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On the concentration of divisors of powers

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To János Pintz, on the occasion of his 75th birthday

Abstract. For integer $n \geqslant 1$ and real u, let $\Delta(n,u) := |\{d : d \mid n, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|$. The Erdős–Hooley Delta-function is then defined by $\Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u)$. For any fixed integer $r \geqslant 2$ and any irreducible polynomial $F \in \mathbb{Z}[X]$, we estimate the normal order of $\Delta(|F(n)|^r)$ to within factors that are slowly varying functions of $\log n$. This is then applied to determine with the same precision the normal order of $\Delta(|F(n)|)$ for any polynomial F with integer coefficients. We also evaluate some weighted average orders of $\Delta(n^r)$ and apply the result to bounding short sums of $\Delta(|F(n)|)$ for general $F \in \mathbb{Z}[X]$.

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1. Introduction and statement of results

For integer $n \geqslant 1$ and real u, put

$$\Delta(n,u) := |\{d : d \mid n, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \qquad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u).$$

The Δ-function was introduced by Erdős in 1973 [4] and was highlighted in 1979 by Hooley [7]. It turned out to be a key-concept in many branches of analytic number theory such as Waring type problems, circle method, Diophantine approximation, distribution of prime factors in polynomial sequences, etc.

A famous conjecture of Erdős, confirmed in 1984 by Maier & Tenenbaum [10] may be rephrased as

$$\Delta(n) > 1 \qquad \text{pp},$$

where, here and in the sequel, we use the mention pp to indicate that a formula holds on a sequence of natural density 1.

The study of the concentration of divisors received much attention in recent years. Since the works of Maier & Tenenbaum [11], [12], it is known that, for suitable positive constants γ_1, γ_2 we have

$$(\log_2 n)^{\gamma_1} < \Delta(n) \leqslant (\log_2 n)^{\gamma_2} \qquad \text{pp}, \tag{1·1}$$

where, here and throughout, we let $\log_k$ designate the k-fold iterated logarithm. The latest values are due to Ford, Green & Koukoulopoulos [5], who obtained the lower bound for any $\gamma_1 < 0.35332$, and to the authors [1], who established the upper bound for any $\gamma_2 > (\log 2)/(\log 2 + 1/\log 2 - 1) \approx 0.6102495$.

This work arises from exchanges of the second author with Lebowitz-Lockard [9]. We first investigate the normal behaviour of $\Delta\big(|F(n)|^r\big)$ when $r \geqslant 2$ and $F \in \mathbb{Z}[X]$ is an irreducible polynomial. We obtain the following estimate.

Theorem 1.1. *Let $r \geqslant 2$ and let $F \in \mathbb{Z}[X]$ be irreducible. Given any function $\xi(n)$ tending to infinity with n, we have*

$$\mathrm{e}^{-\xi(n)\sqrt{\log_2 n}} \leqslant \frac{\Delta\big(|F(n)|^r\big)}{(\log n)^{\log(r+1)-1}} \leqslant \mathrm{e}^{8\sqrt{(\log_3 n)\log_2 n}} \quad \text{pp}. \tag{1·2}$$

We did not try to optimise the constant 8 appearing in the above upper bound. The lower bound readily follows from the pigeonhole principle via the inequality

$$\Delta\big(m^r\big) \gg 1 + \tau\big(m^r\big)/\log m \quad (m > 1) \tag{1·3}$$

and a classical extension to irreducible polynomials of Hardy & Ramanujan's estimate stating that the normal order of the total number of distinct prime factors of an integer n is $\log_2 n + O\big(\xi(n)\sqrt{\log_2 n}\big)$ provided $\xi(n) \to \infty$: see (3·1) *infra.*

As it turns out, Theorem 1.1 furnishes, within the same precision, the normal order of $\Delta(|F(n)|)$ for *any* polynomial $F \in \mathbb{Z}[X]$.

Corollary 1.2. *Let $F = \prod_{1\leqslant j\leqslant s} F_j^{r_j} \in \mathbb{Z}[X]$, where the F_j are pairwise coprime irreducible elements of $\mathbb{Z}[X]$. Assume that $\max(r_1, s) \geqslant 2$ and let $\beta := \sum_{1\leqslant j\leqslant s} \log(r_j + 1)$. Given any function $\xi(n)$ tending to infinity with n, we have*

$$\mathrm{e}^{-\xi(n)\sqrt{\log_2 n}} \leqslant \frac{\Delta\big(|F(n)|\big)}{(\log n)^{\beta-1}} \leqslant \mathrm{e}^{\vartheta(n)\sqrt{\log_2 n}} \quad \text{pp} \tag{1·4}$$

with $\vartheta(n) := 8\sqrt{\log_3 n}$. If $s \geqslant 2$ and $r_1 = r_2$, we may take $\vartheta(n) := \xi(n)$.

Note that, here again, the lower bound follows from the pigeonhole principle.

When $s = r_1 = 1$, we state without proof that

$$(\log_2 n)^{\gamma_0} \leqslant \Delta(|F(n)|) \leqslant (\log_2 n)^{\gamma_2} \qquad \text{pp} \tag{1·5}$$

for any $\gamma_0 < (\log 2)/\log b \approx 0.33827$, with $b := (1-1/\log 27)/(1-1/\log 3)$ and γ_2 as in (1·1). This follows by extending to the case of polynomial arguments the respective approaches implemented in [12; th. 1.4] and [1; th. 1.3], appealing to the upper bounds established in [13]. It is not clear to us whether or not the method of [5] can also be extended in order to replace the exponent γ_0 by γ_1.

Taking (1·5) into account, we may state that the normal behaviour of $\Delta\big(|F(n)|\big)$ is known with an incertitude factor $(\log n)^{o(1)}$ for all polynomials.

Next, we investigate weighted averages of $\Delta(n^r)$. The set of weights is described as follows.

Given $z > 0$, we let $\mathcal{M}_z$ denote the class of those non-negative multiplicative functions ϱ that are bounded on the set of prime powers and satisfy, for suitable $c = c(\varrho) > 0$,

$$\sum_{p\leqslant y} \varrho(p)\log p = zy + O\Big(\frac{y}{(\log y)^c}\Big) \qquad (y \geqslant 2). \tag{1·6}$$

We then have

$$\sum_{P^+(n)<x} \frac{\mu(n)^2\varrho(n)}{n} = \prod_{p<x}\Big(1 + \frac{\varrho(p)}{p}\Big) \asymp (\log x)^z \qquad (x \geqslant 2).$$

Here and throughout, $P^+(n)$—resp. $P^-(n)$—denotes the largest—resp. the smallest—prime factor of $n > 1$ with the convention that $P^+(1) := 1$, $P^-(1) = \infty$. Moreover, μ is the Möbius function and the letter p denotes a prime number. A standard instance of an element of $\mathcal{M}_z$ is the function defined by $\varrho(n) := z^{\omega(n)}$, where, here and throughout, $\omega(n)$ stands for the number of distinct prime factors of the integer n.

We define the weighted sums

$$S_{r,\varrho}(x) := S_\varrho(x) = \sum_{n\leqslant x} \varrho(n)\Delta(n^r) \qquad (\varrho \in \mathcal{M}_z,\ x \geqslant 1), \tag{1·7}$$

and focus on the case $r \geqslant 2$, since the case $r = 1$ is dealt with in [2], [3]. We obtain the following result, where we put $\gamma_r(z) := z - 1 + (zr-1)^+$ and $\delta_r(z) := \delta_{rz,1}$ with Kronecker's notation.

Theorem 1.3. *Let $r \geqslant 1$, $z > 0$, and $\varrho \in \mathcal{M}_z$. For a suitable constant $\kappa = \kappa(r,z) > 0$ and uniformly for $x \geqslant 3$, we have*

$$x(\log x)^{\gamma_r(z)}(\log_2 x)^{\delta_r(z)} \ll S_{r,\varrho}(x) \ll x(\log x)^{\gamma_r(z)}\mathrm{e}^{\kappa\sqrt{\log_2 x}}. \tag{1·8}$$

We note that the above lower bound readily follows from (1·3) if $zr \neq 1$ and from (4·7) below if $zr = 1$ by an immediate extension of [6; (6.8)].[1] Thus only the upper bound will be considered in the proof. If $r = 1$, the exponential factor may be replaced by $(\log_2 x)^{2+\delta(1,z)/2}$ where $\delta(1,z) := \mathbf{1}_{z=1}$—see [2; th. 1.1] and [3; th. 1.1].

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[1] This rests on the fact that the left-hand side of (4·8) below is classically $\ll \Delta(n^r)$ for all $n \geqslant 1$.

Exploiting estimates of Nair & Tenenbaum [13], we deduce from the above a corollary concerning short sums of the Delta-function evaluated at polynomial arguments.

Corollary 1.4. *Let* $\alpha\in]0,1[$ *and* $F=\prod_{1\leqslant j\leqslant s}F_j^{r_j}\in\mathbb{Z}[X]$, *where the* F_j *are irreducible elements of* $\mathbb{Z}[X]$ *and* $\{r_j\}_{j=1}^s\in(\mathbb{N}^*)^s$ *is ordered increasingly. Put* $r:=\sum_{1\leqslant j\leqslant s}r_j$. *Assume furthermore that* F *has no fixed divisor. Then*

$$\sum_{x<n\leqslant x+y}\Delta\big(|F(n)|\big)\ll y(\log x)^{r-1}\mathcal{L}(x)\qquad\big(x\geqslant 3,\ x^\alpha<y\leqslant x\big),\tag{1·9}$$

where we have put

$$\mathcal{L}(x):=\begin{cases}(\log_2 x)^{5/2} & \text{if } r_1=1,\\ \mathrm{e}^{\kappa\sqrt{\log_2 x}} & \text{if } r_1\geqslant 2,\end{cases}\tag{1·10}$$

and $\kappa=\kappa(r_1,1)$ *is the constant appearing in the statement of Theorem 1.3.*

Note that in the case $r=s=1$, (1·9) coincides with the estimate given in [3; cor. 1.2].

2. Proof of Theorem 1.1

The starting point is as in [11] or [6; ch. 5]. Let $\{p_j(m)\}_{j=1}^{\omega(m)}$ denote the increasing sequence of distinct prime factors of an integer m. Given an irreducible polynomial $F\in\mathbb{Z}[X]$ and $Z:=Z(x)=\frac{7}{4}\sqrt{\log_2 x\log_3 x}$, we define

$$\begin{aligned}L=L(n)&:=\max\big\{k:1\leqslant k\leqslant\omega\big(|F(n)|\big),\,\log_2 p_k\big(|F(n)|\big)\leqslant 2Z\big\},\\ K=K(n)&:=\max\big\{k:1\leqslant k\leqslant\omega\big(|F(n)|\big),\,\log p_k\big(|F(n)|\big)\leqslant\mathrm{e}^{-Z}\log x\big\},\\ m_k&:=\prod_{L(n)<j\leqslant k}p_j(m)\quad\big(m=|F(n)|,\,1\leqslant k\leqslant K(n)\big).\end{aligned}$$

We first show that, to the aimed precision, it is sufficient to bound $\Delta\big(|F(n)|_K\big)$.

Here and throughout, we use the notation pp$x$ to indicate that a relation holds for $x+o(x)$ integers not exceeding x. We also introduce the number $\varrho_F(n)$ of roots of F in $\mathbb{Z}/n\mathbb{Z}$ and recall the prime ideal theorem in the form

$$\sum_{p\leqslant x}\varrho_F(p)=\mathrm{li}(x)+O\Big(x\mathrm{e}^{-c\sqrt{\log x}}\Big),\tag{2·1}$$

where $c=c(F)>0$—see for instance [16; lemma 2.1].

We finally define $\omega(m,t):=\sum_{p|m,\,p\leqslant t}1$ $(m\geqslant 1,t\geqslant 3)$.

Lemma 2.1. *With the notation defined above, we have*

$$L(n)>\tfrac{3}{2}Z,\qquad K(n)\leqslant\log_2 x\quad\text{pp}x,\tag{2·2}$$

$$\Delta\big(|F(n)|\big)\leqslant\Delta\big(|F(n)|_K\big)\mathrm{e}^{3Z}\qquad\text{pp}x.\tag{2·3}$$

Proof. Since the two inequalities in (2·2) may be proved similarly, let us focus solely on the second. Those integers $n\leqslant x$ contravening the stated upper bound for $K(n)$ must satisfy $\omega\big(|F(n)|,z\big)>\log_2 x$ with $\log z:=\mathrm{e}^{-Z}\log x$. However, by [13; cor. 2] and (2·1), we have, for $v:=1+1/\sqrt{\log_2 x}$,

$$\sum_{n\leqslant x}v^{\omega(|F(n)|,z)-\log_2 x}\ll x(\log z)^{v-1-\log v}\mathrm{e}^{-Z(v-1)}=o(x).$$

Let us now prove (2·3). By the inequality [6; lemma 61.1]

$$\Delta(mn)\leqslant\Delta(m)\tau(n)\qquad(m\geqslant 1,\,n\geqslant 1),\tag{2·4}$$

we have

$$\Delta\big(|F(n)|\big)\leqslant\Delta\big(|F(n)|_K\big)g\big(|F(n)|\big)\qquad(1\leqslant n\leqslant x)$$

where g is the multiplicative function defined by

$$g(p^\nu)=\begin{cases}\nu+1 & \text{if }\nu\geqslant 2,\text{ or }\log_2 p\leqslant Z,\text{ or }\log p>\mathrm{e}^{-Z}\log x,\\ 1 & \text{in all other cases.}\end{cases}$$

By [13; cor. 2] we have, appealing to (2·1),Z

$$\sum_{n\leqslant x} g\big(|F(n)|\big) \ll x\prod_{p\leqslant x}\Big(1-\frac{\varrho_F(p)}{p}\Big)\sum_{n\leqslant x}\frac{g(n)\varrho_F(n)}{n}$$
$$\ll \frac{x}{\log x}\exp\Big\{\sum_{p\leqslant x}\sum_{\nu\geqslant 1}\frac{\varrho_F(p^\nu)g(p^\nu)}{p^\nu}\Big\}\ll x\mathrm{e}^{2Z}.$$

This plainly implies (2·3). □

Consider the set $\mathcal{M}(x)$ comprising all integers $m\leqslant x$ satisfying the conditions

($\mathcal{M}1$) $\mu(m)^2=1,\ P^+(m)\leqslant x^{\exp(-Z)}$

($\mathcal{M}2$) $\log_2 m\leqslant \log_2 x-\frac12 Z$

($\mathcal{M}3$) $\log_2 t-Z-1<\omega(m,t)\leqslant \log_2 t+Z+1\quad (2Z<\log_2 t\leqslant \log_2 x-Z).$

Lemma 2.2. *Put* $\mathcal{N}(x):=\{n\leqslant x:\prod_{1\leqslant j\leqslant K(n)}p_j\big(|F(n)|\big)\in\mathcal{M}(x)\}$. *Then* $n\in\mathcal{N}(x)$ ppx.

Proof. Condition ($\mathcal{M}1$) is satisfied by construction and condition ($\mathcal{M}2$) readily follows by [16; lemma 3.7]. To prove ($\mathcal{M}3$), consider first those $n\leqslant x$ such that $\omega\big(|F(n)|,t\big)>\log_2 t+Z+1$ for some t in the specified range. Since $t\mapsto\omega(m,t)$ in non-decreasing as a function of t, we must have $\omega\big(|F(n)|,t_h\big)>h+Z$ with $t_h:=\exp\exp h$ and some integer $h\in]2Z,\log_2 x-Z]$. Given any $v\geqslant 1$, the number of these integers n does not exceed

$$\sum_{n\leqslant x} v^{\omega(|F(n)|,t_h)-h-Z}\ll xv^{-h-Z}\exp\Big\{\sum_{p\leqslant t_h}\frac{(v-1)\varrho_F(p)}{p}\Big\}$$
$$\ll x\mathrm{e}^{h\{v-1-v\log v\}}\ll x\mathrm{e}^{-Z^2/3h},$$

on selecting $v:=1+Z/h$ and noting that $v\log v-v+1\geqslant (v-1)^2/3$ for $1\leqslant v\leqslant 3/2$. Summing over $h\in]2Z,\log_2 x-Z]$ yields that the number of these exceptional n is $o(x)$. A similar computation enables us to take of those n such that $\omega\big(|F(n)|,t\big)\leqslant\log_2 t-Z-1$ for some relevant t. □

For the sake of further reference, we note that for $n\in\mathcal{N}(x)$, we have

$$\mathrm{e}^{k-Z-1}\leqslant \log p_k\big(|F(n)|\big)\leqslant \mathrm{e}^{k+Z+1}\quad (L(n)\leqslant k\leqslant K(n)).\tag{2·5}$$

We classically bound $\Delta\big(|F(n)|_K\big)$ by estimating moments

$$M_q(m_k):=\int_{\mathbb{R}}\Delta(m_k,u)^q\,\mathrm{d}u\quad (m=|F(n)|,\ L(n)<k\leqslant K(n),\ q\geqslant 1),$$

via an induction on k. From the formula

$$\Delta(\ell^r p^r,u)=\sum_{0\leqslant j\leqslant r}\Delta(\ell^r,u-j\log p)\qquad (p\nmid\ell)\tag{2·6}$$

we get

$$M_q\big(m_{k+1}^r\big)=(r+1)M_q\big(m_k^r\big)+R_q(m_{k+1})\qquad (L<k<K),\tag{2·7}$$

where

$$R_q(m_{k+1}):=\sum_{\substack{j_0+\dots+j_r=q\\ \max(j_h)<q}}\binom{q}{j_0,\dots,j_r}\int_{\mathbb{R}}\prod_{0\leqslant h\leqslant r}\Delta\big(m_k^r,u-h\log p_{k+1}(m)\big)^{j_h}\,\mathrm{d}u.\tag{2·8}$$

Put

$$S_k(x,\ell):=\sum_{\substack{n\in\mathcal{N}(x)\\ |F(n)|_k=\ell}}1\qquad \Big(x\geqslant 3,\ \tfrac32 Z<k\leqslant\log_2 x,\ \ell\geqslant 1\Big).$$

Note that by ($\mathcal{M}3$) and (2·2) this sum is empty unless $k-3Z-1\leqslant\omega(\ell)\leqslant k-\frac32 Z$.

Lemma 2.3. *For $k \leqslant \log_2 x$, $\omega(\ell) \leqslant k$, $p > P^+(\ell)$, we have*

$$S_{k+1}(x, p\ell) \ll \frac{x\mathrm{e}^Z \varrho_F(p\ell)}{\varphi_F(\ell) p \log p}\cdot \tag{2·9}$$

Proof. If n is counted in (2·9), then $F(n) \equiv 0 \,(\mathrm{mod}\, \ell p)$ and any prime divisor of $F(n)/\ell p$ is either $\leqslant \exp\exp Z$ or $> p$. The summation may hence be handled by sieve methods. Let D_F denote the discriminant of F and write $D := D_F|F(1)|$. Consider the multiplicative function φ_F defined by $\varphi_F(p^\nu) := p^\nu$ if $p|D$, and $\varphi_F(p^\nu) := p^\nu(1 - \varrho_F(p)/p)$ otherwise. Then [15; lemma 3.4] furnishes

$$S_{k+1}(x, p\ell) \ll \frac{x\varrho_F(\ell p)}{\varphi_F(\ell p)\log p} \sum_{P^+(d) \leqslant \exp \mathrm{e}^Z} \frac{\varrho_F(d)}{\varphi_F(d)}\cdot$$

Noting that $\varphi_F(\ell p) = \varphi_F(\ell)\varphi_F(p) \gg \varphi_F(\ell)p$ and estimating the latter sum furnishes the stated bound. □

We now state the induction inequality at the root of the method.

Lemma 2.4. *Simultaneously for $L(n) < k \leqslant K(n)$, $1 \leqslant q \leqslant k$, we have*

$$M_q\big(m_{k+1}^r\big) \leqslant (r+1)M_q\big(m_k^r\big) + \mathrm{e}^{3Z/2-k}W_q(m_k) \quad (m := |F(n)|) \quad \mathrm{pp}x, \tag{2·10}$$

with

$$W_q(m_k) := \sum_{1 \leqslant j < q} \binom{q}{j} r^j M_j\big(m_k^r\big)M_{q-j}\big(m_k^r\big). \tag{2·11}$$

Proof. This is similar to [6; (5.69)]. The starting point is the bound

$$T_k(\ell) := \sum_{\substack{n \in \mathcal{N}(x) \\ |F(n)|_k = \ell \\ k < K(n)}} R_q(m_{k+1}) \ll \frac{x\mathrm{e}^Z \varrho_F(\ell)W_q(\ell)}{\varphi_F(\ell)\{\log P^+(\ell)\}^2}\Big\{1 + \frac{(2r+2)^q}{\mathrm{e}^{c\sqrt{\log P^+(\ell)}}}\Big\}, \tag{2·12}$$

where $c = c(F) > 0$.

To prove (2·12), write $m_{k+1} = \ell p$ in the left-hand side and appeal to (2·9). This yields

$$T_k(\ell) \ll \frac{x\varrho_F(\ell)\mathrm{e}^Z}{\varphi_F(\ell)\{\log P^+(\ell)\}^2} \sum_{\substack{j_0 + \cdots + j_r = q \\ \max(j_h) < q}} \binom{q}{j_0, \ldots, j_r} \sum_{P^+(\ell) < p \leqslant x} \frac{U(\ell, p; \boldsymbol{j})\varrho_F(p)\log p}{p} \tag{2·13}$$

with

$$U(\ell, p; \boldsymbol{j}) := \int_{\mathbb{R}} \prod_{0 \leqslant h \leqslant r} \Delta\big(\ell^r, u - h\log p\big)^{j_h}\,\mathrm{d}u \qquad \big(\boldsymbol{j} = (j_0, \ldots, j_r)\big). \tag{2·14}$$

To evaluate the inner sum of (2·13), we expand the product in (2·14) as a multiple sum over divisors of ℓ^r, invert integration and summation, and apply (2·1). The summation range for p is included in an intersection of intervals of logarithmic lengths less than 1. Put

$$M_j^*(n) := \sideset{}{^*}\sum_{d_1, \ldots, d_j | n} 1 \leqslant 2^j M_j(n) \qquad (n \geqslant 1) \tag{2·15}$$

where the star indicates that summation is restricted to j-tuples of divisors such that $\log(\max d_h / \min d_h) \leqslant 1$ and the inequality is [6; (5.67)]. Since there are at most

$$M(\ell^r; \boldsymbol{j}) := \prod_{0 \leqslant h \leqslant r} M_{j_h}^*(\ell^r)$$

limit points in the summation over p, we get, by partial summation using (2·1),

$$\sum_{P^+(\ell)<p\leqslant x}\frac{U(\ell,p;\boldsymbol{j})\varrho_F(p)\log p}{p}=\int_{\mathbb{R}}\int_{\mathbb{R}}\prod_{0\leqslant h\leqslant r}\Delta\big(\ell^r,u-hv\big)^{j_h}\,\mathrm{d}v\,\mathrm{d}u+O\bigg(\frac{M(\ell^r;\boldsymbol{j})}{\mathrm{e}^{c\sqrt{\log P^+(\ell)}}}\bigg).$$

Up to a change of variables $u\mapsto u-sv$ for suitable $s\geqslant 0$, we may assume $j_0\geqslant 1$ and h runs in an set of $r+1$ consecutive integers.

Let $j:=\sum_{h\neq 0}j_h=q-j_0$. By Hölder's inequality and (2·15), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\prod_h\Delta\big(\ell^r,u-hv\big)^{j_h}\,\mathrm{d}v\,\mathrm{d}u\leqslant M_{q-j}(\ell^r)M_j(\ell^r), \tag{2·16}$$

$$M(\ell^r;\boldsymbol{j})\leqslant 2^qM_{q-j}(\ell^r)M_j(\ell^r). \tag{2·17}$$

Summing over $\boldsymbol{j}$, we obtain (2·12).

From (2·12), we deduce that, for $n\in\mathbb{N}$, $|F(n)|=m$, $L(n)<k\leqslant K(n)$, $\ell\geqslant 1$, $1\leqslant q\leqslant k$, we have

$$\sum_{\substack{n\in\mathcal{N}(x)\\ |F(n)|_k=\ell}}\frac{\{M_q(m_{k+1}^r)-(r+1)M_q(m_k^r)\}^+}{W_q(m_k)}\ll\frac{x\varrho_F(\ell)\mathrm{e}^{2Z-2k}}{\varphi_F(\ell)}\Big(1+\frac{(2r+2)^q}{\exp(3/2)^k}\Big),$$

with the standard notation $z^+:=\max(z,0)$ $(z\in\mathbb{R})$, and where we used the inequality $\log_2P^+(\ell)\geqslant k-Z-1$ arising from (2·5). Summing this over ℓ taking into account that

$$\sum\frac{\varrho_F(\ell)}{\varphi_F(\ell)}\leqslant\prod_{2Z<\log_2p\leqslant k+Z}\Big(1+\frac{\varrho_F(p)}{p-\varrho_F(p)}\Big)\ll\mathrm{e}^{k-Z},$$

we get

$$\sum_{n\in\mathcal{N}(x)}\frac{\{M_q(m_{k+1}^r)-(r+1)M_q(m_k^r)\}^+}{W_q(m_k)}\ll x\mathrm{e}^{Z-k}.$$

We can hence infer that, for each $k\in[\frac{3}{2}Z,\log_2x]$ the following set of inequalities holds simultaneously for all but at most $\ll x\mathrm{e}^{-Z/2}$ integers $n\in\mathcal{N}(x)$:

$$M_q(m_{k+1}^r)\leqslant(r+1)M_q(m_k^r)+\mathrm{e}^{3Z/2-k}W_q(m_k)\qquad(1\leqslant q\leqslant k).$$

After consideration of all the exceptions for different k in the admissible range, we get (2·10) as desired. □

We are now in a position to complete the proof of Theorem 1.1. Let us show by induction on k that for $1\leqslant q\leqslant k$ and $n\in\mathcal{N}(x)$, we have

$$M_q(m_k^r)\leqslant\frac{(r+1)^{q^2+qk}}{\mathrm{e}^{(k-3Z/2)(q-1)}}\qquad\big(m=|F(n)|,\ \max(L,q)\leqslant k\leqslant K\big). \tag{2·18}$$

The initial step is provided by selecting $k=L(n)$ since $m_L=1$, and hence $M_q(m_L)=1$ for all q. Note moreover that (2·18) holds trivially when $q=1$ since $M_1(m_k^r)\leqslant(r+1)^k$. Next, let $q\geqslant 2$ and assume that (2·18) is satisfied for k. For $\max(q,L)\leqslant k\leqslant K$, we have

$$M_j\big(m_k^r\big)M_{q-j}\big(m_k^r\big)\leqslant\frac{(r+1)^{j^2+(q-j)^2+qk}}{\mathrm{e}^{(k-3Z/2)(q-2)}}\qquad(1\leqslant j\leqslant q-1).$$

Carrying back into (2·11), we get

$$W_q(m_k)\leqslant\sum_{1\leqslant j<q}\binom{q}{j}r^j\frac{(r+1)^{j^2+(q-j)^2+qk}}{\mathrm{e}^{(k-3Z/2)(q-2)}}\qquad\big(\max(q,L)\leqslant k\leqslant K\big).$$

Since $j^2+(q-j)^2\leqslant q^2-2(q-1)$ for $1\leqslant j\leqslant q-1$, we obtain, for any $k\in[\max(q,L),K]$,

$$W_q(m_k)\leqslant\frac{(r+1)^{q^2-(q-2)+qk}}{e^{(k-3Z/2)(q-2)}}\leqslant\frac{(r+1)^{q^2+qk}}{e^{(k-3Z/2)(q-2)}}.$$

From (2·10) and (2·18), we deduce that, for $\max(q, L) \leqslant k \leqslant K$,

$$\begin{aligned} M_q\big(m_{k+1}^r\big) &\leqslant (r+1)M_q\big(m_k^r\big) + \frac{W_q(m_k)}{\mathrm{e}^{k-3Z/2}} \\ &\leqslant \frac{(r+1)^{q^2+qk+1}}{\mathrm{e}^{(k-3Z/2)(q-1)}}\Big(1+\frac{1}{r+1}\Big) \leqslant \frac{(r+1)^{q^2+q(k+1)}}{\mathrm{e}^{(k-3Z/2)(q-1)}} \end{aligned}$$

since $(r+1)^{q-1} \geqslant r+1 \geqslant 1+1/(r+1)$. The required bound hence holds at rank $k+1$ provided $q \leqslant k$.

If $q = k+1$, we can still appeal to the induction hypothesis to bound $W_q(m_k)$ because it only involves moments $M_j(m_k^r)$ with $j \leqslant k$, but we need some extra information to bound $M_{k+1}(m_k^r)$. This is provided by the inequality

$$M_{k+1}(m_k^r) \leqslant M_k(m_k^r)\Delta(m_k^r) \leqslant 2M_k(m_k^r)^{1+1/k},$$

following from [6; (5.56)]. This leads to

$$\begin{aligned} M_{k+1}(m_{k+1}^r) &\leqslant \frac{2(r+1)^{k(2k+1)+1}}{\mathrm{e}^{q(k-3Z/2)}} + \frac{(r+1)^{q(q+k)}}{\mathrm{e}^{(q-1)(k-3Z/2)}} \\ &= \frac{(r+1)^{q^2+(k+1)q}(4C)^{q-1}}{\mathrm{e}^{(q-1)(k+1-3Z/2)}}\Big\{\frac{2\mathrm{e}^{k+q-3Z/2}}{(r+1)^{q+2k}} + \frac{\mathrm{e}^{k-3Z/2}}{(r+1)^q}\Big\}, \end{aligned}$$

which implies the required bound.

Since $\Delta(m_K^r) \leqslant 2M_q(m_K^r)^{1/q}$ for $q \geqslant 1$ by [6; (5.56)] and $K = K(n) \leqslant \log_2 x$ pp$x$, we may select $k = K(n)$ and $q = \big\lfloor\sqrt{Z}\big\rfloor$ in (2·18) to get

$$\Delta(m_K^r) \leqslant 2M_q(m_K^r)^{1/q} \ll \frac{(r+1)^{q+k}\mathrm{e}^{k/q}}{\mathrm{e}^{k-3Z/2}} \ll (\log x)^{\log(r+1)-1}\,\mathrm{e}^{cZ} \qquad \mathrm{pp}x$$

provided $c > \frac{3}{2}$. The upper bound stated in (1·2) then follows from (2·3) since $(3+\frac{3}{2})\frac{7}{4} < 8$.

3. Proof of Corollary 1.2

For any irreducible polynomial F and any positive integer r, we have, by a simple sieve argument,

$$1 \leqslant \frac{\tau\big(|F(n)|^r\big)}{(r+1)^{\omega(|F(n)|)}} \leqslant \xi(n) \qquad \mathrm{pp.}$$

However the Hardy-Ramanujan estimate for $\omega(n)$ readily extends to $\omega\big(|F(n)|\big)$ on bounding, for example by [13; cor. 3], sums $\sum_{n\leqslant x} v^{\omega(|F(n)|)}$ with $v = 1 \pm \xi(x)/\sqrt{\log_2 x}$. Thus

$$\mathrm{e}^{-\xi(n)\sqrt{\log_2 n}} \leqslant \frac{\tau\big(|F(n)|^r\big)}{(\log n)^{\log(r+1)}} \leqslant \mathrm{e}^{\xi(n)\sqrt{\log_2 n}} \quad \mathrm{pp.} \tag{3·1}$$

Applying this for $F = F_j$, $r = r_j$ $(1 \leqslant j \leqslant s)$ furnishes the lower bound of (1·4) by (1·3).

To establish the upper bound, we appeal to the inequality (2·4) in the form

$$\Delta\big(|F(n)|\big) \leqslant \Delta\big(|F_1(n)|^{r_1}\big) \prod_{2\leqslant j\leqslant s} \tau\big(|F_j(n)|^{r_j}\big) \qquad (n \geqslant 1).$$

If $r_1 \geqslant 2$, we may invoke Theorem 1.1 to bound the normal order of the first factor on the right. The required estimate then follows by applying (3·1) to the other factors.

If $s \geqslant 2$ and $r_1 = r_2 = r$, let $\varrho_j(m)$ $(j = 1, 2)$ denote the number of roots of $F_j(n)$ in $\mathbb{Z}/m\mathbb{Z}$. A direct application of [13; cor. 2] furnishes

$$\sum_{n\leqslant x} \frac{\Delta\big(|F_1(n)|^r|F_2(n)|^r\big)}{(r+1)^{\omega(|F_1(n)F_2(n)|)}} \ll \frac{xV}{(\log x)^2} \tag{3·2}$$

where

$$V := \sum_{n_1n_2\leqslant x} \frac{\Delta\big(n_1^rn_2^r\big)\varrho_1(n_1)\varrho_2(n_2)}{(r+1)^{\omega(n_1n_2)}n_1n_2} = \sum_{n\leqslant x} \frac{\Delta(n^r)\varrho(n)}{n},$$

with $\varrho(n) := \varrho_1 * \varrho_2(n)/(r+1)^{\omega(n)}$.

Since, classically, $\varrho_j \in \mathcal{M}_1$ $(j = 1, 2)$ and hence $\varrho \in \mathcal{M}_{2/(r+1)}$, we have

$$V \ll \mathrm{e}^{\kappa\sqrt{\log_2 x}} \log x \tag{3·3}$$

by Theorem 1.3 if $r \geqslant 2$ and by [3; th. 1.1] if $r = 1$—in which case the exponential factor could actually be replaced by $(\log_2 x)^3$. Since, by (3·1), we have

$$(r+1)^{\omega(|F_1(n)F_2(n)|)} \leqslant (\log n)^{2\log(r+1)}\mathrm{e}^{\xi(x)\sqrt{\log_2 x}} \quad \mathrm{pp}x,$$

we deduce from (3·2) and (3·3) that

$$\Delta\big(|F_1(n)|^r|F_2(n)|^r\big) \leqslant (\log x)^{2\log(r+1)-1}\mathrm{e}^{\xi(x)\sqrt{\log_2 x}} \qquad \mathrm{pp}x.$$

Applying (2·4) in the form

$$\Delta\big(|F(n)|\big) \leqslant \Delta\big(|F_1(n)|^r|F_2(n)|^r\big)\prod_{2<j\leqslant s}\tau\big(|F_j(n)|^{r_j}\big)$$

and invoking (3·1) to bound the last product furnishes the stated upper bound in the case under consideration.

4. Proof of Theorem 1.3

As indicated in the introduction, we focus solely on the upper bound of (1·8). We follow, in the setting displayed in [3], the method introduced by Koukoulopoulos & Tao in [8].

Define

$$E := \{n \geqslant 1 : \mu(n)^2 = 1\}, \quad n_y := \prod_{p|n,\, p<y} p \quad (n \geqslant 1,\, y \geqslant 1), \tag{4·1}$$

$$E_T := \big\{n \in E : \tau(n_y^r) \leqslant T\log 3y \ (y \geqslant 1)\big\}. \tag{4·2}$$

It is useful to bear in mind that, if $n \in E_T$ then any divisor d of n also belongs to E_T.

Given $x \geqslant 2$, we consider the probability $\mathbb{P}_{x,\varrho}$ defined on E by

$$\mathbb{P}_{x,\varrho}(\{n\}) := \frac{\varrho(n)}{n}\prod_{2\leqslant p<x}\Big(1+\frac{\varrho(p)}{p}\Big)^{-1} \asymp \frac{\varrho(n)}{n(\log x)^z} \qquad \big(P^+(n) < x\big),$$

and let $\mathbb{E}_{x,\varrho}$ denote the corresponding expectation.

Throughout this section, the symbol $\sum^x$ indicates a summation over squarefree integers whose prime factors are restricted to the interval $[2, x[$. For the sake of future reference, we note that, if $\varrho \in \mathcal{M}_z$, then

$$\sum_{n\geqslant 1}^{x} \frac{\mu(n)^2\varrho(n)}{n} \asymp (\log x)^z \qquad (x \geqslant 2). \tag{4·3}$$

Lemma 4.1. *Let $r \geqslant 1$, $z > 0$ and $\varrho \in \mathcal{M}_z$. We have*

$$S_{r,\varrho}(x) \asymp x(\log 3x)^{z-1}\,\mathbb{E}_{x,\varrho}(\Delta(n^r)) \qquad (x \geqslant 1). \tag{4·4}$$

Proof. This is a trivial extension of [6; th. 61]. We omit the details. □

In view of the above estimate and previous remarks, we note that, in the case $zr \geqslant 1$, Theorem 1.3 is an immediate consequence of the following statement. We shall see that the case $zr < 1$ will follow in an easy way.

Proposition 4.2. *For $r \geqslant 1$, $z > 0$, $\varrho \in \mathcal{M}_z$, suitable $\kappa = \kappa(r, z) > 0$, and uniformly for $x \geqslant 3$, we have*

$$\mathbb{E}_{x,\varrho}\big(\Delta(n^r)\big) \ll (\log x)^{(zr-1)^+}\mathrm{e}^{\kappa\sqrt{\log_2 x}}. \tag{4·5}$$

As in [8], [2] and [3], the first step of the proof consists in defining a set of integers with useful multiplicative constraints.

The next lemma is analogous to [8; prop. 4.1] and [2; lemma 3.2]. The essential feature consists in bounding $\mathbb{P}_{x,\varrho}(E \smallsetminus E_T)$ by a multiple of $1/T$. The fact that the resulting estimate is trivial for small T will have no consequence.

Lemma 4.3. *Let* $r \geqslant 1$, $z \geqslant 1/r$, *and* $\varrho \in \mathcal{M}_z$. *We have*

$$\mathbb{P}_{x,\varrho}\big(E \smallsetminus E_T\big) \ll \frac{(\log x)^{rz-1}}{T\sqrt{\log_2 x}} \qquad (x \geqslant 2,\ T \geqslant 2). \tag{4·6}$$

Proof. This may be established by following closely the proof of [3; lemma 3.2]. We omit the details. □

Lemma 4.4. *Let* $r \geqslant 1$, $z \geqslant 1/r$, *and* $\varrho \in \mathcal{M}_z$. *We have*

$$\mathbb{E}_{x,\varrho}\big(M_2(n^r)/\tau(n^r)\big) \asymp (\log x)^{rz-1}(\log_2 x)^{\delta_r(z)} \qquad (x \geqslant 3). \tag{4·7}$$

Proof. Put $\tau(n,\vartheta) := \sum_{d\mid n} d^{i\vartheta}$ $(n \geqslant 1, \vartheta \in \mathbb{R})$. By Parseval's formula (see [6; (6.23)]) we have

$$\frac{M_2(n^r)}{\tau(n^r)} \asymp \int_0^1 \frac{|\tau(n^r,\vartheta)|^2}{\tau(n^r)}\,\mathrm{d}\vartheta. \tag{4·8}$$

For any prime number p, we have

$$\frac{|\tau(p^r,\vartheta)|^2}{\tau(p^r)} = 1 + 2\sum_{1\leqslant j\leqslant r}\Big(1 - \frac{j}{r+1}\Big)\cos(j\vartheta\log p).$$

Splitting the integral at $\vartheta = 1/\log x$, a standard manipulation (see, e.g., [6; lemma 30.3]) furnishes

$$\mathbb{E}_{x,\varrho}\Big(\frac{M_2(n^r)}{\tau(n^r)}\Big) \asymp (\log x)^{rz}\int_0^{1/\log x}\mathrm{d}\vartheta + \int_{1/\log x}^1 \frac{\mathrm{d}\vartheta}{\vartheta^{rz}} \asymp (\log x)^{rz-1}(\log_2 x)^{\delta_r(z)}.$$ □

Let $\{\vartheta_{j,T}\}_{j=0}^{\infty} \in (\mathbb{R}^+)^{\mathbb{N}}$ denote a sequence at our disposal. Define further

$$H_T^q := \big\{n \in E_T : M_j(n^r) \leqslant \tau(n^r)\vartheta_{j,T}\ (1 \leqslant j \leqslant q)\big\}. \tag{4·9}$$

Note that $M_j(n^r)/\tau(n^r)$ is multiplicatively increasing, so $n \in H_T^q$ implies that any divisor of n also lies in H_T^q. The quantities $\vartheta_{j,T}$ will be formally defined later—see (4·19). For now, we only specify that

$$\vartheta_{0,T} = \vartheta_{1,T} = 1, \qquad \vartheta_{2,T} \asymp T\log T, \tag{4·10}$$

so that $H_T^0 = H_T^1 = E_T$ and $H_T^q \subset H_T^{q-1}$.

Recall notation $\sum^x$ and put

$$\begin{aligned}\mathcal{T}_z(x;q) &:= \mathbb{P}_{x,\varrho}(H_T^{q-1})\mathbb{E}_{x,\varrho}\Big(\frac{M_q(n^r)}{\tau(n^r)}\Big|H_T^{q-1}\Big)\\ &= \prod_{p<x}\Big(\frac{1}{1+\varrho(p)/p}\Big)\sideset{}{^x}\sum_{n\in H_T^{q-1}}\frac{\varrho(n)M_q(n^r)}{n\tau(n^r)}.\end{aligned} \tag{4·11}$$

Proposition 4.5. *Let* $r \geqslant 1$, $z \geqslant 1/r$, $\varrho \in \mathcal{M}_z$, *and let* C_0 *be a sufficiently large absolute constant. Recall that* $\vartheta_{1,T} = 1$, *and assume*

$$\vartheta_{q-1,T}(2r+2)^{2q} \leqslant \frac{\vartheta_{q,T}}{C_0 q^2 T \log T} \qquad (q \geqslant 2). \tag{4·12}$$

Then

$$\mathcal{T}_z(x;q) \leqslant \frac{C_0 \vartheta_{q,T}(\log x)^{rz-1}}{q^2 T} \qquad \big(q \geqslant 3,\ x \geqslant 2,\ T \geqslant 3\big). \tag{4·13}$$

Proof. Let $q \geqslant 3$. Applying (2·6) as we did for (2·7), we get

$$\frac{M_q\big(m^r p^r\big)}{\tau(m^r p^r)} = \frac{M_q\big(m^r\big)}{\tau(m^r)} + \frac{w_q(m,p)}{(r+1)\tau(m^r)} \qquad (p \nmid m), \tag{4·14}$$

where

$$w_q(m,p) := \sum_{\substack{j_0+\ldots+j_r=q\\ \max(j_h)<q}} \binom{q}{j_0,\ldots,j_r} \int_{\mathbb{R}} \prod_{0\leqslant h\leqslant r} \Delta\big(m^r, u - h\log p_{k+1}(n)\big)^{j_h}\,\mathrm{d}u. \tag{4·15}$$

Now [3; lemma 5.1] yields

$$\mathcal{T}_z(x;q) \ll 1 + \int_1^{x^2} \sum_{m\in H_T^{q-1}}^{y} \frac{\varrho(m)W_q(m,y)}{m\tau(m^r)} \frac{\mathrm{d}y}{y(\log 3y)^{1+z}}, \tag{4·16}$$

with

$$W_q(m,y) := \sum_{\sqrt{y}\leqslant p<y} \frac{\varrho(p)w_q(m,p)}{p}.$$

By partial integration and appeal to the Brun-Titchmarsh inequality as in [8; (6.9)] or [3; (8.5)], we get, for all m satisfying $P^+(m) < y$,

$$W_q(m,y) \ll \frac{1+(2r+2)^q(\log 3y)/y^{1/4}}{\log y} \sum_{\substack{j_0+\ldots+j_r=q\\ \max(j_h)<q}} \binom{q}{j_0,\ldots,j_r} I(m^r,\boldsymbol{j}), \tag{4·17}$$

where

$$I(m,\boldsymbol{j}) := \int_{\mathbb{R}}\int_{\mathbb{R}} \prod_{0\leqslant h\leqslant r} \Delta\big(m, u-hv\big)^{j_h}\,\mathrm{d}u\,\mathrm{d}v \quad \big(m \geqslant 1,\ \boldsymbol{j} = (j_0,\ldots,j_r)\big).$$

Up to a change of variables, we may assume that $j := j_0 \geqslant 1$. Now, (2·16) furnishes

$$I(m^r,\boldsymbol{j}) \leqslant M_j(m^r)M_{q-j}(m^r) \leqslant \tau(m^r)M_{q-1}(m^r).$$

Carrying back into (4·17), we get, for $m \in H_T^{q-1}$,

$$\begin{aligned} W_q(m,y) &\ll \Big(1+(2r+2)^q\frac{\log 3y}{y^{1/4}}\Big)(r+1)^q\frac{\tau(m^r)M_{q-1}(m^r)}{\log y}\\ &\ll (2r+2)^{2q}\frac{\tau(m^r)M_{q-1}(m^r)}{\log y} \leqslant (2r+2)^{2q}\vartheta_{q-1,T}\frac{\tau(m^r)^2}{\log y}. \end{aligned} \tag{4·18}$$

Inserting back into (4·16), we get, if $rz > 1$,

$$\mathcal{T}_z(x;q) \leqslant C_0(2r+2)^{2q}\vartheta_{q-1,T}(\log x)^{rz-1}.$$

In the case when $rz = 1$, we exploit (4·18) in the form that, for $m \in H_T^{q-1}$, $P^+(m) < y$, we have

$$W_q(m, y) \ll (2r+2)^{2q}\vartheta_{q-1,T}\frac{\tau(m^r)^{3/2}\sqrt{\min\{\tau(m^r), T\log y\}}}{\log y}.$$

Splitting the integral in (4·16) at $y = Y \geqslant 2$, we derive

$$\mathcal{T}_z(x;q) \ll (2r+2)^{2q}\vartheta_{q-1,T}\big(\log_2 Y + T^{1/2}(\log Y)^{-c_r}\big),$$

with $c_r := \big(\frac{1}{2}r + 1 - \sqrt{r+1}\big)/r > 0$. Selecting $\log Y = T^{1/2c_r}$ yields

$$\mathcal{T}_z(x;q) \ll (2r+2)^{2q}\vartheta_{q-1,T}x\log T.$$

In view of hypothesis (4·12), this implies (4·13) and hence completes the proof. □

We may now embark on majorising $\mathbb{E}_{x,\varrho}(\Delta(n^r))$.

First assume $z \geqslant 1/r$. Observe that the sequence defined by

$$\vartheta_{q,T} = C_0^{q-1}(2r+2)^{2q^2}(q!)^2(T\log T)^{q-1} \quad (q \geqslant 2) \tag{4·19}$$

satisfies (4·12).

The contribution of those n such that $\Delta(n^r) > (\log x)^{zr(r+1)+1}$ is easily bounded by

$$\ll \mathbb{E}_{x,\varrho}\big(\tau(n^r)^2\big)(\log x)^{-zr(r+1)-1} \ll (\log x)^{zr-1}.$$

Let $E' := \{n \in E : \Delta(n^r) \leqslant (\log x)^{zr(r+1)+1}\}$. Assuming momentarily that, for $A := 2\log(2r+2) + 1$ and suitable $C \geqslant 1$, we have

$$\mathbb{P}_{x,\varrho}\Big(n \in E' : \Delta(n^r) > CT\mathrm{e}^{A\sqrt{\log_2 x}}\Big) \ll \frac{(\log x)^{zr-1}\log_2 x}{T}, \tag{4·20}$$

we see by summing over dyadic intervals that

$$\mathbb{E}_{x,\varrho}(\Delta(n^r)) \ll (\log x)^{zr-1}\mathrm{e}^{A\sqrt{\log_2 x}}(\log_2 x)^2,$$

hereby confirming (1·8).

Let us now prove (4·20). Put $Q := \lfloor\sqrt{\log x}\rfloor$. By (4·13) for $q \geqslant 3$ and (4·7) for $q = 2$, we have

$$\mathbb{P}_{x,\varrho}\Big(n \in H_{q-1}^T \smallsetminus H_q^T\Big) \ll \frac{(\log x)^{zr-1}\log_2 x}{q^2 T} \qquad (2 \leqslant q \leqslant Q).$$

Summing over $q \leqslant Q$ furnishes

$$\mathbb{P}_{x,\varrho}\Big(n \in E \smallsetminus H_Q^T\Big) \ll \frac{(\log x)^{zr-1}\log_2 x}{T}.$$

However, for $n \in H_Q^T \cap E'$, we have

$$\Delta(n^r) \leqslant 2M_Q(n^r)^{1/Q} \leqslant 2\big\{\vartheta_{Q,T}\tau(n^r)\big\}^{1/Q} \ll \vartheta_{Q,T}^{1/Q}(T\log x)^{1/Q} \ll T\mathrm{e}^{A\sqrt{\log x}}.$$

This completes the proof of (4·5) in the case $z \geqslant 1/r$.

Assume now that $0 < z < 1/r$. Put $\varrho_1(n) := (1/r - z)^{\omega(n)}$. We have

$$\begin{aligned}(\log x)^{1/r-z}\sum_{m\geqslant 1}^{x}\frac{\varrho(m)\Delta(m^r)}{m} &\ll \sum_{d\geqslant 1}^{x}\frac{\varrho_1(d)}{d}\sum_{m\geqslant 1}^{x}\frac{\varrho(m)\Delta(m^r)}{m}\\ &\ll \sum_{n\geqslant 1}^{x}\frac{\varrho*\varrho_1(n)\Delta(n^r)}{n}.\end{aligned} \tag{4·21}$$

Now $\varrho * \varrho_1(p) = \varrho(p) - z + 1/r$ for all p, so $\varrho * \varrho_1 \in \mathcal{M}_{1/r}$. By the first part of the proof, the right-hand side of (4·21) is hence $\ll (\log x)^{1/r}\mathrm{e}^{\kappa\sqrt{\log_2 x}}$. This yields (4·5) when $0 < z < 1/r$, and thus completes the proof of Theorem 1.3.

5. Proof of Corollary 1.4

Denote by $\varrho(m)$—resp. $\varrho_j(m)$—the number of solutions of $F(n)\equiv 0\,(\operatorname{mod} m)$—resp. $F_j(n)\equiv 0\,(\operatorname{mod} m)$. A direct application of [13; cor. 2] furnishes

$$\sum_{x<n\leqslant x+y}\Delta(|F(n)|)\ll y\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\sum_{P^+(n_1\cdots n_s)\leqslant x}\Delta(n_1^{r_1}\cdots n_s^{r_s})\prod_{1\leqslant j\leqslant s}\frac{\varrho_j(n_j)}{n_j}\cdot$$

The last sum may does not exceed

$$\Bigg(\sum_{P^+(n_1)\leqslant x}\frac{\varrho_1(n_1)\Delta\big(n_1^{r_1}\big)}{n_1}\Bigg)\prod_{2\leqslant j\leqslant s}\sum_{P^+(n_j)\leqslant x}\frac{\varrho_j(n_j)\tau\big(n_j^{r_j}\big)}{n_j}\ll(\log x)^{r+s-1}\mathcal{L}(x)\tag{5·1}$$

with $\mathcal{L}(x)$ defined by (1·10). Here we took into account that, classically, $\varrho_j\in\mathcal{M}_1$ for $1\leqslant j\leqslant s$, and estimated the sum over n_1 by [3; th. 1.1] if $r_1=1$ and by Theorem 1.3 with $z=1$ if $r_1\geqslant 2$.

Observing that

$$\prod_{p\leqslant x}\Big(1-\frac{\varrho(p)}{p}\Big)\asymp\frac{1}{(\log x)^s}$$

completes the proof.

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