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On Behrend sequences

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1. *Introduction*

Let $\mathcal{A}$ denote a sequence of integers exceeding 1, and let $\tau(n,\mathcal{A})$ be the number of those divisors of n which belong to $\mathcal{A}$. We say that $\mathcal{A}$ is a *Behrend sequence* if

$$\tau(n,\mathcal{A}) \geqslant 1, \qquad \text{pp}, \tag{1.1}$$

where, here and in the sequel, we use the notation pp to indicate that a relation holds on a set of asymptotic density one.

This terminology was introduced only recently by Hall [8], but the underlying concept has been a constant concern for Erdős in the past fifty years. For instance, he writes in [5] : " *It seems very difficult to obtain a necessary and sufficient condition that, if $a_1 < a_2 < \ldots$ is a sequence of integers, then almost all integers n should be a multiple of one of the a's.*" Indeed, if the corresponding problem for sequences of prime numbers is essentially trivial, the required criterion being

$$\sum_{j=1}^{\infty} p_j^{-1} = +\infty, \tag{1.2}$$

it turns out that the general case leads to delicate and interesting questions.

Given an integer sequence $\mathcal{A}$, we denote by $\mathrm{d}\mathcal{A}$ (resp. $\overline{\mathrm{d}}\mathcal{A}$, $\underline{\mathrm{d}}\mathcal{A}$) its asymptotic (resp. upper, lower asymptotic) density and by $\mathcal{M}(\mathcal{A}) := \{ma : m \geqslant 1, \, a \in \mathcal{A}\}$ its set of multiples. A deep result of Davenport and Erdős [2,3] (see also [13] ex.5, p.312) states that for any increasing sequence $\mathcal{A} = \{a_1, a_2, \ldots\}$ one has

$$\lim_{k\to+\infty} \mathrm{d}\mathcal{M}\big(\{a_1,\ldots,a_k\}\big) = \underline{\mathrm{d}}\mathcal{M}(\mathcal{A}). \tag{1.3}$$

From Behrend's fundamental inequality [1] valid for finite sequences, we hence deduce that

$$1 - \underline{\mathrm{d}}\mathcal{M}(\mathcal{A}\cup\mathcal{B}) \geqslant \big(1-\underline{\mathrm{d}}\mathcal{M}(\mathcal{A})\big)\big(1-\underline{\mathrm{d}}\mathcal{M}(\mathcal{B})\big) \tag{1.4}$$

holds for all sequences $\mathcal{A}$, $\mathcal{B}$. It follows in particular from this that any tail $\mathcal{A}^{(k)} := \{a_j : j > k\}$ of a Behrend sequence $\mathcal{A}$ is still a Behrend sequence.

Another interesting general feature of Behrend sequences lies in the fact that (1.1) is actually equivalent to

$$\tau(n, \mathcal{A}) \to +\infty, \qquad \text{pp} \tag{1.5}$$

This has probably been known to Erdős and a few others for several years, but has never been explicitely stated in the literature — although it makes the notion of a Behrend sequence even more attractive. This follows almost immediately from the tail property recalled above and (1.3). Indeed, if $\mathcal{A}$ is Behrend, then, for any fixed $\varepsilon > 0$, we may find a k_1 such that the right-hand side of (1.3) is $\geqslant 1 - \varepsilon/2$; but, since $\mathcal{A}^{(k_1)}$ is still Behrend, we may plainly find a further k_2 such that

$$1 - \mathrm{d}\mathcal{M}\big(\{a_j : k_1 < j \leqslant k_2\}\big) \leqslant \varepsilon/4.$$

Continuing this process, we see that, given an arbitrary $R \geqslant 1$, the upper density of those integers n which do not have a divisor in each of the finite sequences $\{a_j : k_r < j \leqslant k_{r+1}\}$ $(0 \leqslant r < R)$ does not exceed

$$\sum_{r=0}^{R-1} \varepsilon 2^{-r-1} < \varepsilon.$$

This all we need.

For sequences $\mathcal{A}$ with a special structure, it is sometimes possible to give criteria for deciding whether $\mathcal{A}$ is or not Behrend. We give two examples.

(i) The letters p, q being restricted to denote prime numbers, Erdős proved in [4] that the sequence

$$\mathcal{A} := \{pq : p < q \leqslant p^{1+\varepsilon_p}\} \tag{1.6}$$

is Behrend if and only if

$$\sum_p \frac{\min(1, \varepsilon_p)}{p} = +\infty. \tag{1.7}$$

(ii) A long standing conjecture of Erdős, eventually established by Maier and Tenenbaum in 1983 [11], states that the sequence $\mathcal{A} := \{ab : a < b \leqslant 2a\}$ is Behrend. Actually, a more precise result holds. The sequence

$$\mathcal{A}(\alpha) := \big\{ab : a < b \leqslant a\big(1 + (\log a)^{-\alpha}\big)\big\}$$

is Behrend for all $\alpha < \log 3 - 1$ and is not Behrend for all $\alpha > \log 3 - 1$, the case of equality being left in doubt. The first statement follows directly from Theorem 1 of [11] where it is shown that

$$\min_{ab|n, a\neq b} |\log(b/a)| \leqslant (\log n)^{1-\log 3+o(1)}, \qquad \text{pp} \tag{1.8}$$

The second statement may be established by a straight-forward adaptation of the argument of Erdős-Hall in [6] where it is proved that (1.8) is actually an equality.

Another class of special sequences for which some non-trivial information is available is provided by the so-called *block sequences*, i.e. sequences of the form

$$\mathcal{A} := \bigcup_{j=1}^{\infty} \mathcal{A}_j, \qquad \mathcal{A}_j := (T_j, H_jT_j] \cap \mathbb{Z}^+ \qquad (j = 1, 2, \ldots), \tag{1.9}$$

satisfying a growth condition that guarantees some local regularity, namely, for some fixed parameter $\eta > 0$,

$$H_j \geqslant 1 + (T_j)^{\eta - 1}, \qquad T_{j+1} \geqslant H_jT_j \qquad (j = 1, 2, \ldots). \tag{1.10}$$

Furthermore, we may always suppose, by introducing new T_j if necessary, that

$$H_j \leqslant T_j \qquad (j = 1, 2, \ldots) \tag{1.11}$$

In this case, the (sharp) necessary condition of Hall [8] takes the simple shape

$$\sum_{a \in \mathcal{A}} a^{-1}(\log a)^{-\beta} = +\infty \qquad (\beta < 1 - \log 2), \tag{1.12}$$

which, for comparison with further results to be stated below, we rewrite as

$$\sum_{j=1}^{\infty} \frac{\log H_j}{(\log T_j)^{\beta}} = +\infty \qquad (\beta < 1 - \log 2). \tag{1.13}$$

Our first aim in this article is to improve on this by giving a new necessary condition which is strictly stronger than (1.13) for the sub-sum corresponding to "large" H_j. In the following statement and in the sequel of the paper, we understand that a block sequence is defined by (1.9) and satisfies (1.10) and (1.11).

Theorem 1. *Put* $\delta := 1 - (1 + \log_2 2)/\log 2 = \cdot 08607\ldots$, *and let* $\beta < 1 - \log 2$. *For any block sequence* $\mathcal{A}$, *and given an arbitrary function* $\xi(j)$ *tending to infinity with* j, *define*

$$\beta(j) := \begin{cases} \beta & (\textit{if } \log H_j \leqslant \log T_j/(\log_2 T_j)^{\xi(j)}) \\ \delta & (\textit{ otherwise}). \end{cases} \tag{1.14}$$

Then, if $\mathcal{A}$ *is a Behrend sequence, we must have*

$$\sum_{j=1}^{\infty} \frac{\log H_j}{1 + \log H_j} \left(\frac{1 + \log H_j}{\log T_j} \right)^{\beta(j)} = +\infty. \tag{1.15}$$

In the case when $H_j = 2$ for all j, we clearly have that the condition

$$\sum_{j=1}^{\infty} \frac{1}{\log T_j} = +\infty \tag{1.16}$$

is sufficient for $\mathcal{A}$ to be Behrend, since the primes of $\mathcal{A}$ then satisfy (1.2). It is remarkable that Theorem 1 implies that (1.16) cannot be weakened in the form

$$\sum_{j=1}^{\infty} \frac{\psi(T_j)}{\log T_j} = +\infty \tag{1.17}$$

for any function $\psi(t) \to +\infty$. Indeed, we may plainly assume that $\psi(t)$ is non-decreasing and has a rate of growth as slow as we wish. We then define T_k^* to be the smallest solution to $\psi(T_k^*) = k^{2/\delta} \quad (k = 1, 2, \ldots)$ and set

$$j_0 := 1, \quad j_{k+1} := j_k + 1 + \left[k^{-2/\delta} \log T_k^*\right] \qquad (k = 0, 1, 2, \ldots).$$

We have $T_{k+1}^*/T_k^* > 2^{j_{k+1}-j_k}$ provided $\psi(t)$ increases sufficiently slowly, and we can put

$$T_j := T_k^* 2^{j-j_k} \quad (j_k \leqslant j < j_{k+1},\ k \geqslant 0),$$

so that

$$\mathcal{A} = \bigcup_j (T_j, 2T_j] \cap \mathbb{Z}^+ \subset \bigcup_k \left(T_k^*, (T_k^*)^{1+k^{-2/\delta}}\right] \cap \mathbb{Z}^+.$$

By Theorem 1, $\mathcal{A}$ is certainly not Behrend ; however, we have

$$\begin{aligned} \sum_{j_k \leqslant j < j_{k+1}} \frac{\psi(T_j)}{\log T_j} &\gg \psi(T_k^*) \sum_{0 \leqslant h < j_{k+1} - j_k} \frac{1}{\log T_k^* + h} \\ &\gg \psi(T_k^*) \log\left(1 + \frac{j_{k+1} - j_k}{\log T_k^*}\right) \gg \psi(T_k^*) k^{-2/\delta} \gg 1 \end{aligned}$$

and hence (1.17) holds.

Thus it is clear that, even in the case $H_j = 2 \quad (j \geqslant 1)$, our necessary condition (1.15) is not sufficient. However, if the T_j are somewhat well distributed (so as to avoid a counter-example of the type described above), we believe that (1.15) is very close to optimality. In support of this, we return to the so-called $\mathcal{B}(\lambda)$-conjecture of Erdős, dating at least from the seventies (private communication) and referred to in [10], pp.49 & 63, which states that the block sequence

$$\mathcal{A}(\lambda) := \bigcup_{j=1}^{\infty} (\exp j^\lambda, 2 \exp j^\lambda] \cap \mathbb{Z}^+ \tag{1.18}$$

is Behrend for some $\lambda > 1$. (The name of the conjecture coming from the former notation $\mathcal{B}(\lambda) = \mathcal{M}\big(\mathcal{A}(\lambda)\big)$.) This was proved in [9] for all values of $\lambda < 1{\cdot}31457\ldots$;

furthermore the obvious heuristic argument assuming even distribution mod 1 of $(\log d)^{1/\lambda}$ as d runs through the $\tau(n) = (\log n)^{\log 2+o(1)}$ (pp) divisors of n leads to the conjecture that the upper bound λ_0 of the set of values of λ for which $\mathcal{A}(\lambda)$ is Behrend is equal to $1/(1-\log 2)$. Hall's necessary condition (and *a fortiori* our Theorem 1) gives that $\lambda_0 \leqslant 1/(1-\log 2)$. In the following theorem, we actually show that equality holds, and we also decide on the nature of the sequence when $\lambda = 1/(1-\log 2)$.

Theorem 2. *The sequence $\mathcal{A}(\lambda)$ defined by* (1.18) *is a Behrend sequence if, and only if, $\lambda \leqslant 1/(1-\log 2)$.*

This is proved by the technique of Maier-Tenenbaum [11]. The harder part of the argument is of course the limit case $\lambda = \lambda_0$. Here, we use the fact that the local distribution of the prime factors of a normal integer necessarily presents some rather large concentrations.

It is clear from Theorem 2 that, in the statement of Theorem 1, the bound $1-\log 2$ for β cannot be replaced by any smaller one. Our last theorem is devoted to showing that the other value, δ, occurring in (1.14) is also optimal.

Theorem 3. *Let $\mathcal{A}$ be a block sequence. Suppose that, for some $\varepsilon > 0$, we have*

$$\big(\log H_{j+1}\big)^{1+\varepsilon} > 2\big(\log T_{j+1}\big)^{\varepsilon} \log T_j \qquad (j = 1, 2, \ldots), \tag{1.19}$$

and

$$\sum_{j=1}^{\infty} \Big(\frac{\log H_j}{\log T_j}\Big)^{\delta+\varepsilon} = +\infty. \tag{1.20}$$

Then $\mathcal{A}$ is a Behrend sequence.

Corollary. *For positive λ, define the block sequence*

$$\mathcal{E}_\lambda := \bigcup_{j=1}^{\infty} \Big(\exp\{e^j\}, \exp\{e^j(1+j^{-\lambda})\}\Big] \cap \mathbb{Z}^+.$$

Then $\mathcal{E}_\lambda$ is a Behrend sequence for all $\lambda < 1/\delta$ and is not a Behrend sequence for all $\lambda > 1/\delta$.

Indeed, Theorem 1 implies, on the one hand, that $\lambda \leqslant 1/\delta$ is necessary for $\mathcal{E}_\lambda$ to be Behrend ; on the other hand, the fact that $\lambda < 1/\delta$ is sufficient follows immediately from Theorem 3, on observing in this case that, for small enough ε, the subset of $\mathcal{E}_\lambda$ composed of the blocks of indices $j = \big[m^{1+\varepsilon}\big]$ $\big(m > m_0(\varepsilon)\big)$ satisfies (1.19) and (1.20). We leave open the case $\lambda = 1/\delta$, but, in view of Theorem 2, we conjecture that the answer is still positive in this circumstance.

One of Erdős' main concerns in this field has been to understand how one has to strengthen the (trivially necessary) hypothesis

$$\sum_{j=1}^{\infty} \mathrm{d}\mathcal{M}(\mathcal{A}_j) = +\infty \tag{1.21}$$

in order to obtain a criterion that a block sequence $\mathcal{A}$ be Behrend. We now summarize the present state of knowledge.

Given a block sequence $\mathcal{A}$ and a function $\xi(j) \to \infty$, we split the blocks $\mathcal{A}_j$ into three classes and write

$$\mathcal{A} = \mathcal{A}' \cup \mathcal{A}'' \cup \mathcal{A}''', \tag{1.22}$$

where $\mathcal{A}'$ contains those $\mathcal{A}_j$ with $H_j \leqslant 2$, $\mathcal{A}''$ corresponds to the condition

$$2 < H_j \leqslant \exp\big\{(\log T_j)/(\log_2 T_j)^{\xi(j)}\big\}, \tag{1.23}$$

and $\mathcal{A}'''$ is the union of all remaining $\mathcal{A}_j$.

If $\mathcal{A}_j$ appears in $\mathcal{A}'$ we have (see [12] or [10], chap.2)

$$\mathrm{d}\mathcal{M}(\mathcal{A}_j) = \big(\log T_j\big)^{-E(\alpha_j)+o(1)} \qquad (j \to +\infty), \tag{1.24}$$

where α_j is defined by $H_j = 1 + (\log T_j)^{-\alpha_j}$ and

$$E(\alpha) := \begin{cases} \dfrac{1+\alpha}{\log 2} \log\Big(\dfrac{1+\alpha}{\log 2}\Big) - \dfrac{1+\alpha}{\log 2} + 1 & (0 \leqslant \alpha \leqslant \log 4 - 1) \\ \alpha & (\alpha > \log 4 - 1) \end{cases} \tag{1.25}$$

(note that $E(\alpha)$ is continuous at $\alpha = \log 4 - 1$ and that $E(0) = \delta$). In particular, we always have $E(\alpha) \leqslant \alpha + \delta$ $(\alpha \geqslant 0)$, so (1.21) for $\mathcal{A}'$ is certainly weaker than

$$\sum_{j=1}^{\infty} \frac{\log H_j}{\big(\log T_j\big)^{\delta+o(1)}} = +\infty,$$

which is in turn much weaker than (1.13) or (1.15).

If $\mathcal{A}_j$ appears in $\mathcal{A}''$ or $\mathcal{A}'''$, we have [12]

$$\Big(\frac{\log H_j}{\log T_j}\Big)^{\delta+o(1)} \ll \mathrm{d}\mathcal{M}(\mathcal{A}_j) \ll \Big(\frac{\log H_j}{\log T_j}\Big)^{\delta} \qquad (j \to +\infty). \tag{1.26}$$

In particular, we see again that (1.21) is much weaker than (1.15) for $\mathcal{A}''$. However, Theorem 3 seems to indicate that, in the case of $\mathcal{A}'''$, condition (1.21) is "close" to being necessary and sufficient.

A heuristic explanation for this phenomenon can be found in the fact that the property that a normal integer has a divisor in $\mathcal{A}_j$ depends essentially only on its prime factors in the range $(H_j, H_jT_j]$. When the H_j are "small" these intervals overlap a great deal and the corresponding properties become strongly dependent. Condition (1.21), which is based on a Borel–Cantelli type model, is then far from sufficiency. On the contrary, when the H_j are "large", the events "$n \in \mathcal{M}(\mathcal{A}_j)$" are determined by disjoint or almost disjoint intervals and (1.21) gets closer to the desired criterion.

We should like to thank here the referee for his careful reading of the paper and pertinent remarks.

2. *Proof of Theorem* 1

We introduce the arithmetic functions

$$\Omega(n;w,z) := \sum_{p^\nu \| n,\, w<p\leqslant z} \nu \qquad (1 \leqslant w \leqslant z)$$

and write, as usual, $\Omega(n) := \Omega(n;1,n)$.

Lemma 2.1. *Let $0<\varepsilon<\frac{1}{2}$ be fixed and $\xi(z)\to+\infty$. Then we have*

$$\max_{\substack{z_0<z\leqslant n \\ 2\leqslant w\leqslant \exp\{\log z/(\log_2 z)^{\xi(z)}\}}} \left| \frac{\Omega(n;w,z)-\log\left(\frac{\log z}{\log w}\right)}{\log\left(\frac{\log z}{\log w}\right)} \right| \leqslant \varepsilon \tag{2.1}$$

unless n belongs to a sequence $\mathcal{Y}(z_0)$ such that

$$\lim_{z_0\to+\infty} \overline{\mathrm{d}}\Big(\mathcal{Y}(z_0)\Big) = 0. \tag{2.2}$$

Proof. We introduce the checkpoints

$$t_j := \exp\exp\{6\varepsilon^{-2} j \log j\} \qquad (j=1,2,\ldots) \tag{2.3}$$

and, given an integer n, consider (w,z) realizing the maximum in the left–hand side of (2.1). We have for some $j,\ k \geqslant 1$

$$t_j < w \leqslant t_{j+1}, \ \ t_k < z \leqslant t_{k+1},$$

and note, for large enough z_0, that $k \geqslant j+2$, since we may then write

$$\log_2 z \geqslant \log_2 w + \xi(z)\log_3 z > \log_2 t_j + 20\varepsilon^{-2}\log(j+2) > \log_2 t_{j+2}.$$

Hence, if n does not satisfy (2.1), we must have either

$$\Omega(n;t_j,t_{k+1}) > (1+\varepsilon)\log\Big(\frac{\log t_k}{\log t_{j+1}}\Big), \tag{2.4}$$

or

$$\Omega(n; t_{j+1}, t_k) < (1-\varepsilon)\log\Big(\frac{\log t_{k+1}}{\log t_j}\Big). \tag{2.5}$$

The number of integers n satisfying (2.4) does not exceed

$$\begin{aligned}\sum_{n\leqslant x}\sum_{j\geqslant 1}\sum_{\substack{k\geqslant j+2\\ t_k>z_0}}&(1+\varepsilon)^{\Omega(n;t_j,t_{k+1})}\Big(\frac{\log t_k}{\log t_{j+1}}\Big)^{-(1+\varepsilon)\log(1+\varepsilon)}\\ &\ll x\sum_{j\geqslant 1}\sum_{\substack{k\geqslant j+2\\ t_k>z_0}}\Big(\frac{\log t_k}{\log t_{j+1}}\Big)^{-Q(1+\varepsilon)}\end{aligned} \tag{2.6}$$

with $Q(v) := v\log v - v + 1$. The above estimate follows immediately from the Halberstam–Richert inequality [7] for sums of nonnegative multiplicative functions — see also Theorem 01 of [10] — and we leave out the details.

Using the elementary lower bound

$$\frac{\log t_k}{\log t_{j+1}} \geqslant \exp\{6\varepsilon^{-2}(k-j-1)\log(j+1)\} \qquad (k > j+1)$$

and observing that $Q(1+\varepsilon) > \frac{1}{3}\varepsilon^2$ for $0 < \varepsilon < \frac{1}{2}$, we see that the right–hand side of (2.6) is

$$\ll x\sum_{j\geqslant 1}\sum_{h\geqslant \max(1,\, j_0-j)}(j+1)^{-2h}$$

for some $j_0 = j_0(z_0)$ tending to infinity with z_0. Thus the above double sum is $o(1)$ as $z_0 \to +\infty$ and we obtain that the upper density of the sequence defined by (2.4) tends to 0 as $z_0 \to +\infty$. Condition (2.5) may be treated similarly, and we hence deduce the required result.

Lemma 2.2. *We have uniformly for $x \geqslant z \geqslant y \geqslant 2$*

$$\Big|\Big\{n \leqslant x : \prod_{\substack{p^\nu || n\\ p\leqslant y}} p^\nu > z\Big\}\Big| \ll x\exp\Big\{-\frac{\log z}{2\log y}\Big\}.$$

This is a slightly improved version of Theorem 07 of [10], where a similar result is stated with an unspecified constant in the exponential. The details corresponding to the above bound may be found in [13] (exercise 5, p.437).

Completion of the proof of Theorem 1.

Let $\mathcal{A}$ be a Behrend block sequence, which we split as in (1.22). By Behrend's inequality (1.4), at least one of the three subsequences $\mathcal{A}'$, $\mathcal{A}''$, $\mathcal{A}'''$, must be itself

Behrend. If $\mathcal{A}'$ is a Behrend sequence, then Hall's Theorem implies that (1.13) holds for $\mathcal{A}'$ and this plainly implies (1.15). If now $\mathcal{A}'''$ is Behrend, then (1.4) implies by iteration that the series of the corresponding $\mathrm{d}\mathcal{M}(\mathcal{A}_j)$ diverges and by the upper bound of (1.26) we get

$$\sum\Big(\frac{\log H_j}{\log T_j}\Big)^{\delta} = +\infty,$$

where the summation is restricted to those indices j for which $\mathcal{A}_j$ appears in $\mathcal{A}'''$. This again yields the desired conclusion (1.15).

We may therefore assume that neither $\mathcal{A}'$ nor $\mathcal{A}'''$ is Behrend, or simply that (1.23) holds for all blocks $\mathcal{A}_j$ of $\mathcal{A}$.

Since any tail of $\mathcal{A}$ is still Behrend, there is no loss of generality in assuming T_1 is large. Also, by (1.3), we plainly have, for large enough J,

$$\mathrm{d}\mathcal{M}\Big(\bigcup_{1\leqslant j\leqslant J}\mathcal{A}_j\Big)\geqslant \tfrac{3}{4}. \tag{2.7}$$

Thus, if $\mathcal{N} = \mathcal{N}(\varepsilon, T_1)$ denotes the sequence of integers n satisfying (2.1) for $z_0 = T_1$, we must have

$$\underline{\mathrm{d}}\Big\{\mathcal{N}\cap\mathcal{M}\Big(\bigcup_{1\leqslant j\leqslant J}\mathcal{A}_j\Big)\Big\}\geqslant \tfrac{1}{2} \tag{2.8}$$

and *a fortiori*

$$\sum_{1\leqslant j\leqslant J}\overline{\mathrm{d}}\mathcal{N}_j\geqslant \tfrac{1}{2}\qquad \big(\mathcal{N}_j := \mathcal{N}\cap\mathcal{M}(\mathcal{A}_j),\ j=1,2,\ldots\big). \tag{2.9}$$

Put $u_j := (\log H_j)/\log T_j$ $(j\geqslant 1)$. We are going to prove that, for arbitrary $\beta < 1-\log 2$ and sufficiently small $\varepsilon > 0$, we have

$$\overline{\mathrm{d}}\mathcal{N}_j \ll (u_j)^{\beta}\qquad (j=1,2,\ldots). \tag{2.10}$$

This implies that (2.9) cannot hold for large T_1 if the series $\sum (u_j)^{\beta}$ converges and therefore yields the required conclusion that if the subsums of (1.15) corresponding to $\mathcal{A}'$ and $\mathcal{A}'''$ converge then the subsum corresponding to $\mathcal{A}''$ must diverge.

We now set out to prove (2.10). Put $v_j := 2\log(1/u_j)$. Next, write $\mathcal{N}_j = \mathcal{N}_{j1}\cup\mathcal{N}_{j2}$ where $\mathcal{N}_{j1}$ is the subset of those n in $\mathcal{N}_j$ such that

$$\prod_{\substack{p^{\nu}\|n\\ p\leqslant H_j}} p^{\nu} > (H_j)^{v_j}. \tag{2.11}$$

By Lemma 2.2, we have

$$\overline{\mathrm{d}}\mathcal{N}_{j1}\ll u_j, \tag{2.12}$$

which is plainly compatible with (2.10).

If n is counted by $\mathcal{N}_{j2}$, any divisor d of n in $(T_j, H_jT_j]$ may be written in the form ab where a divides the left–hand side of (2.11) and all prime factors of b exceed H_j. Since $1 \leqslant a \leqslant (H_j)^{v_j}$, we deduce that

$$T_j(H_j)^{-v_j} < b \leqslant T_jH_j \tag{2.12}$$

and we obtain that

$$\sum_{\substack{n\leqslant x\\ n\in\mathcal{N}_j}} 1 \leqslant \sum_{n\leqslant x} (\tfrac{1}{2})^{\Omega(n;H_j,T_j)}\Big(\frac{\log T_j}{\log H_j}\Big)^{(1+\varepsilon)\log 2}\sum_{b|n} 1$$

with the temporary convention that the letter b denotes generically an integer lying in the range (2.12) and free of prime factors $\leqslant H_j$. Using twice Theorem 01 of [10], we estimate the double sum above by

$$\ll x(u_j)^{(1/2)-(1+\varepsilon)\log 2}\sum_b 2^{-\Omega(b)}b^{-1}$$
$$\ll x(1+v_j)(u_j)^{1-(1+\varepsilon)\log 2} \ll x(u_j)^{\beta}$$

provided ε is sufficiently small. This yields (2.10) and completes the proof of Theorem 1.

3. *Proof of Theorem 2*

In the case of the sequence $\mathcal{A}(\lambda)$ defined by (1.18), the necessary condition (1.15) of Theorem 1 becomes

$$\sum_{j=1}^{\infty} j^{-\beta\lambda} = +\infty \qquad (\beta < 1-\log 2). \tag{3.1}$$

Hence $\mathcal{A}(\lambda)$ is certainly not a Behrend sequence when $\lambda > 1/(1-\log 2)$.

It remains to show that $\mathcal{A}(\lambda)$ is Behrend for all $1 \leqslant \lambda \leqslant 1/(1-\log 2)$, the result being trivial when $\lambda < 1$. We begin by introducing some notation.

For $k \geqslant 0$, we put $Z_k := \exp\exp k$, and define for every integer n

$$n_k := \prod_{\substack{p|n\\ p\leqslant Z_k}} p.$$

Note that n_k is a multiplicative function of n.

We let $R \geqslant 1$ be a parameter to be specified later, and put

$$J_1 = J_1(k) := 2R\mathrm{e}^{k/\lambda}, \qquad J_2 = J_2(k) := 3R\mathrm{e}^{k/\lambda}.$$

We next define

$$G(\vartheta) := \sum_{J_1 < j \leqslant J_2} \exp\{ij^\lambda \vartheta\}, \qquad H(\vartheta) := \int_0^{|\vartheta|} |G(\varphi)|^2 \,\mathrm{d}\varphi. \tag{3.2}$$

We also consider for every pair of integers $m \geqslant 1$, $k \geqslant 0$, the set

$$\mathcal{L}(m,k) := \bigcup_{\substack{d|m \\ J_1 < j \leqslant J_2}} \Big(j^\lambda - \log d +]0, \tfrac{1}{2}]\Big) \tag{3.3}$$

whose Lebesgue measure we denote by $\lambda(m,k)$.

For integer $m \geqslant 1$ and real $u > 0$, $\vartheta > 0$, we write

$$\omega(m,u) := \sum_{\substack{p|m \\ p \leqslant u}} 1, \qquad \omega_\vartheta(m) := \omega\big(m, \exp\{1/\vartheta\}\big). \tag{3.4}$$

and, as usual, $\omega(m) := \omega(m,m)$. Finally, we introduce the arithmetical functions

$$\tau(m,\vartheta) := \sum_{d|m} d^{i\vartheta}, \qquad I(m) := \int_0^1 |G(\vartheta)|^2 \frac{|\tau(m,\vartheta)|^2}{2^{2\omega(m)}} \,\mathrm{d}\vartheta. \tag{3.5}$$

We shall need a number of lemmas, the first two of which concern the distribution of the prime factors of normal integers.

Lemma 3.1. *Let $0 < \sigma < \frac{1}{10}$, $T \geqslant 1$, $x \geqslant 1$. Then we have*

$$\max_{2 \leqslant u \leqslant n} \Big\{\omega(n,u) - (1+\sigma)\log_2 u\Big\} \leqslant T \tag{3.6}$$

for all but at most $\ll \sigma^{-2} x (1+\sigma)^{-T}$ of the integers $n \leqslant x$.

This is a special case of Lemma 50.1 of [10].

Lemma 3.2. *Let $0 < \alpha < 1$. For integer $D > D(\alpha)$, and $x > x(\alpha, D)$, put*

$$t_0 := \left[\sqrt{\frac{\log_3 x}{\log D}}\right], \quad t_1 := \left[\frac{\log_3 x}{2\log D}\right], \quad k_t := D^t \quad (t_0 < t \leqslant t_1).$$

Then there is a constant $c = c(D) > 0$ such that

$$\max_{t_0 < t \leqslant t_1} \frac{\omega(n_{k_t})}{k_t + \sqrt{2\alpha k_t \log_2 k_t}} \geqslant 1 \tag{3.7}$$

holds for all but at most $\ll x \exp\{-c\sqrt{\log_4 x}\}$ of the integers $n \leqslant x$.

This is essentially established in the course of the proof of the the law of the iterated logarithm for the distribution of prime factors — Theorem 11 of [10]. The analysis p.20 of [10] yields the result immediately : we obtain that the number of exceptional n is

$$\ll x \prod_{t_0 < t \leqslant t_1} \Big(1 - \frac{c_0}{t \log D \sqrt{\log(t \log D)}}\Big) \ll x \exp\{-c_1 \sqrt{\log t_1}\}$$

for some positive constants c_0, c_1, such that c_0 is absolute and $c_1 = c_1(D)$. This implies the required estimate.

Lemma 3.3. *We have uniformly for $\vartheta \in \mathbb{R}$*

$$H(\vartheta) \ll R\mathrm{e}^{k/\lambda}\Big(|\vartheta| + \mathrm{e}^{-k(1-1/\lambda)}\Big). \tag{3.8}$$

Proof. Introduce the weight function

$$w(\varphi) := \frac{1}{2\pi}\left(\frac{\sin(\frac12\varphi)}{\frac12\varphi}\right)^2 \tag{3.9}$$

with Fourier transform $\widehat{w}(\vartheta) = \int_{-\infty}^{+\infty} \mathrm{e}^{-i\vartheta\varphi} w(\varphi)\,\mathrm{d}\varphi = (1-|\vartheta|)^+$. We have

$$\begin{aligned} H(\vartheta) \ll \int_{-\infty}^{+\infty} w(\tfrac{\varphi}{\vartheta})|G(\varphi)|^2\,\mathrm{d}\varphi = |\vartheta| \int_{-\infty}^{+\infty} w(\varphi)|G(\vartheta\varphi)|^2\,\mathrm{d}\varphi \\ =|\vartheta| \sum_{J_1<j,h\leqslant J_2} \widehat{w}\big(\big(\vartheta(j^\lambda - h^\lambda)\big). \end{aligned}$$

Now, we observe that $|j^\lambda - h^\lambda| \asymp |j-h|R^{\lambda-1}\mathrm{e}^{k(1-1/\lambda)}$. Hence, for a suitable absolute constant $c_2 > 0$, we have

$$\begin{aligned} H(\vartheta) \ll& |\vartheta| \sum_{|j-h|\leqslant c_2|\vartheta|^{-1}R^{1-\lambda}\mathrm{e}^{-k(1-1/\lambda)}} 1 \\ \ll& |\vartheta| R\mathrm{e}^{k/\lambda}\big(1 + |\vartheta|^{-1}R^{1-\lambda}\mathrm{e}^{-k(1-1/\lambda)}\big) \\ \ll& R\mathrm{e}^{k/\lambda}\big(|\vartheta| + \mathrm{e}^{-k(1-1/\lambda)}\big), \end{aligned}$$

as required.

Given $\alpha \in]0,1[$, $D > D(\alpha)$ and $x > x(\alpha, D)$, we denote by $\mathcal{S}(x,D)$ the set of integers $n \leqslant x$ that satisfy (3.7). For each $n \in \mathcal{S}(x,D)$, we let $t(n)$ be the smallest index t realizing (3.7), and we define $t(n) := t_0$ when $n \in [1,x] \smallsetminus \mathcal{S}(x,D)$. Further, we put

$$h := D^{t_0/10}. \tag{3.10}$$

We observe that it follows from (3.7) that, whenever $n \in \mathcal{S}(x,D)$,

$$\omega(n_k) > k + \sqrt{k} \qquad (k_{t(n)} < k \leqslant k_{t(n)} + h). \tag{3.11}$$

Lemma 3.4. *Let α, σ, D be fixed with $0 < \alpha < 1$, $0 < \sigma < \frac{1}{10}$, and $D \in \mathbb{Z}^+$, $D > D(\sigma)$. Then there is a positive constant $c_3 = c_3(\alpha, D)$ such that we have*

$$\max_{k_{t(n)} < k \leqslant k_{t(n)}+h} \mathrm{e}^{-k(2/\lambda-1)} I(n_k) \ll R \tag{3.12}$$

for all but at most

$$\ll x\bigg(\exp\Big\{-c_3\sqrt{\log_4 x}\Big\} + \sigma^{-2}(1+\sigma)^{-h}\bigg) \tag{3.13}$$

of the integers $n \leqslant x$.

Proof. Let $I_1(n_k)$ (resp. $I_2(n_k)$) denote the contribution to the integral $I(n_k)$ of the range $|\vartheta| \leqslant \mathrm{e}^{-k(1-1/\lambda)}$ (resp. $\mathrm{e}^{-k(1-1/\lambda)} < |\vartheta| \leqslant 1$). Estimating trivially $|\tau(n_k,\vartheta)|^2/2^{2\omega(n_k)} \leqslant 1$ and applying Lemma 3.3, we see that

$$I_1(n_k) \ll R\mathrm{e}^{k(2/\lambda-1)} \tag{3.14}$$

uniformly for all k, n. Thus we only need to show that the same upper bound remains valid for $I_2(n_k)$ with the uniformity required in (3.12) and all integers $n \leqslant x$ but at most the indicated number of exceptions.

To this end, we first set

$$v(\vartheta,k) := k + \sqrt{k} - (1+\sigma)\log(1/\vartheta) - h, \tag{3.14}$$

and notice that we may deduce from Lemma 3.1 and (3.11) that

$$\min_{k_{t(n)} < k \leqslant k_{t(n)}+h} \ \inf_{\mathrm{e}^{-k} < \vartheta \leqslant 1} \frac{\omega(n_k) - \omega_\vartheta(n_k)}{v(\vartheta,k)} \geqslant 1 \tag{3.15}$$

holds for all integers $n \leqslant x$ but at most those belonging to an exceptional set of size bounded by (3.13).

Denote by $\mathcal{G}(x)$ the set of integers $n \leqslant x$ that satisfy (3.15). For $n \in \mathcal{G}(x)$, we have

$$I_2(n_k) \leqslant \int_{\exp\{-k(1-1/\lambda)\}}^1 \frac{|G(\vartheta)|^2|\tau(n_k,\vartheta)|^2}{2^{\omega(n_k)+\omega_\vartheta(n_k)}} 2^{-v(\vartheta,k)}\,\mathrm{d}\vartheta. \tag{3.16}$$

Hence

(3.17)
$$\sum_{n\in\mathcal{G}(x)} \max_{k_{t(n)} < k \leqslant k_{t(n)}+h} \mathrm{e}^{-k(2/\lambda-1)} I_2(n_k)$$
$$\leqslant \sum_{t_0 < t \leqslant t_1} \sum_{k_t < k \leqslant k_t+h} \mathrm{e}^{-k(2/\lambda-1)} \int_{\exp\{-k(1-1/\lambda)\}}^1 \frac{H'(\vartheta)}{2^{v(\vartheta,k)}} \sum_{n\leqslant x} \frac{|\tau(n_k,\vartheta)|^2}{2^{\omega(n_k)+\omega_\vartheta(n_k)}}\,\mathrm{d}\vartheta.$$

The inner n-sum may be easily majorized by Theorem 01 of [10]. We find that it is

$$\ll x\exp\Big\{\sum_{\exp(1/\vartheta)<p\leqslant Z_k}\frac{\cos(\vartheta\log p)}{p}\Big\}\ll x$$

where the second estimate follows from Lemma 30.1 of [10]. Next, we have

$$\begin{aligned}\int_{\exp\{-k(1-1/\lambda)\}}^{1} & H'(\vartheta)2^{-v(\vartheta,k)}\,\mathrm{d}\vartheta\\ &\ll H(1)2^{-v(1,k)}+\int_{\exp\{-k(1-1/\lambda)\}}^{1}H(\vartheta)\vartheta^{-1}2^{-v(\vartheta,k)}\,\mathrm{d}\vartheta\\ &\ll R2^{-k}\mathrm{e}^{k/\lambda-\sqrt{k}/2}.\end{aligned}$$

For $\lambda\leqslant 1/(1-\log 2)$, we have $1/\lambda-\log 2\leqslant 2/\lambda-1$, hence

$$\mathrm{e}^{k/\lambda}2^{-k}\leqslant\mathrm{e}^{k(2/\lambda-1)}.$$

This implies that the right-hand side of (3.17) is

$$\ll xRht_1\mathrm{e}^{-2h}\ll xR\,\mathrm{e}^{-h}. \tag{3.18}$$

Thus, the number of $n\in\mathcal{G}(x)$ which do not satisfy (3.12) is $\ll x\mathrm{e}^{-h}$. This is of smaller order of magnitude than (3.13), and this estimate thereby completes the proof of Lemma 3.4.

Lemma 3.5. *For squarefree m and large k, we have*

$$\lambda(m,k)\geqslant\tfrac{1}{11\pi}R^2\mathrm{e}^{2k/\lambda}I(m)^{-1}. \tag{3.19}$$

Proof. Let w be defined by (3.9), and put

$$F(z):=\sum_{J_1<j\leqslant J_2}\ \sum_{\substack{d\mid m\\ j^\lambda-\log d<z\leqslant j^\lambda-\log d+\frac12}}1$$

so that $\lambda(m,k)$ is exactly the measure of the set of real z such that $F(z)\neq 0$. Now, we have for large k

$$\int_{-\infty}^{+\infty}F(z)\,\mathrm{d}z=\Big\{J_2-J_1+O(1)\Big\}2^{\omega(m)-1}>\tfrac49R\,\mathrm{e}^{k/\lambda}\,2^{\omega(m)}. \tag{3.20}$$

On the other hand, we have for all z

$$F(z)\leqslant\frac{1}{w(\frac12)}\sum_{\substack{d\mid m\\ J_1<j\leqslant J_2}}w(j^\lambda-\log d-z)=\frac{1}{2\pi w(\frac12)}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\vartheta z}G(\vartheta)\overline{\tau(m,\vartheta)}\widehat{w}(\vartheta)\,\mathrm{d}\vartheta.$$

By Plancherel's formula we thence infer

$$\int_{-\infty}^{+\infty} F(z)^2 \,\mathrm{d}z \leqslant \frac{1}{2\pi w(\frac12)^2} \int_{-1}^{1} |G(\vartheta)|^2 |\tau(m,\vartheta)|^2 \,\mathrm{d}\vartheta. \tag{3.21}$$

The required lower bound (3.19) follows from this and (3.20), in view of the Cauchy–Schwarz inequality

$$\Big(\int_{-\infty}^{+\infty} F(z)\,\mathrm{d}z\Big)^2 \leqslant \lambda(m,k) \int_{-\infty}^{+\infty} F(z)^2 \,\mathrm{d}z.$$

Lemma 3.6. *There is an absolute constant* $c_4 > 0$ *such that*

$$\min_{k_{t(n)}<k\leqslant k_{t(n)}+h} \lambda(n_k,k)\mathrm{e}^{-k} > c_4 R \tag{3.22}$$

holds for all integers $n \leqslant x$ *except those that belong to a set of cardinality majorized by* (3.13).

This follows immediately from Lemmas 3.4 and 3.5.

Lemma 3.7. *For* $R := 4\log h$*, we have*

$$\max_{k_{t(n)}<k\leqslant k_{t(n)}+h} \mathrm{e}^{-k}\log n_k \leqslant R \tag{3.23}$$

for all integers $n \leqslant x$ *except at most*

$$\ll x/\sqrt{h}. \tag{3.24}$$

Proof. From Lemma 2.2, we know that for *fixed* k the number of integers n such that $\log n_k > R\,\mathrm{e}^k$ is $\ll x\exp\{-R/2\} = xh^{-2}$. Since we want uniformity in a set of indices k of size $\leqslant (t_1 - t_0 + 1)h < h^{3/2}$, we obtain the estimate stated.

Completion of the proof of Theorem 2.

We use an inductive argument similar to that of Theorem 51 of [10], where a result stronger than (1.8) was established.

We assume throughout that $\lambda \geqslant 1$, since the result is otherwise trivial. Given a large real number x, and h being defined by (3.10), we consider for $1 \leqslant s \leqslant h$ the quantity

$$N_s := \Big|\Big\{n \leqslant x : \max_{\substack{d|n_{k_{t(n)}+s}\\ j^\lambda<\log d}} (j^\lambda - \log d) < -\tfrac12\Big\}\Big|.$$

This is plainly a decreasing function of s and we set out to show that $N_h = o(x)$ whenever $\lambda \leqslant 1/(1-\log 2)$. Indeed, this implies that all but at most $o(x)$ of the integers $n \leqslant x$ belong to $\mathcal{M}\big(\mathcal{A}(\lambda)\big)$.

Let α, σ, D be fixed as in Lemma 3.3. By Lemmas 3.6 and 3.7, there is a constant $c_5 = c_5(\alpha, \sigma, D) > 0$ such that, for suitable absolute K and $\eta := K \exp\{-c_5\sqrt{\log_4 x}\}$, relations (3.22) and (3.23) are satisfied for all but at most ηx of the integers $n \leqslant x$.

For each s, let N'_s denote the number of those n counted in N_s and which also satisfy (3.22) and (3.23). We obviously have

$$N_s \leqslant N'_s + \eta x \qquad (1 \leqslant s \leqslant h). \tag{3.25}$$

We may assume that $N_h > 2\eta x$, since otherwise there is nothing to prove. By (3.25), we then have, for every s, that $N_s \leqslant N'_s + \frac{1}{2}N_h \leqslant N'_s + \frac{1}{2}N_s$, whence

$$\tfrac{1}{2}N_s \leqslant N'_s. \tag{3.26}$$

Let $\mathcal{M}'_s$ denote the set of all integers m of the form $m = n_{k_{t(n)+s}}$ for some n counted by N'_s. We have a natural partition

$$\mathcal{M}'_s = \bigcup_{t_0 \leqslant t \leqslant t_1} \mathcal{M}'_{s,t}$$

where $\mathcal{M}'_{s,t}$ contains those m of the form $n_{k_{t(n)}+s}$ with $t(n) = t$. By the definition of $t(n)$, a given m may be obtained from several n but then all corresponding $t(n)$ must be equal. Hence every m belongs to exactly one $\mathcal{M}'_{s,t}$.

By the sieve, we have for all $s \quad (1 \leqslant s \leqslant h)$

$$\tfrac{1}{2}N_s \leqslant N'_s \ll \sum_{t_0 \leqslant t \leqslant t_1} \sum_{m \in \mathcal{M}'_{s,t}} \frac{x}{\varphi(m)} \mathrm{e}^{-k_t - s}. \tag{3.27}$$

Indeed condition (3.23) and the definition of t_1 in Lemma 3.2 imply that any m in $\mathcal{M}_s$ is less than $x^{o(1)}$ — which is more than sufficient for the Fundamental Lemma of sieve theory to be applicable.

Next, let D_s denote the number of those $n \leqslant x$ which can be written in the form $n = m\ell pb$ for some $m \in \mathcal{M}'_s$, with $\ell | m$ and

(i) $\quad k_t + s < \log_2 p \leqslant k_t + s + r$

(ii) $\quad \log p \in \mathcal{L}(m, k_t + s)$

(iii) $\quad q|b \quad (q \text{ prime}) \Rightarrow q > Z_{k_t+s+r}$

where $r := 1 + \big[\lambda \log(4R)\big]$ and t is uniquely determined by the property that $m \in \mathcal{M}'_{s,t}$. Condition (ii) implies that, for some $d|m$ and some $j \geqslant 0$,

$$j^\lambda < \log(pd) \leqslant j^\lambda + \tfrac{1}{2}.$$

Thus, any n couted by D_s is counted by N_s but not by N_{s+r}, and we deduce that

$$D_s \leqslant N_s - N_{s+r}. \tag{3.28}$$

We now have

$$D_s \geqslant \sum_{t_0\leqslant t\leqslant t_1} \sum_{m\in\mathcal{M}'_{s,t}} \sum_{\ell\mid m} \sum_{\substack{Z_{k_t+s}<p\leqslant Z_{k_t+s+r}\\ \log p\in\mathcal{L}(m,k_t+s)}} \sum_{\substack{b\leqslant x/m\ell p\\ q\mid b\Rightarrow q>Z_{k_t+s+r}}} 1.$$

By the sieve, the inner b-sum is $\gg \mathrm{e}^{-k_t-s-r}x/m\ell p$. We next estimate the p-sum observing that condition (3.23) implies

$$\mathcal{L}(m,k_t+s)\subset\Big[\big((2R)^\lambda-R\big)\mathrm{e}^{k_t+s},(3R)^\lambda\mathrm{e}^{k_t+s}+\tfrac12\Big]$$

so that our choice of r guarantees that the first summation condition, viz

$$Z_{k_t+s}<p\leqslant Z_{k_t+s+r},$$

is redundant. Thus, we find that the p-sum is

$$\geqslant \mathrm{e}^{-k_t-s-r}\sum_{\log p\in\mathcal{L}(m,k_t+s)}\frac{\log p}{p}\gg \mathrm{e}^{-k_t-s-r}\lambda(m,k_t+s), \tag{3.29}$$

by the prime number theorem. Indeed, $\mathcal{L}(m,k_t+s)$ is a union of intervals which can be assumed to be disjoint and of type $]a,a+b]$, with $b\geqslant\frac12$. For each such interval, the corresponding subsum over p is $b+o(1)\gg b$, and adding the b's gives (3.29). Now, we see that (3.22) implies that the right–hand side of (3.29) is $\gg R^{1-\lambda}$.

Hence we finally obtain that

$$\begin{aligned} D_s &\gg R^{1-2\lambda}x\sum_{t_0\leqslant t\leqslant t_1}\sum_{m\in\mathcal{M}'_{s,t}}\mathrm{e}^{-k_t-s}\varphi(m)^{-1}\\ &\gg N_sR^{1-2\lambda}, \end{aligned}$$

in view of (3.27). From this and (3.28), it follows that, for a suitable absolute $c_6>0$, we have

$$N_{s+r}\leqslant N_s\big(1-c_6R^{1-2\lambda}\big).$$

Iterating, we get

$$N_h\ll N_1\big(1-c_6R^{1-2\lambda}\big)^{h/r}\ll x\exp\big\{-c_6hr^{-1}R^{1-2\lambda}\big\}.$$

Since this is of smaller order of magnitude than ηx, we have shown that the estimate

$$N_h\ll\eta x=o(x)$$

holds in any circumstance. This completes the proof.

4. *Proof of Theorem* 3

We denote by $\chi(n;y,z)$ the characteristic function of $\mathcal{M}(\{d : y < d \leqslant z\})$. Furthermore, we shall use systematically the quantity u, implicitely defined by the relation

$$z = y^{1+u}.$$

Lemma 4.1. *Let $\varepsilon > 0$ be fixed. Uniformly in the range*

$$y \geqslant 2, \qquad 2y \leqslant z \leqslant \min\left(y^2, x^{1/\log_2 x}\right) \tag{4.1}$$

we have

$$\sum_{\substack{m\leqslant x\\ p|m\Rightarrow y^v<p\leqslant z}} \frac{\chi(m;y,z)}{m} \gg u^{\delta-1} \quad (v := u^{1+\varepsilon}). \tag{4.2}$$

Proof. We may assume $u \leqslant u_0 = u_0(\varepsilon)$ since the left–hand side of (4.2) is always as large as $\sum_{y<p\leqslant y^{1+u}} p^{-1}$.

We decompose generically the integers $n \leqslant x$ in the form $n = amb$, where all prime factors of a (resp. b) are $\leqslant y^v$ (resp. $> z$) and m satisfies the property indicated in (4.2). By Theorem 21 of [10], we have

$$\sum_{n\leqslant x} \chi(n;y,z) \gg xu^{\delta(1+\varepsilon/4)}. \tag{4.3}$$

Our first aim is to show that this still holds if $\chi(n;y,z)$ is replaced by $\chi(m;y,z)$. To this end, we observe that if $\chi(n;y,z) = 1$ but $\chi(m;y,z) = 0$ then $\chi(m;y/a,y) = 1$. Indeed, any divisor of n in the range $(y,z]$ may be written as a_1m_1 with $a_1|a$, $m_1|m$, and, since $m_1 \notin (y,z]$, we must have $y/a \leqslant y/a_1 < m_1 \leqslant y$. Now Lemma 2.2 implies that, for $w := u^{1+\varepsilon/2}$, we have $a \leqslant y^w$ for all but at most

$$x\exp\{-\tfrac{1}{2}u^{-\varepsilon/2}\} \ll xu \tag{4.4}$$

of the integers $n \leqslant x$. For non–exceptional n, we hence have $\chi(n;y^{1-w},y) = 1$ and Theorem 21 of [10] then yields that the number of such integers is at most

$$\ll xw^\delta = xu^{\delta(1+\varepsilon/2)}. \tag{4.5}$$

As (4.4) and (4.5) are of smaller order of magnitude than (4.3), we obtain as desired that, for suitable $u_0(\varepsilon)$, we have

$$\sum_{n\leqslant x} \chi(m;y,z) \gg xu^{\delta(1+\varepsilon/4)}. \tag{4.6}$$

Next we note that Lemma 2.2 implies that $m \leqslant z^{(1/2)\log(1/u)}$ for all but at most $O(xu^{1/4})$ of the integers $n \leqslant x$. Since $u \geqslant \log 2/\log y > 1/\log x$ under condition (4.1), we see that (4.6) still holds with the extra summation condition $m \leqslant \sqrt{x}$.

Thus, we may write

$$\sum_{m\leqslant\sqrt{x}} \chi(m;y,z) \sum_{ab\leqslant x/m} 1 \gg xu^{\delta(1+\varepsilon/4)}. \tag{4.7}$$

Since $x/m > \sqrt{x} > z^2$ (say), we may apply the sieve to estimate the inner sum from above. We find that it is $\ll vx/m$ and this yields the required result.

Completion of the proof of Theorem 3.

We define

$$u_j := \frac{\log H_j}{\log T_j}, \quad v_j := u_j^{1+\varepsilon}, \quad H_j^* := T_j^{v_j} = \exp\left\{(\log H_j)^{1+\varepsilon}(\log T_j)^{-\varepsilon}\right\}$$

and write, for every integer $n \leqslant x$,

$$n_j := \prod_{\substack{p^\nu \| n \\ H_j^* < p \leqslant H_j T_j}} p^\nu.$$

Moreover, we denote generically by m_j an integer all of whose prime factors lie in the range $I_j := (H_j^*, H_j T_j]$. By (1.19), we have $H_{j+1}^* > T_j^2 > H_j T_j$ for all j, so the intervals I_j are pairwise disjoint. For fixed k and large x, we may write

$$\begin{aligned}\sum_{\substack{n\leqslant x \\ n\notin\mathcal{M}(\mathcal{A})}} 1 &\leqslant \sum_{n\leqslant x} \prod_{j=1}^k \Big(1 - \chi(n;T_j,H_jT_j)\Big) \\ &\leqslant \sum_{n\leqslant x} \prod_{j=1}^k \Big(1 - \chi(n_j;T_j,H_jT_j)\Big) \\ &\ll x \sum_{m_1,\ldots,m_k} \prod_{j=1}^k \Big(1 - \chi(m_j;T_j,H_jT_j)\Big) \frac{w_j}{m_j}\end{aligned} \tag{4.8}$$

by the sieve, with

$$w_j := \prod_{H_j^* < p \leqslant H_j T_j} (1 - \frac{1}{p}) \asymp v_j.$$

Now Lemma 4.1 enables us to write, for some positive constant $c_8 = c_8(\varepsilon)$,

$$\sum_{m_1,\dots,m_k} \prod_{j=1}^{k} \Big(1-\chi(m_j;T_j,H_jT_j)\Big)\frac{w_j}{m_j}$$
$$= \prod_{j=1}^{k} \left\{ w_j \sum_{m_j} \frac{1}{m_j} - w_j \sum_{m_j} \frac{\chi(m_j;T_j,H_jT_j)}{m_j} \right\}$$
$$\leqslant \prod_{j=1}^{k} \Big(1-c_8 u_j^{\delta+\varepsilon}\Big).$$

By (1.20), this tends to 0 as $k \to +\infty$. Inserting in (4.8), we deduce that $\mathcal{A}$ is Behrend.

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