Math. Proc. Cambridge Phil. Soc.
(1996) **120** 355–367.

On block Behrend sequences

G. Tenenbaum

1. Introduction

Let $\mathcal{A}$ be a strictly increasing sequence of integers exceeding 1 and let

$$\mathcal{M}(\mathcal{A}) := \{ma : a \in \mathcal{A},\, m \geqslant 1\}$$

denote its set of multiples. We say that $\mathcal{A}$ is a Behrend sequence if $\mathcal{M}(\mathcal{A})$ has asymptotic density 1. The theory of sets of multiples was first developed in the late thirties, under the influence of Erdős, Besicovitch, and others. An account of the classical notions is presented in the book of Halberstam & Roth (1966), and recent progress in the area may be found in Hall & Tenenbaum [7], Erdős, Hall & Tenenbaum [4], Ruzsa & Tenenbaum [9]. As underlined by Erdős in [3], one of the central problems in the field is that of finding general criteria to decide whether a given sequence $\mathcal{A}$ is Behrend.

A particularly straightforward case arises when the elements of $\mathcal{A}$ are pairwise coprime. Indeed, the simple condition

$$\sum_{a\in\mathcal{A}} 1/a = \infty$$

then turns out to be the required criterion for $\mathcal{A}$ to be Behrend. This follows immediately from the Davenport-Erdős theorem [1], [2], which yields in this circumstance that

$$\underline{\mathbf{d}}\mathcal{M}(\mathcal{A}) = \lim_{z\to\infty} \mathbf{d}\mathcal{M}(\mathcal{A}\cap]1,z]) = 1 - \prod_{a\in\mathcal{A}}(1-1/a),$$

where $\mathbf{d}$ (resp. $\underline{\mathbf{d}}$) denotes asymptotic (resp. lower asymptotic) density.

In the present state of available techniques, it seems very difficult, if not hopeless, to obtain effective general criteria. Recent research has been developed in the direction of finding necessary and sufficient conditions (or pairs of conditions of each kind which are not too far apart) that are valid for sequences with certain special structures. Among these, that of Erdős' block sequences is certainly one of the most natural. As in [7], we formally define a block sequence by the property that it can be written in the form

$$\mathcal{A} := \bigcup_{j=1}^{\infty} \mathcal{A}_j, \qquad \mathcal{A}_j := (T_j, H_jT_j] \cap \mathbb{Z}^+ \qquad (j = 1, 2, \ldots),$$

where blocks $\mathcal{A}_j$ satisfy a growth condition that guarantees some local regularity, namely that, for some fixed parameter $\eta > 0$,

$$1 + (T_j)^{\eta-1} \leqslant H_j \leqslant \min\{T_j, T_{j+1}/T_j\} \quad (j = 1, 2, \ldots).$$

The underlying idea here is to impose such a link between the lengths of the intervals and the sizes of their elements that the arithmetical properties of the integers involved be statistically preserved. The lower bound for H_j is indeed sufficient to ensure that the arithmetical irregularities of the elements of $\mathcal{A}$ have little influence on average : see e.g.Shiu's theorem [10]. The upper bound $H_j \leqslant T_j$ can be assumed without loss of generality, since we can always introduce new blocks if necessary. However, this is a natural restriction because a single block $(H_j, T_j]$ is already 'almost' Behrend if $\log T_j / \log H_j$ is large—see e.g.[6], theorem 21.

Let $\xi(j)$ be an arbitrary function tending to infinity. Any block sequence $\mathcal{A}$ may be split in the form $\mathcal{A} = \mathcal{A}^* \cup \mathcal{A}^\dagger$ where $\mathcal{A}^*$ comprises all blocks $\mathcal{A}_j = (T_j, H_jT_j]$ with

$$\log H_j \leqslant (\log T_j)/(\log_2 T_j)^{\xi(j)} \tag{1·1}$$

and $\mathcal{A}^\dagger$ is the union of all remaining blocks. We shall say that $\mathcal{A}^*$ is *sawn* and that $\mathcal{A}^\dagger$ is *stretched* (with respect to ξ). As we observed in [7], it is a straightforward consequence of Behrend's inequality and of the Davenport-Erdős theorem that

$$1 - \underline{\mathbf{d}}\mathcal{M}(\mathcal{A}^* \cup \mathcal{A}^\dagger) \geqslant \big(1 - \underline{\mathbf{d}}\mathcal{M}(\mathcal{A}^*)\big)\big(1 - \underline{\mathbf{d}}\mathcal{M}(\mathcal{A}^\dagger)\big)$$

hence $\mathcal{A}$ is Behrend if, and only if, $\mathcal{A}^*$ or $\mathcal{A}^\dagger$ is a Behrend sequence. In other words, any criterion for block Behrend sequences may be split into a criterion for sawn sequences and a criterion for stretched sequences.

Hall & Tenenbaum [7] obtained the following result.

Theorem A. *Put $\delta := 1 - (1 + \log_2 2)/\log 2 \approx 0 \cdot 08607$, and let $\beta < 1 - \log 2$ be given. Furthermore, let $\xi(j)$ be any function tending to infinity as $j \to \infty$. Let $\mathcal{A} = \cup_j \mathcal{A}_j$ be a block sequence which is either sawn or stretched with respect to ξ, and put $\beta(\mathcal{A}) := \beta$ in the former case, $\beta(\mathcal{A}) := \delta$ in the latter. Then a necessary condition that $\mathcal{A}$ should be a Behrend sequence is*

$$\sum_{j=1}^{\infty} \frac{\log H_j}{1 + \log H_j} \left(\frac{1 + \log H_j}{\log T_j}\right)^{\beta(\mathcal{A})} = \infty. \tag{1·2}$$

Moreover, in the case when $\mathcal{A}$ is stretched, then $\mathcal{A}$ is Behrend provided there exists a positive real number ε such that

$$\sum_{j \in \mathcal{J}} \left(\frac{\log H_j}{\log T_j}\right)^{\delta+\varepsilon} = \infty \tag{1·3}$$

where the summation is restricted to any subsequence $\mathcal{J}$ of indexes satisfying

$$(1{\cdot}4)\qquad \big(\log H_{j_1}\big)^{1+\varepsilon} > 2\big(\log T_{j_1}\big)^{\varepsilon}\log T_j \qquad (j\in\mathcal{J}),$$

with $j_1 := \min\{h\in\mathcal{J} : h > j\}$.

In particular, if $\mathcal{A}$ is stretched and satisfies (1·4) with $j_1 = j+1$, conditions (1·2) and (1·3) are 'adjacent' and the situation is rather satisfactory inasmuch as Theorem A will provide a definite answer in non-pathological cases. Thus the sequence

$$\mathcal{E}_\lambda := \bigcup_{j=1}^{\infty}\Big(\exp\{\mathrm{e}^j\},\exp\{\mathrm{e}^j(1+j^{-\lambda})\}\Big]\cap\mathbb{Z}^+$$

is a Behrend sequence for all $\lambda < 1/\delta$ and is not a Behrend sequence for all $\lambda > 1/\delta$.

The main purpose of this article is to achieve corresponding sufficient conditions, adjacent to (1·2), that are valid for sawn block sequences. We obtain the following result.

Theorem 1. *Let $\mathcal{A} = \cup_{j=1}^{\infty}(T_j, H_jT_j]$ be a block sequence which satisfies the following conditions for some $\beta > 1-\log 2$*

(i) $T_{j+1} < T_j^2 \quad (j=1,2,\dots)$

(ii) $\log H_j \asymp \log H_i \qquad (T_i \leqslant T_j \leqslant T_i^2)$

(iii) $\log(T_{j+1}/T_j) \asymp \log(T_{i+1}/T_i) \qquad (T_i \leqslant T_j \leqslant T_i^2)$

(iv) $\exists\varrho\in\big(0,\min\{\frac{3}{5},\frac{3}{2}(\beta-1+\log 2)\}\big)$ *such that*
$H_j > 1+\exp\big\{-(\log T_j)^{\varrho}\big\} \quad (j=1,2,\dots)$

(v) $\displaystyle\sum_{j=1}^{\infty}\frac{\log H_j}{1+\log H_j}\left(\frac{1+\log H_j}{\log T_j}\right)^{\beta} = \infty.$

Then $\mathcal{A}$ is a Behrend sequence.

The technique involved for the proof is much more sophisticated than that needed for Theorem A. It is built upon that of Maier & Tenenbaum [8], also described in chapter 5 of [6]. Basically, it consists of considering, for a given integer n, the product of its k smallest prime factors, say $n_{(k)}$, and finding an upper bound for the conditional probability that $n_{(k+1)} \notin \mathcal{M}(\mathcal{A})$ knowing already that $n_{(k)} \notin \mathcal{M}(\mathcal{A})$. One then proceeds to show first that this is not too close to 1, and next to prove that the $k+1$th prime factor is sufficiently independent of $n_{(k)}$, the required conclusion being finally derived from an iterative argument. As in some of the other works based upon this idea, a convenient first technical reduction is to replace $n_{(k)}$ by

$$(1{\cdot}5)\qquad n_k := \prod_{p\mid n,\ p\leqslant\exp\exp k} p,$$

which has the advantage of being multiplicative in n and yet behaves, to all intents and purposes, as a good approximation to $n_{(k)}$.

Conditions (i)-(iii) are not very restrictive and indeed they are realised in most natural instances. Moreover, as shown by the counter-example constructed in [7], we know that some hypotheses of this type are inevitable. Condition (iv) is much more stringent inasmuch as it rules out very short intervals, for which, e.g.$H_j \leqslant 1+T_j^{c-1}$. This limitation is inherent to the technique applied here which, even under the Riemann hypothesis, will only yield that (iv) can be relaxed to $H_j > 1 + T_j^{-\varepsilon_j}$ for any $\varepsilon_j \to 0$. We show in another work [12] that shorter blocks may be dealt with by divisor density techniques, but, at present, only in the case of very regular sequences T_j, for instance $T_j = j^2$, $H_j = 1 + 1/j(\log j)^\alpha$.

Apart from the possibility of choosing $\beta = 1 - \log 2$, assumption (v) coincides, in the case of sawn sequences, with the necessary condition (1·2) of Theorem A. It is hence essentially sharp. We conjecture that the conclusion still holds true when $\beta = 1 - \log 2$. This is supported by theorem 2 of [7], which asserts that $\mathcal{A}(\lambda) := \cup_{j=1}^{\infty}(\exp j^\lambda, 2\exp j^\lambda] \cap \mathbb{Z}^+$ is a Behrend sequence if, and only if, $0 < \lambda \leqslant 1/(1 - \log 2)$. In this instance, we have $H_j = 2$ for all j so (1·2) and (v) both take the form

$$\sum_{j=1}^{\infty} j^{-\lambda\beta} = \infty$$

for all β less than, or some β larger than $1 - \log 2$, respectively. The conclusion hence follows from Theorem A or Theorem 1 for all λ except in the limit case $\lambda = 1/(1 - \log 2)$. The proof for $\lambda = 1/(1 - \log 2)$ turns out to be much more difficult and uses the finer behaviour of the distribution of prime factors—namely the occasional, but necessary, occurrence of large concentrations.

The above described results provide in practice highly applicable criteria for block sequences. By way of examples, we state the following two corollaries, the second of which is a very special case of the first and yet essentially solves a conjecture of Erdős [3]. These are immediate consequences of Theorems A & 1, and we omit the (obvious) details of the verification.

Corollary 1. *Let $\mathcal{A} = \cup_j (T_j, H_jT_j] \cap \mathbb{Z}^+$ be a block sequence such that, for suitable real constants α, γ, σ, τ with $\sigma > -1$, we have*

$$\log(T_{j+1}/T_j) \asymp j^\sigma(\log 2j)^\tau, \quad \log H_j \asymp j^{-\alpha}(\log 2j)^\gamma \qquad (j = 1, 2, \ldots).$$

Put $\sigma_0 := \log 2/(1 - \log 2)$ and define

$$\alpha_0(\sigma) := \begin{cases} (1 - \log 2)(\sigma_0 - \sigma) & \text{if } -1 < \sigma \leqslant \sigma_0 \\ \sigma_0 - \sigma & \text{if } \sigma > \sigma_0. \end{cases}$$

Then $\mathcal{A}$ is a Behrend sequence if $\alpha < \alpha_0(\sigma)$ and is not a Behrend sequence if $\alpha > \alpha_0(\sigma)$.

Corollary 2. *Let $\mathcal{A} = \cup_j (T_j, H_jT_j] \cap \mathbb{Z}^+$ be a block sequence satisfying for some positive constants c_1, c_2,*

$$1 + c_1 \leqslant T_{j+1}/T_j \leqslant 1 + c_2 \qquad (j = 1, 2, \ldots).$$

Assume furthermore that $H_j = 1 + j^{-\alpha}\,(j \geqslant 1)$ with $\alpha > 0$. Then $\mathcal{A}$ is a Behrend sequence if $\alpha < \log 2$ and is not a Behrend sequence if $\alpha > \log 2$.

Erdős' original claim was the existence of a critical value $\alpha_0 \in (0, 1)$, under the sole condition $T_{j+1}/T_j \geqslant 1 + c_1$. This is certainly not true as it stands, since Theorem A implies that $\mathcal{A}$ is non-Behrend for all α if we choose for instance $T_j = \exp\exp j$. However, we understand from subsequent discussions with Erdős that he had actually in mind a two-sided condition on the ratios T_{j+1}/T_j, so that the above corollary confirms his conjecture exactly, with the extra information that $\alpha_0 = \log 2$.

The probabilistic discussion started in [7] may be pursued somewhat further in the light of Theorem 1. The issue is rather puzzling. As observed in [7], Behrend's inequality immediately implies that the condition

$$\sum_{j=1}^{\infty} \mathbf{d}\mathcal{M}(\mathcal{A}_j) = \infty$$

is necessary for $\mathcal{A} = \cup_j \mathcal{A}_j$ to be Behrend. Moreover this condition is in general much weaker than (1·2). Assume for instance that $\mathcal{A} = \cup_j \mathcal{A}_j$ is a block sequence composed exclusively of blocks $\mathcal{A}_j$ satisfying

$$2 \leqslant H_j \leqslant \exp\left\{(\log T_j)/(\log_2 T_j)^{\xi(j)}\right\} \tag{1·6}$$

and that the growth conditions (i)-(iv) of Theorem 1 are fulfilled. Then we have [11],

$$\left(\frac{\log H_j}{\log T_j}\right)^{\delta + o(1)} \ll \mathrm{d}\mathcal{M}(\mathcal{A}_j) \ll \left(\frac{\log H_j}{\log T_j}\right)^{\delta} \qquad (j \to +\infty). \tag{1·7}$$

Theorems A & 1 may hence be merged into an 'pseudo' necessary and sufficient condition of the shape

$$\sum_{j=1}^{\infty} \left\{\mathbf{d}\mathcal{M}(\mathcal{A}_j)\right\}^{\alpha + o(1)} = \infty,$$

with $\alpha = (1 - \log 2)/\delta \approx 3.56509$. This is a Borel-Cantelli type condition, but where the probabilities of the individual events are raised to some (essentially) fixed power > 1. It follows from Theorem A that a similar pseudo-criterion holds, but with $\alpha = 1$, in the case of stretched sequences. It would be of interest to find a purely probabilistic model which leads to similar situations : the value of the "Borel-Cantelli exponent" α measures in some sense the level of mutual dependence of the events involved, but the reason why the criteria take such simple forms remains quite mysterious.

2. Lemmas

The proof of Theorem 1 depends on five lemmas. The first of these concerns the local distribution of the prime factors of a normal integer. We recall the definition of the multiplicative function n_k in (1·5), and set

$$\omega(n) := \sum_{p|n} 1, \qquad \omega_\vartheta(n) := \sum_{p|n,\, p\leqslant \exp(1/\vartheta)} 1 \quad (\vartheta > 0).$$

We also put

$$Q(v) := v\log v - v + 1 \qquad (v > 0).$$

Lemma 1. *Let $S \geqslant 1$, $k \geqslant 1$, $x \geqslant \exp\exp k$, $0 < \varepsilon < 1$. Then we have*

$$\inf_{\vartheta > Se^{-k}} \frac{\omega(n_k) - \omega_\vartheta(n_k)}{k - \log^+(1/\vartheta)} \geqslant 1 - \varepsilon$$

for all except $\ll \varepsilon^{-2} S^{-Q(1-\varepsilon)} x$ of the integers $n \leqslant x$.

This is a reformulation of lemma 51.2 of [6] and we omit the details.

For the next lemmas and the proof itself of Theorem 1, we need to introduce some further notation.

We give ourselves a large parameter, R, which will ultimately tend to infinity, but may be viewed as fixed for the whole duration of the proof. We shall consider those integers n such that $n_k \notin \mathcal{M}(\mathcal{A})$ for integers $k \leqslant K$, where $K = K(R)$ is fixed, but large in terms of R. We evaluate the density of these integers in $(1, x]$, and so x may be thought of as arbitrarily large in terms of K and R.

For each k, we define $J_1 = J_1(k)$ and $J_2 = J_2(k)$ by the inequalities

$$T_{J_1} \leqslant \exp(2Re^k) < T_{J_1+1} < \cdots < T_{J_2-1} < \exp(6Re^k) \leqslant T_{J_2}.$$

We note that condition (i) of Theorem 1 guarantees that

$$J = J(k) := J_2(k) - J_1(k) \geqslant 2,$$

and also that

$$T_{J_1} > \exp(Re^k), \quad T_{J_2} < \exp(12Re^k). \tag{2·1}$$

We next define, for $k \geqslant 1$,

$$G_k(\vartheta) := \sum_{J_1 < j \leqslant J_2} T_j^{i\vartheta}, \qquad \mathcal{H}_k(\vartheta) := \int_0^\vartheta |G_k(\varphi)|^2 \,\mathrm{d}\varphi,$$

and put

$$\sigma_k := \frac{1}{\min_{J_1<j\leqslant J_2} \log(T_{j+1}/T_j)}, \qquad h_k := \frac{1}{\min_{J_1<j\leqslant J_2} \log H_j}.$$

We note right away that, from hypothesis (iii),

$$J \asymp \sigma_k Re^k \tag{2·2}$$

and of course that $h_k \geqslant \sigma_k$. We also write

$$\gamma_k := \sum_{J_1<j\leqslant J_2} \frac{\log H_j}{1+\log H_j}\Big(\frac{1+\log H_j}{\log T_j}\Big)^{\beta}, \tag{2·3}$$

and remark that hypothesis (v) of Theorem 1 implies

$$\sum_{k=1}^{\infty} \gamma_k = \infty. \tag{2·4}$$

We set, for positive integers m, k,

$$\begin{aligned}\mathcal{L}_k(m) &:= \bigcup_{\substack{d\mid m\\ J_1<j\leqslant J_2}} \Big(\log(T_j/d) + (0, 1/h_k]\Big)\\ &\subset \bigcup_{\substack{d\mid m\\ J_1<j\leqslant J_2}} \Big(\log(T_j/d), \log(T_jH_j/d)\Big],\end{aligned}$$

and denote by $\lambda_k(m)$ the Lebesgue measure of $\mathcal{L}_k(m)$. Finally, we introduce the arithmetic functions

$$\tau(n,\vartheta) := \sum_{d\mid n} d^{i\vartheta}, \qquad I_k(n) := \int_{-h_k}^{h_k} |G_k(\vartheta)|^2 \frac{|\tau(n,\vartheta)|^2}{\tau(n)^2}\,\mathrm{d}\vartheta.$$

Lemma 2. *We have uniformly for $\vartheta \geqslant 0$, $k \geqslant 1$,*

$$\mathcal{H}_k(\vartheta) \ll (\vartheta + \sigma_k)J.$$

Proof. We make use of the classical weight function

$$w(\varphi) := \frac{1}{2\pi}\Big(\frac{\sin(\varphi/2)}{\varphi/2}\Big)^2 \tag{2·5}$$

with Fourier transform

$$\widehat{w}(\vartheta) = \int_{-\infty}^{+\infty} e^{-i\vartheta\varphi} w(\varphi)\,\mathrm{d}\varphi = (1-|\vartheta|)^+.$$

We have

$$
\begin{aligned}
\mathcal{H}_k(\vartheta) \leqslant & \frac{1}{w(1)} \int_{-\infty}^{+\infty} w\Big(\frac{\varphi}{\vartheta}\Big) |G_k(\varphi)|^2 \,\mathrm{d}\varphi = \frac{\vartheta}{w(1)} \int_{-\infty}^{+\infty} w(\varphi) |G_k(\vartheta\varphi)|^2 \,\mathrm{d}\varphi \\
= & \frac{\vartheta}{w(1)} \sum_{J_1 < i,j \leqslant J_2} \widehat{w}\big(\vartheta \log(T_j/T_i)\big) \ll \vartheta \sum_{\substack{J_1 < i,j \leqslant J_2 \\ |\log(T_j/T_i)| \leqslant 1/\vartheta}} 1.
\end{aligned} \tag{2·6}
$$

In the last double sum, we write $j = i + \ell$. Then, by hypothesis (iii),

$$
\log(T_j/T_i) = \sum_{0 \leqslant m < \ell} \log(T_{m+i+1}/T_{m+i}) \asymp \ell/\sigma_k,
$$

so, for fixed i, there are $\ll (\sigma_k/\vartheta) + 1$ values of j such that $|\log(T_j/T_i)| \leqslant 1/\vartheta$. Inserting this in (2·6), we obtain the required estimate.

Lemma 3. *We have, uniformly for $m \geqslant 1$ and $k \geqslant 1$,*

$$
\lambda_k(m) \gg J^2 I_k(m)^{-1}. \tag{2·7}
$$

Proof. Let w be defined by (2·5), and put

$$
F_k(z) := \sum_{J_1 < j \leqslant J_2} \sum_{\substack{d \mid m \\ 0 < z - \log(T_j/d) \leqslant 1/h_k}} 1,
$$

so that $\mathcal{L}_k(m)$ is precisely the set of real z such that $F_k(z) \neq 0$. We have

$$
\int_{-\infty}^{+\infty} F_k(z) \,\mathrm{d}z = J\tau(m) h_k^{-1}. \tag{2·8}
$$

However, we also have, for all z,

$$
\begin{aligned}
F_k(z) \leqslant & w(1)^{-1} \sum_{\substack{d \mid m \\ J_1 < j \leqslant J_2}} w\Big(h_k\big\{z - \log(T_j/d)\big\}\Big) \\
& = (2\pi w(1))^{-1} \int_{-\infty}^{+\infty} e^{i\vartheta h_k z}\, \overline{G_k(\vartheta h_k)} \tau(m, \vartheta h_k)\, \widehat{w}(\vartheta) \,\mathrm{d}\vartheta \\
& = (2\pi h_k w(1))^{-1} \int_{-\infty}^{+\infty} e^{i\vartheta z}\, \overline{G_k(\vartheta)} \tau(m, \vartheta)\, \widehat{w}(\vartheta/h_k) \,\mathrm{d}\vartheta.
\end{aligned}
$$

By Plancherel's formula, we hence deduce that

$$
\int_{-\infty}^{+\infty} F_k(z)^2 \,\mathrm{d}z \ll h_k^{-2} \int_{-h_k}^{h_k} |G_k(\vartheta)|^2 |\tau(m, \vartheta)|^2 \,\mathrm{d}\vartheta.
$$

The required lower bound (2·7) follows from this and (2·8), in view of the Cauchy–Schwarz inequality

$$\left(\int_{-\infty}^{+\infty} F_k(z)\,\mathrm{d}z\right)^2 \leqslant \lambda_k(m)\int_{-\infty}^{+\infty} F_k(z)^2\,\mathrm{d}z.$$

Lemma 4. *Let $a < 3/5$. Then, uniformly for $k \geqslant 1$, $x \geqslant \exp\exp k$ and $0 \leqslant \vartheta \leqslant \exp(e^{ak})$, we have*

$$\sum_{n\leqslant x} \frac{|\tau(n_k,\vartheta)|^2}{2^{\omega(n_k)+\omega_\vartheta(n_k)}} \ll \begin{cases} x & \text{if } \vartheta \leqslant 1,\\ x|\zeta(1+i\vartheta)| & \text{if } \vartheta > 1.\end{cases}$$

Proof. By theorem 01 of [6], the sum to be estimated is

$$\ll x\exp\Big\{\sum_{\exp(1/\vartheta)<p\leqslant\exp\exp k} \frac{\cos(\vartheta\log p)}{p}\Big\},$$

since $|\tau(p,\vartheta)|^2 = 2\big(1+\cos(\vartheta\log p)\big)$. When $\vartheta \leqslant 1$, the p-sum is $O(1)$ by lemma 30.1 of [6], and hence the required estimate holds. When $\vartheta > 1$, we use the prime number theorem in a strong form to write

$$\begin{aligned}\sum_{p\leqslant\exp\exp k} \frac{\cos(\vartheta\log p)}{p} &= \Re e\sum_p p^{-1-i\vartheta} - \sum_{p>\exp\exp k}\frac{\cos(\vartheta\log p)}{p}\\ &= \log|\zeta(1+i\vartheta)| + O(1) - \int_{\exp\exp k}^{\infty} \frac{\cos(\vartheta\log t)}{t\log t}\,\mathrm{d}t\\ &\quad - \int_{\exp\exp k}^{\infty}\frac{\cos(\vartheta\log t)}{t}\,\mathrm{d}R(t),\end{aligned}$$

where $R(t) \ll t\exp\big\{-(\log t)^{a_1}\big\}$ for some $a_1 > a$. In the first integral, we make the change of variables $v = \vartheta\log t$ and apply the second mean value theorem. We obtain that it is $\ll e^{-k}$, uniformly in ϑ. The second integral is treated by partial summation. We get the bound

$$\ll \exp\big\{-e^{a_1k}\big\} + |\vartheta|\int_{\exp\exp k}^{\infty}\exp\big\{-(\log t)^{a_1}\big\}\frac{\mathrm{d}t}{t} \ll |\vartheta|\exp\big\{-e^{ak}\big\} \ll 1.$$

This completes the proof of the lemma.

Lemma 5. *Let γ_k be defined by (2·3). There exists a positive constant $c_0 = c_0(\beta)$ such that for $R \geqslant 1$, $k \geqslant 1$, and large x, the bound*

$$\lambda_k(n_k) \gg e^k R^{-\beta-2}\frac{\gamma_k}{1+\gamma_k}$$

holds uniformly for all but at most $\ll_\beta xR^{-c_0}$ of the integers $n \leqslant x$.

Proof. Let $\varepsilon = \varepsilon(\beta) > 0$ be so small that

$$\alpha := (1-\varepsilon)\log 2 > 1-\beta. \tag{2·9}$$

We want to apply Lemma 3, and hence need an upper bound for $I_k(n_k)$. Let $I_k^{(1)}(n_k)$ and $I_k^{(2)}(n_k)$ denote the respective contributions to the integral $I_k(n_k)$ of the ranges $0 \leqslant \vartheta \leqslant R^2\sigma_k$ and $R^2\sigma_k < \vartheta \leqslant h_k$ (with the convention that $I_k^{(2)}(n_k) = 0$ if $h_k < R^2\sigma_k$). We plainly have

$$I_k(n_k) \leqslant 2I_k^{(1)}(n_k) + 2I_k^{(2)}(n_k). \tag{2·10}$$

Moreover, using the trivial bound $|\tau(n_k,\vartheta)| \leqslant \tau(n_k)$ in $I_k^{(1)}(n_k)$ and applying Lemma 2, we obtain

$$I_k^{(1)}(n_k) \ll R^2\sigma_k J. \tag{2·11}$$

We now turn our attention to the estimation of $I_k^{(2)}(n_k)$. By (2·2), we have $R^2\sigma_k \gg Re^{-k}$, so we may apply Lemma 1 with $S = cR$ for some absolute constant c to infer that, for all $n \leqslant x$ but an acceptable number of exceptions (namely $\ll_\varepsilon xR^{-Q(1-\varepsilon)}$), we have

$$\omega(n_k) - \omega_\vartheta(n_k) \geqslant (1-\varepsilon)\big(k - \log^+(1/\vartheta)\big)$$

for all ϑ in the integration range $R^2\sigma_k < \vartheta \leqslant h_k$. This implies

$$2^{-\omega(n_k)} \leqslant 2^{-\omega_\vartheta(n_k)}\, e^{-\alpha k}(1+\vartheta^{-\alpha})$$

and in turn

$$I_k^{(2)}(n_k) \leqslant e^{-\alpha k} I_k^*(n_k), \tag{2·12}$$

with

$$I_k^*(n_k) := \int_{R^2\sigma_k}^{h_k} |G_k(\vartheta)|^2 \frac{|\tau(n_k,\vartheta)|^2}{2^{\omega(n_k)+\omega_\vartheta(n_k)}}(1+\vartheta^{-\alpha})\,\mathrm{d}\vartheta.$$

Here, we also make the convention that $I_k^*(n_k) := 0$ if $h_k < R^2\sigma_k$.

We apply Lemma 4 to bound $I_k^*(n_k)$ on average. Let $b \in (2/3, 1)$ satisfy

$$b\varrho < \beta - 1 + \alpha. \tag{2·13}$$

By hypothesis (iv), this is possible provided ε is sufficiently small. By a classical estimate of Vinogradov, we have

$$|\zeta(1+i\vartheta)| \ll_b 1 + (\log\vartheta)^b \qquad (\vartheta > 1).$$

This and Lemma 4 yield

$$
(2{\cdot}14)\qquad \begin{aligned} x^{-1}\sum_{n\leqslant x} I_k^*(n_k) &\ll \int_{R^2\sigma_k}^{h_k} |G_k(\vartheta)|^2\big(1+|\zeta(1+i\vartheta)|\big)(1+\vartheta^{-\alpha})\,\mathrm{d}\vartheta \\ &\ll \int_{R^2\sigma_k}^{h_k} |G_k(\vartheta)|^2\big(1+\log^+\vartheta\big)^b(1+\vartheta^{-\alpha})\,\mathrm{d}\vartheta. \end{aligned}
$$

We claim that the right-hand side is

$$
(2{\cdot}15)\qquad \ll Jh_k(1+h_k^{-\alpha})\big(1+\log^+ h_k\big)^b.
$$

To see this, we consider two cases, according as $h_k \leqslant 1$ or not. In the first instance, the integral of (2·14) is

$$
(2{\cdot}16)\qquad \begin{aligned} &\ll \int_{R^2\sigma_k}^{h_k} |G_k(\vartheta)|^2\vartheta^{-\alpha}\,\mathrm{d}\vartheta \ll \Big[\mathcal{H}_k(\vartheta)\vartheta^{-\alpha}\Big]_{R^2\sigma_k}^{h_k} + \int_{R^2\sigma_k}^{h_k} \mathcal{H}_k(\vartheta)\vartheta^{-\alpha-1}\,\mathrm{d}\vartheta \\ &\ll \Big[J\vartheta^{1-\alpha}\Big]_{R^2\sigma_k}^{h_k} + \int_{R^2\sigma_k}^{h_k} J\vartheta^{-\alpha}\,\mathrm{d}\vartheta \ll Jh_k^{1-\alpha}, \end{aligned}
$$

by Lemma 2, which, for the relevant values of ϑ, yields that $\mathcal{H}_k(\vartheta) \ll J\vartheta$. This is compatible with (2·15). When $h_k > 1$, we split the integral of (2·14) into two parts, at $\vartheta = 1$. The contribution of the lower range (empty if $R^2\sigma_k > 1$) is, by a computation parallel to (2·16),

$$
\ll \int_{R^2\sigma_k}^{1} |G_k(\vartheta)|^2\vartheta^{-\alpha}\,\mathrm{d}\vartheta \ll J \ll Jh_k^{1-\alpha}.
$$

The complementary contribution is

$$
\int_1^{h_k} |G_k(\vartheta)|^2\big(1+\log^+\vartheta\big)^b\,\mathrm{d}\vartheta \ll Jh_k\big(1+\log^+ h_k\big)^b,
$$

also compatible with (2·15). This establishes our claim.

Next, we observe that

$$
(2{\cdot}17)\qquad e^{-\alpha k}(1+h_k^{-\alpha})\big(1+\log^+ h_k\big)^b \ll R^{\beta+1}e^{-(1-\beta)k}(1+h_k^{\beta-1}).
$$

Indeed, when $h_k > 1$, the left-hand side is

$$
\ll e^{-\alpha k}\big(Re^k\big)^{b\varrho} \ll Re^{-(1-\beta)k},
$$

by hypothesis (iv) and (2·13), and, when $h_k \leqslant 1$, it is

$$
\ll R^\alpha\big(Rh_ke^k\big)^{-\alpha} \ll R^\alpha\big(Rh_ke^k\big)^{-(1-\beta)} \ll R^\beta e^{-(1-\beta)k}(1+h_k^{\beta-1}),
$$

where we have used (2·9) and $Rh_ke^k \geqslant R\sigma_ke^k \gg 1$, which follows from (2·2).

From (2·14), (2·15), and (2·17), we infer that

$$e^{-\alpha k} I_k^*(n_k) \ll JR^{\beta+2} e^{-(1-\beta)k} h_k (1 + h_k^{\beta-1})$$

holds for all but $\ll xR^{-1}$ of the integers $n \leqslant x$. In view of (2·10), (2·11), and (2·12), this implies that, for all $n \leqslant x$ but an acceptable number of exceptions, we have

$$I_k(n_k) \ll JR^{\beta+2} \Big\{ \sigma_k + e^{-(1-\beta)k} h_k (1 + h_k^{\beta-1}) \Big\}. \tag{2·18}$$

Now, by hypotheses (ii) and (iii),

$$\gamma_k \asymp J h_k^{-1} (1 + h_k^{\beta-1})^{-1} (Re^k)^{-\beta} \asymp \sigma_k (Re^k)^{1-\beta} h_k^{-1} (1 + h_k^{\beta-1})^{-1}, \tag{2·19}$$

where we have used (2·2). Hence

$$e^{-(1-\beta)k} h_k (1 + h_k^{\beta-1}) \asymp \sigma_k \gamma_k^{-1} R^{1-\beta}.$$

Inserting this in (2·18), we get

$$I_k(n_k) \ll JR^{\beta+3} \sigma_k \frac{1+\gamma_k}{\gamma_k}, \tag{2·20}$$

and, applying Lemma 3, we deduce that

$$\lambda_k(n_k) \gg J \sigma_k^{-1} R^{-\beta-3} \frac{\gamma_k}{1+\gamma_k}.$$

The required result now follows from (2·2). This finishes the proof of Lemma 5.

3. Proof of Theorem 1

We are now in a position to complete the proof of our main result. We use an inductive argument and set out to find an upper bound for the quantity

$$N_k := \Bigg| \Big\{ n \leqslant x : \max_{\substack{j \geqslant 1,\, d \mid n_k \\ T_j < d}} \frac{\log(T_j/d)}{\log H_j} < -1 \Big\} \Bigg|.$$

This is plainly a non-increasing function of k, and any integer not counted in N_k has a divisor d such that, for some $j \geqslant 1$,

$$-1 \leqslant \frac{\log(T_j/d)}{\log H_j} < 0, \quad \text{that is} \quad T_j < d \leqslant H_j T_j.$$

We shall show that

$$\lim_{k\to+\infty}\limsup_{x\to+\infty} x^{-1}N_k = 0, \tag{3·1}$$

which plainly implies the conclusion of the theorem.

Let $R \geqslant 1$ be given and, for each k, let N'_k denote the number of those integers n counted in N_k which also satisfy

$$\begin{aligned} &\text{(a)} \quad \log n_k \leqslant Re^k, \\ &\text{(b)} \quad \lambda_k(n_k) \geqslant c_1 e^k R^{-2-\beta}\frac{\gamma_k}{1+\gamma_k}, \end{aligned}$$

where c_1 is a positive absolute constant, so small that Lemma 5 is applicable to bound the number of integers n contravening (b). By this and theorem 07 of [6] (which provides an uppper bound for the number of those n which do not satisfy (a)), we have

$$N_k \leqslant N'_k + \eta x, \tag{3·2}$$

with $\eta := c_2(\beta)R^{-c_0(\beta)}$.

We want to bound N_k for $1 \leqslant k \leqslant K$. We may assume $N_K > 2\eta x$, otherwise (3·1) follows immediately and there is nothing to prove. By (3·2), we then have, for all $k \leqslant K$,

$$N_k \leqslant N'_k + \tfrac{1}{2}N_K \leqslant N'_k + \tfrac{1}{2}N_k,$$

whence

$$\tfrac{1}{2}N_k \leqslant N'_k.$$

Let $\mathcal{M}'_k$ denote the set of all integers m of the form $m = n_k$ for at least one n counted in N'_k. We now have

$$\tfrac{1}{2}N_k \leqslant N'_k \leqslant \sum_{m\in\mathcal{M}'_k}\ \sum_{\substack{mh\leqslant x \\ p|h\Rightarrow p>\exp\exp k \text{ or } p|m}} 1 \ll \sum_{m\in\mathcal{M}'_k}\frac{x}{\varphi(m)}e^{-k}. \tag{3·3}$$

This is an immediate consequence of a classical sieve result (see, e.g.Halberstam & Richert [5], theorem 3.5), noticing that condition (a) above guarantees that m is bounded in terms of k and R only. However, for the sake of completeness, we note that the following simple estimate suffices here

$$\begin{aligned} \sum_{\substack{n\leqslant x \\ (n,M)=1}} 1 &= \sum_{n\leqslant x}\sum_{d|(n,M)}\mu(d) = \sum_{d|M}\mu(d)\Big[\frac{x}{d}\Big] \\ &= x\frac{\varphi(M)}{M} + O\big(\tau(M)\big) \asymp x\prod_{p|M}\big(1-1/p\big) \qquad (x > x_0(M)). \end{aligned} \tag{3·4}$$

This is valid for any integer M, and, for (3·3), we select $M := \prod_{\substack{p\leqslant\exp\exp k \\ p\nmid m}} p$.

Next, let D_k denote the number of those $n \leqslant x$ which can be written in the form $n = m\ell ph$ for some $m \in \mathcal{M}'_k$, with $\ell | m$ and

$$\begin{aligned} &\text{(c)} \quad \exp\exp k < p \leqslant \exp\exp(k+r),\\ &\text{(d)} \quad \log p \in \mathcal{L}_k(m),\\ &\text{(e)} \quad P^-(h) > \exp\exp(k+r), \end{aligned}$$

where we have set $r := 1 + \big[\log(24R)\big]$. Here and in the sequel, we let $P^-(h)$ denote the smallest prime factor of a positive integer h, with the convention that $P^-(1) = 1$. Condition (d) implies that, for some $d|m$ and some $j \in (J_1(k), J_2(k)]$, we have

$$\log(T_j/d) < \log p \leqslant \log(T_j/d) + 1/h_k,$$

whence, by the definition of h_k,

$$T_j < pd \leqslant H_j T_j.$$

Thus, any n counted by D_k is counted by N_k but not by N_{k+r}, and we deduce that

$$D_k \leqslant N_k - N_{k+r}. \tag{3·5}$$

We now have

$$D_k \geqslant \sum_{m \in \mathcal{M}'_k} \sum_{\ell | m} \sum_{\substack{k < \log_2 p \leqslant k+r \\ \log p \in \mathcal{L}_k(m)}} \sum_{\substack{h \leqslant x/m\ell p \\ P^-(h) > \exp\exp(k+r)}} 1. \tag{3·6}$$

By (3·4)—or, of course, any classical sieve estimate for integers free of small prime factors—the inner sum is $\gg R^{-1}e^{-k}x/m\ell p$. We next estimate the p-sum, observing that condition (a) and (2·1) imply

$$\mathcal{L}_k(m) \subset \big(Re^k, 24Re^k\big],$$

so our choice of r makes the size restriction $k < \log_2 p \leqslant k + r$ redundant in (3·6). Hence we obtain that the p-sum is

$$\gg R^{-1}e^{-k} \sum_{\log p \in \mathcal{L}_k(m)} \frac{\log p}{p} \gg R^{-1}e^{-k}\lambda_k(m), \tag{3·7}$$

by the prime number theorem. Indeed, $\mathcal{L}_k(m)$ is a union of intervals which can be assumed to be disjoint and of the type $(u, u+v]$, with

$$u \asymp Re^k \quad \text{and} \quad v \gg h_k^{-1} \gg \exp\big(-(12Re^k)^{\varrho}\big) \gg \exp\{-c_3(\log u)^{\varrho}\},$$

for some absolute $c_3 > 0$. For such an interval $(u, u+v]$, the corresponding subsum over p is $v + O_\varepsilon\big(\exp\{-(\log u)^{3/5-\varepsilon}\}\big) \sim v$ (since $\varrho < \frac{3}{5}$), and adding the v's gives (3·7).

Put $\gamma_k^* := \gamma_k/(1+\gamma_k)$, so that, by (2·4)

$$\sum_{k=1}^{\infty} \gamma_k^* = \infty. \tag{3·8}$$

Condition (b) implies that the right-hand side of (3·7) is $\gg R^{-3-\beta}\gamma_k^*$. Inserting in (3·6), we finally obtain that

$$\begin{aligned} D_k &\gg xR^{-4-\beta}e^{-k}\gamma_k^* \sum_{m\in\mathcal{M}'_k} m^{-1} \sum_{\ell\mid m} \ell^{-1} \\ &\gg xR^{-4-\beta}e^{-k}\gamma_k^* \sum_{m\in\mathcal{M}'_k} \varphi(m)^{-1} \gg N_k R^{-4-\beta}\gamma_k^*, \end{aligned}$$

where the last estimate follows from (3·3). By (3·5), we infer that there exists a positive absolute constant c_4 such that

$$N_{k+r} \leqslant N_k\big(1 - c_4 R^{-4-\beta}\gamma_k^*\big).$$

Iterating, we get

$$N_K \ll N_1 \exp\{-c_4 R^{-4-\beta} \sum_{\ell\leqslant K/r} \gamma_{\ell r}^*\}. \tag{3·9}$$

At this stage, we note that conditions (i), (ii), and (iii) of the theorem imply, for all k,

$$h_{k+1}/h_k \asymp 1, \quad \sigma_{k+1}/\sigma_k \asymp 1,$$

so, by (2·19),

$$\gamma_{k+1}^*/\gamma_k^* \asymp 1.$$

It follows that, for suitable absolute constants $c_5 > 0$, $c_6 > 0$, we have

$$\gamma_{\ell r}^* \geqslant e^{-c_5 r} \max_{\ell r < k \leqslant (\ell+1)r} \gamma_k^* \geqslant R^{-c_6} \sum_{\ell r < k \leqslant (\ell+1)r} \gamma_k^*.$$

Inserting in (3·9), we get

$$N_K \ll x \exp\Big\{ - c_4 R^{-4-\beta-c_6} \sum_{k\leqslant r[K/r]} \gamma_k^*\Big\}.$$

By (3·8), the last sum tends to infinity as $K \to +\infty$ for each fixed R. Hence, for suitable $K = K(R)$, we have

$$N_K \leqslant 2\eta x, \tag{3·10}$$

contradicting our assumption $N_K > 2\eta x$. Hence (3·10) holds unconditionally, and (3·1) follows. This completes the proof of Theorem 1.

References

[1] H. Davenport & P. Erdős, On sequences of positive integers, *Acta Arith.* **2** (1937), 147–151.

[2] H. Davenport & P. Erdős, On sequences of positive integers, *J. Indian Math. Soc.* **15** (1951), 19–24.

[3] P. Erdős, Some unconventional problems in number theory, *Astérisque* **61** (1979), 73–82.

[4] P. Erdős, R.R Hall & G. Tenenbaum, On the densities of sets of multiples, *J. reine angew. Math.* **454** (1994), 119–141.

[5] H. Halberstam & H.-E. Richert, *Sieve methods*, Academic Press, London, New York, San Franscisco, 1974.

[6] R.R Hall & G. Tenenbaum, *Divisors*, Cambridge University Press, 1988.

[7] R.R Hall & G. Tenenbaum, On Behrend sequences, *Math. Proc. Camb. Phil. Soc.* **112** (1992), 467–482.

[8] H. Maier & G. Tenenbaum, On the set of divisors of an integer, *Invent. Math.* **76** (1984), 121–128.

[9] I. Ruzsa & G. Tenenbaum, A note on Behrend sequences, *Acta Math. Hung.*, to appear.

[10] P. Shiu, A Brun-Titchmarsh theorem for multiplicative functions, *J. reine angew. Math.* **313** (1980), 161–170.

[11] G. Tenenbaum, Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné, *Compositio Math.* **51** (1984), 243–263.

[12] G. Tenenbaum, Uniform distribution on divisors and Behrend sequences, preprint.

Institut Élie Cartan
Université Henri Poincaré–Nancy 1
BP 239
54506 Vandœuvre Cedex
France