**4**, Lemme 10]:

$$\sum_{\substack{d \leqq w \\ P^+(d) < \exp\exp v}} z^{\Omega(d)} \ll_z w (\log w)^{z-1} \exp(-c e^{-v} \log w), \quad w \geqq 2,$$

where c is a positive absolute constant. Selecting $z = \frac{1}{3}$, we find that (8) is

$$\ll x(3^{1+\varepsilon}/e)^v (3/e_1)^{-v} = x(e^2/3)^{-\varepsilon v}.$$

This implies the stated result.

COROLLARY. *We have*

$$d, d' \mid n_k, \quad d \neq d' \Rightarrow |\log(d/d')| > (3/e_2)^{-k}, \quad \xi < k \leqq K, \quad \text{p.p.}x,$$

where $e_2 = e_2(\varepsilon) \to e$ *as* $\varepsilon \to 0$.

Proof. By Lemma 4, we have that $n_k \mid n((1+\varepsilon)k)$, $\xi < k \leqq K$, p.p.x. Choosing successively, in Lemma 7, $v = v_j = (1+\varepsilon)^j$ for all possible $j > (\log \xi)/\log(1+\varepsilon)$, we see that the total number of exceptional integers is $o(x)$. Since for every k, $\xi < k \leqq K$, there is v_j such that $(1+\varepsilon)k < v_j \leqq (1+\varepsilon)^2 k$, this proves the corollary.

LEMMA 8. *Let e_2 be as in the corollary above. We have*

$$\Delta(n_k) \leqq 1+(3/e_2)^{k/q} M_q(n_k)^{1/q}, \quad \xi < k \leqq K, q \geqq 1, \quad \text{p.p.}x.$$

Proof. Let u_0 be such that $\Delta(n_k, u_0) = \Delta(n_k)$, and let d_1, d_2, $d_1 < d_2$, be the two smaller divisors of n_k in $(e^{u_0}, e^{u_0+1}]$. By the above corollary, we have

$$\log d_2 > \log d_1 + (3/e_2)^{-k}, \quad \xi < k \leqq K, \quad \text{p.p.}x.$$

Thus $\Delta(n_k, u) \geqq \Delta(n_k)-1$ for $\log d_1 \leqq u < \log d_2$, that is on an interval of length $\geqq (3/e_2)^{-k}$. This implies that

$$M_q(n_k) \geqq (\Delta(n_k)-1)^q (3/e_2)^{-k}, \quad \xi < k \leqq K, \; q \geqq 1, \quad \text{p.p.}x.$$

This is all that is required.

4. *Proof of the theorem*

The trivial inequality (see [**10** p. 119]) $\Delta(ab) \leqq \Delta(a)\tau(b)$, for $a, b \geqq 1$, and (2) imply that

$$\Delta(n) \leqq \Delta(n_K) 4^{(1+o(1))\xi}, \quad \text{p.p.}x. \tag{9}$$

We are going to prove that

$$\Delta(n_K) \ll K, \quad \text{p.p.}x.$$

Together with (3) and the fact that the growth of ξ is arbitrary, this will be sufficient to yield the result wanted.

The starting point is the identity

$$\Delta(n_{k+1}, u) = \Delta(n_k, u) + \Delta(n_k, u-\log p_{k+1}(n)), \quad \xi < k < K,$$

from which the following formula is immediately derived:

$$M_q(n_{k+1}) = 2M_q(n_k) + \sum_{j=1}^{k} \binom{q}{j} \int_{-\infty}^{+\infty} \Delta(n_k, u)^j \Delta(n_k, u-\log p_{k+1}(n))^{q-j}\, du. \tag{10}$$

The method we then use is rather novel. It consists of averaging this relation over all numbers n with fixed n_k and variable $p_{k+1}(n)$. This gives a set of inequalities relating $M_q(n_{k+1})$ and the $M_j(n_k)$, $1 \leqq j \leqq q$. The proof can then be completed by a simple recurrence procedure.

For $\xi < k \leqq L$, $a \in A_k$, $1 \leqq j \leqq q-1$, we put

$$\begin{aligned} T_j(a,x) :&= \sum_{\substack{n \in A,\, n_k = a \\ K(n,x) > k}} \int_{-\infty}^{+\infty} \Delta(a,u)^j \Delta(a, u-\log p_{k+1}(n))^{q-j}\, du \\ &\leqq \sum_{\substack{p > P^+(a) \\ ap \in A_{k+1}}} S_{k+1}(x, ap) \int_{-\infty}^{+\infty} \Delta(a,u)^j \Delta(a, u-\log p)^{q-j}\, du \\ &\ll \frac{x}{a} \exp((1+\varepsilon)\xi - (1-\varepsilon)k) \int_{-\infty}^{+\infty} \Delta(a,u)^j \sum_{p > P^+(a)} \frac{\Delta(a, u-\log p)^{q-j}}{p}\, du \end{aligned}$$

by Lemma 5. Expanding Δ^{q-j} as a multiple sum, we find that the p-sum is equal to

$$\sum_{d_1,\dots,d_{q-j}|a}^{*} \sum \left\{\frac{1}{p} : p > P^+(a), u - \log \min d_i < \log p \leqq u - (\log \max d_i) + 1\right\}, \quad (11)$$

where the star indicates that the summation is restricted to those $(q-j)$-uples of divisors of a such that $\log((\max d_i)/(\min d_i)) \leqq 1$. In the inner sum p covers an interval with bounds e^α, e^β, say. By the prime number theorem, this is $\int_\alpha^\beta dv/v + O(\exp(-c\sqrt{\alpha}))$. We then rearrange the main terms and add the remainders, noticing that $\alpha > \log P^+(a) > e^{(1-\varepsilon)k}$. This shows that (11) is equal to

$$\int_{\log P^+(a)}^{+\infty} \Delta(a, u-v)^{q-j} \frac{dv}{v} + O(M_{q-j}^*(a) \exp(-c\lambda^k))$$

with $\lambda = e^{\frac{1}{2}(1-\varepsilon)}$. Estimating $M_{q-j}^*(a)$ by Lemma 6 and using the fact that $v \geqq \log P^+(a) > e^{(1-\varepsilon)k}$, we get

$$T_j(a,x) \ll \frac{x}{a} \exp((1+\varepsilon)\xi - 2(1-\varepsilon)k)\, M_j(a)\, M_{q-j}(a)\, (1 + 2^q \exp(k - c\lambda^k)). \quad (12)$$

Put $R_q(n) := \sum_{j=1}^{q-1} \binom{q}{j} M_j(n) M_{q-j}(n)$. By (10) and (12) we have, uniformly in k with $\xi < k \leqq L$ and $q \geqq 2$, that

$$\sum_{\substack{n\in A\\ n_k = a}} \frac{(M_q(n_{k+1}) - 2M_q(n_k))^+}{R_q(n_k)} \ll \frac{x}{a} \exp((1+\varepsilon)\xi - 2(1-\varepsilon)k)(1 + 2^q \exp(k - c\lambda^k)).$$

Next we sum this estimate over $a \in A_k$, noticing that

$$\sum_{a\in A_k} \frac{1}{a} \leqq \prod_{(1-\varepsilon)\xi < \log\log p < (1+\varepsilon)k} (1 + 1/p) \ll \exp((1+\varepsilon)k - (1-\varepsilon)\xi).$$

We obtain

$$\sum_{n\in A} \frac{(M_q(n_{k+1}) - 2M_q(n_k))^+}{R_q(n_k)} \ll x \exp(-(1-5\varepsilon)k)(1 + 2^q \exp(k - c\lambda^k)).$$

By a standard argument, this implies that for each k, $\xi < k \leqq L$, the following set of inequalities hold for all but at most $O(x e^{-\varepsilon k})$ integers n of A:

$$M_q(n_{k+1}) \leqq 2M_q(n_k) + e^{-(1-7\varepsilon)k} R_q(n_k), \quad 1 \leqq q \leqq k. \quad (13)$$

Summing the number of exceptional integers for $\xi < k \leqq L$, we obtain that (13) holds uniformly in k, $\xi < k \leqq K$, p.p.x.

We shall prove by induction on k, $\xi < k \leqq K$, that

$$M_q(n_k) \leqq 2^{\delta k} q!, \quad 1 \leqq q \leqq k, \quad \text{p.p.}x, \quad (14)$$

where δ is a constant such that

$$1 < \delta < (1-7\varepsilon)/\log 2.$$

To lighten the presentation, we put $e_3 := e^{(1-7\varepsilon)}$.

For $k = \xi + 1$, we have $M_q(n_k) = 2$ for every $q \geqq 1$ and every n in A, thus (14) is verified. Suppose it is still true for k. If $q \leqq k$, we can use (14) to estimate the right-hand side of (13). It becomes

$$M_q(n_{k+1}) \leqq 2^{\delta(k+1)}\, q!(2^{1-\delta}+k(2^\delta/e_3)^k) \leqq 2^{\delta(k+1)}\, q!$$

if ξ, and therefore k, is large enough.

If $q = k+1$, we can still use (14) to bound $R_q(n_k)$ which only contains $M_j(n_k)$ with $j \leqq q-1 = k$, but we need some extra information to estimate $M_q(n_k)$. We have

$$M_{k+1}(n_k) \leqq \Delta(n_k)\, M_k(n_k) \leqq M_k(n_k)+(3/e_2)\, M_k(n_k)^{(k+1)/k}$$

by Lemma 8 with $q = k$. Whence

$$\begin{aligned} M_{k+1}(n_k) &\leqq 2^{\delta(k+1)}(k+1)!\left(\frac{2^{-\delta}}{k+1}+\frac{3}{e_2}\frac{(k!)^{1/k}}{(k+1)}\right) \\ &\leqq 2^{\delta(k+1)}(k+1)!\left(\frac{3}{e{\cdot}e_2}+o(1)\right) \end{aligned}$$

by Stirling's formula. Eventually we obtain, by (13), that

$$\begin{aligned} M_{k+1}(n_{k+1}) &\leqq 2^{\delta(k+1)}(k+1)!\left(\frac{6}{e{\cdot}e_2}+o(1)+k(2^\delta/e_3)^k\right) \\ &\leqq 2^{\delta(k+1)}(k+1)! \end{aligned}$$

if ε is small enough and ξ large enough. This completes the proof of (14). In particular

$$M_K(n_K) \leqq 2^{\delta K}\, K!, \quad \text{p.p.}x,$$

whence by (5)

$$\Delta(n_K) \ll K, \quad \text{p.p.}x.$$

This completes the proof.

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Department of Mathematics
University of Georgia
Athens
Georgia 30602
USA

UER Sciences Mathématiques
Université de Nancy I
Boîte Postale 239
54506 Vandoeuvre les Nancy Cedex
France