On Arithmetic Functions Involving Consecutive Divisors

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Dedicated to Paul Bateman

§ 1. Introduction.

This article is motivated by several questions posed in [3], which we can now at least partially answer.

Let $1 = d_1 < d_2 < \ldots < d_{\tau(n)} = n$ denote the increasing sequence of divisors of a general integer n. A quantitative measure of the growth of the d_i is provided by the arithmetic function

$$H(n) := \sum_{1 \le i < \tau(n)} (d_{i+1} - d_i)^{-1}.$$

It is established in [3] that

$$H(n) \ll \tau(n)(\log \tau(n))^{-\frac{1}{3}+\varepsilon} \qquad (n \ge 2) \tag{1.1}$$

holds for any fixed $\varepsilon > 0$ and that

$$\max_{n \le x} H(n) > \exp\left\{(\log x)^{\frac{1}{2}+o(1)}\right\} \qquad (x \to \infty). \tag{1.2}$$

Our first result is an upper bound for the left hand side of (1.2) in terms of the quantity

$$D(x) := \max_{n \le x} \tau(n) = 2^{(1+o(1))\frac{\log x}{\log_2 x}} \qquad (x \to \infty).$$

(Here and in the sequel we let $\log_k$ denote the k-fold iterated logarithm.)

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Theorem 1. *Set* $c = \frac{5}{3} - \frac{\log 3}{\log 2} = 0.08170$. *Then we have*

$$\max_{n \le x} H(n) \le D(x)^{1-c+o(1)} \qquad (x \to \infty). \tag{1.3}$$

An analogous bound, with an unspecified constant c, has been independently obtained by Erdös and Sárközy, with a different method (unpublished). It emerges from (1.3) that (1.1) can be significantly improved in the case when $\tau(n)$ is "large". On the other hand, it follows from Theorem 9 of [3] that

$$H(n) \ll \tau(n)^{1-c} \log(1 + \omega(n)) \tag{1.4}$$

holds, with the same value of c, whenever n is squarefree. (We let $\omega(n)$ denote the number of distinct prime factors of n.) This leads to the conjecture that a bound of the type

$$H(n) \ll \tau(n)^{1-\delta} \tag{1.5}$$

with an absolute $\delta > 0$ could hold unconditionally. We haven't been able up to now to prove or disprove this hypothesis.

The lower bound (1.2) is probably not optimal, but it seems difficult to make a reasonable guess concerning the maximal order of $H(n)$. One trivially has

$$H(n) \ge \kappa(n) := \sum_{d(d+1)|n} 1 \tag{1.6}$$

and the function $\kappa(n)$ raises an interesting open problem. We certainly believe that

$$\kappa(n) \ll_\varepsilon \tau(n)^\varepsilon$$

holds for any $\varepsilon > 0$, but no upper estimate is actually available other than those which follow, via (1.6), from the results on $H(n)$. Erdös and Hall established in [2] the asymptotic inequality

$$\max_{n \le x} \kappa(n) > (\log x)^{\sqrt{e}+o(1)} \qquad (x \to \infty).$$

We can strengthen this estimate in the following way.

Theorem 2. *We have*

$$\max_{n \le x} \kappa(n) > (\log x)^{\frac{\log_3 x}{9 \log_4 x}} \qquad (x \to \infty). \tag{1.7}$$

This confirms a conjecture of Erdös. The analogous problem for the counting function of those divisors of the form $d(d+1)\dots(d+t-1)$ with fixed $t > 2$ seems much more difficult and we do not know in this case whether the maximal order exceeds an arbitrary power of $\log n$. Erdös and Hall prove in [2] that a power α_t is acceptable provided $\alpha_t < e^{1/t}$.

Our proof of Theorem 2 rests on an effective version of a result of Hildebrand [8] which is of independent interest. Let $P^+(n)$ (resp. $P^-(n)$) denote the largest (resp. the smallest) prime factor of n, with the convention $P^+(1) = 1$, $P^-(1) = +\infty$. Moreover, let us systematically put

$$u := \frac{\log x}{\log y} \qquad (x \geq y \geq 2).$$

The key to Theorem 2 is the following

Theorem 3. *The estimate*

$$\sum_{\substack{n \leq x \\ P^+(n(n+1)) \leq y}} 1 \gg x u^{-u^{7u}} \tag{1.8}$$

holds uniformly in the range

$$x \geq 3, \quad \max\{2, x^{\frac{8 \log_3 x}{\log_2 x}}\} \leq y \leq x. \tag{1.9}$$

Let $\rho(u)$ denote Dickman's function. It is known [9,10] that one has

$$\Psi(x,y) := \sum_{\substack{n \leq x \\ P^+(n) \leq y}} 1 \sim x\rho(u) \tag{1.10}$$

as x, y tend to infinity in the range

$$\exp\left\{(\log_2 x)^{\frac{5}{3}+\varepsilon}\right\} \leq y \leq x,$$

and the Riemann Hypothesis implies the persistence of (1.10) in any region of the type

$$(\log x)^{\xi(x)} \leq y \leq x \tag{1.11}$$

where $\xi(x) \to \infty$, see [7,13]. It is hence natural to conjecture that the left hand side of (1.8) is asymptotically $(1 + o(1))x\rho(u)^2$ when $x, y \to \infty$ in the range (1.11). Such a result would provide a strong measure of the multiplicative independence of n and $n + 1$, but seems at present very difficult, if not out of reach, even in a more modest region like $x^\varepsilon \leq y \leq x$.

In order to prove Theorem 3, we establish an elementary lower bound for the quantity

$$\Psi(x, y; a, q) := \operatorname{card}\{n \leq x : P^+(n) \leq y,\ n \equiv a (\operatorname{mod} q)\} \tag{1.12}$$

under the hypotheses

$$(\log x)^3 \le y \le x, \quad 1 \le q \le y^{1-\epsilon}, \ (a,q) = 1.$$

This is the content of Lemma 3.2 below. In this context it is also natural to conjecture that

$$\Psi(x,y;a,q) \sim \frac{x}{q}\rho(u) \tag{1.13}$$

holds uniformly as $x, y \to \infty$ in the range (1.11) and $q = o(y)$. Fouvry and Tenenbaum have shown in [4] that (1.13) actually holds when

$$\exp\{c_0(\log_2 x)^2\} \le y \le x, \ 1 \le q \le e^{c_1\sqrt{\log y}} \tag{1.14}$$

where c_0, c_1 are absolute constants.

The bounds (1.2) and (1.3) summarize our knowledge on the maximal order of $H(n)$. The average behaviour of this function is given by the formula

$$\sum_{n \le x} H(n) = Bx + O\Big(x\frac{(\log_2 x)^3}{\log x}\Big), \tag{1.15}$$

proved in [3], sharpening an estimate of Ivič and De Koninck [12]. As for the normal behaviour, it is established in [3] that $H(n)$ has a distribution function. We are now able to provide some extra information.

Theorem 4. *The arithmetic function $H(n)$ has a distribution function which is everywhere continuous on the real line.*

We derive this result in Sect.4 from a theorem of Behrend concerning primitive sequences and an inequality proved in [3] which is essentially equivalent to (1.15).

§ 2. Proof of Theorem 1.

Let $\omega_1(n)$ denote the number of prime factors p of n such that $p^2 \nmid n$. From [3] (Th. 9) we have

$$H(n) \ll \tau(n) B_1^{\omega_1(n)} \log_2 n \qquad (n \ge 3) \tag{2.1}$$

with $B_1 = 3.2^{-\frac{5}{3}} = 0.94494$.

Lemma 2.1. *Set $\lambda := \frac{\log 3}{\log 4}$. We have*

$$\tau(n) \le D(n)^{\lambda + o(1)} 2^{(1-\lambda)\omega_1(n)} \qquad (n \to \infty). \tag{2.2}$$

Proof. Consider the canonical decomposition $n = ab$ where $a = \prod_{p\|n} p$. We have

$$\tau(n) = \tau(a)\tau(b) = 2^{\omega_1(n)}\tau(b). \tag{2.3}$$

Let t be a parameter chosen freely in the range $2 \le t \le b$. Then

$$\tau(b) = \prod_{p^\nu\|b}(1+\nu) \le \prod_{\substack{p^\nu\|b\\ p\le t}}\left(1+\frac{\log b}{\log 2}\right)\prod_{\substack{p^\nu\|b\\ p>t}} 2^{\lambda\nu}$$

where we have used the fact that $p^\nu\|b$ implies $\nu \ge 2$, whence $1+\nu \le 2^{\lambda\nu}$. If $b > 1$, it follows that

$$\tau(b) \le \left(1+\frac{\log b}{\log 2}\right)^{\pi(t)} 2^{\lambda\frac{\log b}{\log t}} \le D(b)^{\lambda+o(1)}$$

for the choice $t = \log b/(\log_2 b)^2$, where the quantity $o(1)$ above is defined for all $b > 1$ and is bounded for bounded b. Taking into account the easy estimate

$$D(a)D(b) \le D(ab)^{1+o(1)} \qquad (ab \to \infty)$$

we infer that

$$\tau(b) \le \left(\frac{D(n)}{D(a)}\right)^{\lambda+o(1)} \le \left(D(n)2^{-\omega(a)}\right)^{\lambda+o(1)}.$$

Inserting this in (2.3), we get (2.2).

Theorem 1 is an immediate consequence of (2.2) and (2.1), in view of the estimate $\omega_1(n) \le \omega(n) \le (1+o(1))\frac{\log n}{\log_2 n}$.

§ 3. Proof of Theorem 3.

We use two auxiliary results. We say that an ordered set of integers $S = \{m_1 < m_2 < \ldots < m_R\}$ is *special* if we have

$$m_j - m_i = (m_i, m_j) \qquad (1 \le i < j \le R). \tag{3.1}$$

Lemma 3.1 (Heath–Brown, [6]). *There exists an absolute constant α such that, for every integer $R \ge 1$, there is a special set of R elements such that*

$$m_R \le R^{\alpha R^3}. \tag{3.2}$$

Lemma 3.2. *Let $0 < \varepsilon < 1$. With the notation (1.12) the estimate*

$$\Psi(x, y; a, q) \gg_\varepsilon \frac{x}{q}u^{-2u} \tag{3.3}$$

is uniformly valid under the conditions

$$x \geq 2,\ (\log x)^3 \leq y \leq x,\ 1 \leq q \leq y^{1-\epsilon}, (a,q) = 1. \tag{3.4}$$

Proof. Put $a_0 := 1$ if $q = 1$, $a_0 := a - q[a/q]$ otherwise. Then a_0 is counted by $\Psi(x, y; a, q)$, and this is always ≥ 1. We may therefore suppose without loss of generality that $x \geq x_0(\epsilon)$, whence $y \geq y_0(\epsilon)$.

Put $k := [u] - 1$, $\eta := \frac{1}{2}\varepsilon(1 - \frac{1}{6}\varepsilon)$. We obtain a lower bound for $\Psi(x, y; a, q)$ by counting all the integers n not exceeding x which have a representation in the form $n = mh\ell$ with the following conditions

(a) $p \mid m \Rightarrow p \in I_q := \{p : p \nmid q,\ y^{1-1/u} < p \leq y\},\ \Omega(m) = k;$

(b) $p \mid h \Rightarrow p \in J_q := \{p : p \nmid q,\ y^{\eta} < p \leq y^{\frac{1}{2}\varepsilon}\};$

(c) $xy^{-1} \leq mh \leq xy^{\varepsilon - 1};$

(d) $\ell \equiv a\overline{m}\overline{h} \pmod q$.

Here and in the sequel, the letter p denotes exclusively a prime number. The symbols $\overline{m}, \overline{h}$ refer to the respective inverses of m, h modulo q.

When $1 \leq u \leq 2$, we have $m = 1$. Otherwise, I_q and J_q are disjoint. Thus, in any case $(m, h) = 1$. Furthermore, condition (c) implies

$$y^{1-\epsilon} \leq x/mh \leq y. \tag{3.5}$$

Since $l \leq x/mh$, it follows that the number of prime factors, counted with multiplicity, of (ℓ, mh) is at most $1/\eta$. But they must be chosen among the prime factors of n which belong to $]y^\eta, x]$ — and these are not more than $1 + [u/\eta]$ in number. Hence the total number of representations of a given n in the form $mh\ell$ is $\leq \binom{1+[u/\eta]}{[1/\eta]} \ll_\varepsilon u^{1/\eta}$.

Now, inequality (3.5) shows that for fixed m, h there are at least

$$\left[\frac{x}{mhq}\right] \geq \frac{x}{2mhq}$$

values of ℓ satisfying (d). Hence we can write

$$\Psi(x, y; a, q) \gg_\varepsilon u^{-\frac{1}{\eta}} \frac{x}{q} \sum \frac{1}{m} \sum \frac{1}{h} \tag{3.6}$$

where, by convention, the letters m, h denote integers subjected to constraints (a), (b), (c).

For each m, put $T := \frac{2\log(x/my)}{\varepsilon \log y}$; then

$$T \leq \frac{2}{\varepsilon}\left\{u - 1 - k\left(1 - \frac{1}{u}\right)\right\} \leq \frac{4}{\varepsilon},$$

and the length of the interval $\left[T/(1-\frac{1}{6}\varepsilon), T+2\right]$ is at least

$$2-T\varepsilon/(6-\varepsilon)\geq 2-4/(6-\varepsilon)\geq 6/5>1.$$

It therefore contains an integer, say s. We restrict h to run through the products of s (not necessarily distinct) primes from J_q. We have in this circumstance

$$\frac{x}{my}\leq y^{\eta T/(1-\frac{1}{6}\varepsilon)}\leq y^{\eta s}\leq h\leq y^{\frac{1}{2}\varepsilon s}\leq y^{\frac{1}{2}\varepsilon(T+2)}\leq\frac{x}{my^{1-\varepsilon}}.$$

This shows that condition (c) is always fulfilled.

As $y\to\infty$, we have

$$\sum_{p\in J_q}\frac{1}{p}\geq\sum_{y^\eta<p\leq y^{\frac{1}{2}\varepsilon}}\frac{1}{p}-\frac{\omega(q)}{y^\eta}\geq\log\left(\frac{1}{1-\frac{1}{6}\varepsilon}\right)+o(1).$$

This sum is hence $\gg_\varepsilon 1$ for $y\geq y_0(\varepsilon)$. Since $s\ll_\varepsilon 1$, it follows that

$$\sum\frac{1}{h}\geq\frac{1}{s!}\Big(\sum_{p\in J_q}\frac{1}{p}\Big)^s\gg_\varepsilon 1. \tag{3.7}$$

It remains to estimate $\sum\frac{1}{m}$. We may plainly suppose that $u\geq 2$. We then have

$$\sum_{p\in I_q}\frac{1}{p}\geq\sum_{y^{1-\frac{1}{u}}<p\leq y}\frac{1}{p}-\frac{\omega(q)}{\sqrt{y}}=L+O\Big(\frac{\log y}{\sqrt{y}}\Big),$$

say. From the prime number theorem

$$L=\log\left(\frac{u}{u-1}\right)+O\Big(e^{-\sqrt{\log y}}\Big).$$

This implies

$$\sum_{p\in I_q}\frac{1}{p}\geq\frac{1}{2u} \tag{3.8}$$

provided $y_0(\varepsilon)$ is sufficiently large and $\log y\geq(\log_2 x)^3$. Moreover,for

$$(\log x)^3\leq y\leq\exp\left\{(\log_2 x)^3\right\}$$

we have

$$1+y^{-\frac{1}{3}}\leq y^{\frac{1}{u}}\leq 1+o(1)\qquad(x\to\infty)$$

and Huxley's theorem [11] on the distribution of primes in short intervals gives the estimate $L \geq (1+o(1))\frac{1}{u}$. Thus (3.8) is again valid. For suitable $y_0(\varepsilon)$ we may therefore write

$$\sum \frac{1}{m} \geq \frac{1}{k!}\Big(\sum_{p\in I_q} \frac{1}{p}\Big)^k \gg \big(\frac{e}{2}\big)^u u^{-2u}.$$

Taking (3.6) and (3.7) into account, we readily obtain the required estimate.

Completion of proof of Theorem 3.

We may suppose x and u sufficiently large. Indeed the left hand side of (1.8) always counts $n=1$, hence is ≥ 1, and is for fixed x a decreasing function of u.

Put $R := [(2u)^{2u}]$, $M := R^{2\alpha R^3}$. From Lemma 3.1, we can find R integers

$$m_1 < m_2 < \ldots < m_R \leq \sqrt{M}$$

satisfying (3.1). Let us now consider the sets of integers

$$T_i := \big\{t \leq \frac{x}{M} : P^+(m_i t + 1) \leq y\big\} \qquad (1 \leq i \leq R).$$

We have $|T_i| = \Psi((xm_i/M)+1, y; 1, m_i)$ and can appeal to Lemma 3.2 to obtain a lower bound for this quantity. Indeed, by (1.9) we have for $x \geq x_0$

$$M \leq y, \tag{3.9}$$

whence

$$m_i \leq \sqrt{y},\ \Big(\log \frac{x}{\sqrt{M}}\Big)^3 \leq y \leq \frac{x}{M}.$$

In order to check (3.9), it is sufficient to observe that $u \leq \frac{1}{8}\frac{\log_2 x}{\log_3 x}$, hence

$$\begin{aligned}\log M &\leq 2\alpha(2u)^{6u+1}\log(2u) \leq 2\alpha \log_3 x \exp\big\{(7/8)\log_2 x\big\}\\ &\leq (\log x)^{\frac{15}{16}} \leq \log y.\end{aligned}$$

With a suitable absolute constant C_0, we therefore have

$$|T_i| \geq C_0 \frac{x}{M} u^{-2u} \qquad (1 \leq i \leq R). \tag{3.10}$$

Furthermore, if $t \in T_i \cap T_j$ with $1 \leq i < j \leq R$, then

$$P^+(tm_i+1) \leq y,\ P^+(tm_j+1) \leq y$$

whence

$$P^+\left(t[m_i,m_j]+\frac{m_j}{(m_i,m_j)}\right)\leq y,\ P^+\left(t[m_i,m_j]+\frac{m_i}{(m_i,m_j)}\right)\leq y.$$

From (3.1), this last condition may be rewritten as $P^+(n(n+1))\leq y$ for

$$n:=t[m_i,m_j]+\frac{m_i}{(m_i,m_j)}.$$

Let N denote the left hand side of (1.8). We deduce from the above reasoning that

$$|T_i\cap T_j|\leq N \qquad (1\leq i<j\leq R).$$

The inclusion-exclusion principle then implies

$$\begin{aligned}\frac{x}{M}\geq\left|\bigcup_{i=1}^{R}T_i\right|&\geq\sum_{1\leq i\leq R}T_i-\sum_{1\leq i<j\leq R}|T_i\cap T_j|\\&\geq C_0\frac{x}{M}Ru^{-2u}-R^2N\end{aligned}$$

where we have taken (3.10) into account. It follows that

$$N\geq\frac{x}{MR^2}\{C_0Ru^{-2u}-1\}\gg xu^{-u^{7u}}$$

since $Ru^{-2u}\gg 2^{2u}$ and, for $u\geq u_0$,

$$MR^2\ll R^{2(\alpha+1)R^3}\ll(2u)^{2(\alpha+1)(2u)^{6u+1}}\ll u^{u^{7u}}.$$

This completes the proof of Theorem 3.

§ 4. Proof of Theorem 2.

Let y_0 be a sufficiently large constant, and suppose $y\geq y_0$. We put

$$x:=y^{\frac{\log_2 y}{8\log_3 y}}$$

so that (1.9) is satisfied. We also have

$$\log_2 y=8u\log_3 y>8u\log u$$

whence

$$y\geq e^{u^{8u}}. \tag{4.1}$$

We define, for each prime $p \le y$, the integer α_p by

$$y^2 < p^{\alpha_p} \le py^2$$

and we put

$$n := \prod_{p \le y} p^{\alpha_p}.$$

Plainly

$$\log n \asymp y. \tag{4.2}$$

Now, we have on the one hand, from Theorem 3,

$$\sum_{\substack{d \le x \\ P^+(d(d+1)) \le y}} 1 \gg xu^{-u^{7u}},$$

and on the other hand

$$\sum_{\substack{d \le x \\ P^+(d(d+1)) \le y \\ d(d+1) \nmid n}} 1 \le \sum_{d \le x} \sum_{\substack{p \le y \\ p^{\alpha_p+1} \mid d(d+1)}} 1 \le 2 \sum_{p \le y} \frac{x+1}{p^{\alpha_p+1}} \ll \frac{x}{y}.$$

Taking (4.1) into account, we get

$$\kappa(n) \gg xu^{-u^{7u}} \gg x^{\frac{9}{10}} \gg (\log n)^{\frac{\log_3 n}{9 \log_4 n}}.$$

This completes the proof.

§ 5. Proof of Theorem 4.

We use three lemmas. The first enunciates a property of primitive sequences — that is sequences no element of which divides any other — due to Behrend [1]. Another proof may be found in [5], Chap. V, Th. 6.

Lemma 5.1. *There exists an absolute constant K such that for every primitive sequence $\mathcal{A} \subseteq \mathbf{Z}^+$ and every $x \ge 3$ we have*

$$\sum_{\substack{a \le x \\ a \in \mathcal{A}}} \frac{1}{a} \le K \frac{\log x}{\sqrt{\log_2 x}}.$$

Lemma 5.2. *Let m, n be positive integers such that $m \mid n$. Then*

$$H(m) \le H(n). \tag{5.1}$$

Proof. The sequence of divisors of m is a subsequence of that of divisors of n. Consider two consecutive divisors of m, say d_j, d_{j+1}. The divisors of n in $[d_j, d_{j+1}]$ are $d_j = d_{j1} < d_{j2} < \dots d_{jr} = d_{j+1}$ and we obviously have

$$(d_{j+1} - d_j)^{-1} \le \sum_{1 \le i < r} (d_{ji+1} - d_{ji})^{-1}.$$

Summing over j, $1 \le j < \tau(m)$, we obtain (5.1).

Lemma 5.3. *Let $y \geq 2$ and define, for every integer n,*

$$a_n := \prod_{\substack{p \leq y \\ p^\nu \| n}} p^\nu. \tag{5.2}$$

There exists an absolute constant C such that the inequalities

$$0 \leq H(n) - H(a_n) \leq y^{-1}(\log y)^4 \tag{5.3}$$

hold for all but at most $Cx(\log y)^{-1}$ integers $n \leq x$.

Proof. The left hand inequality follows from (5.1). The right hand inequality is implied by the estimate

$$\sum_{n \leq x} \{H(n) - H(a_n)\} \leq Cxy^{-1}(\log y)^3$$

established in [3], eq. (7.1).

We are now in a position to embark on the proof of Theorem 4.

We proceed by contradiction. If the required conclusion fails to hold, there is an $\alpha \geq 0$ and a $\delta > 0$ such that for every positive ε

$$\sum_{\substack{n \leq x \\ |H(n) - \alpha| \leq \frac{1}{2}\varepsilon}} 1 \geq \delta x \qquad (x > x_0(\varepsilon)). \tag{5.4}$$

From now on, we suppose that α and δ are given and agree that all the constants, implicit or explicit, may depend on these two quantities.

Put $y = \varepsilon^{-2}$ and define a_n by (5.2). From Lemma 5.3 and (5.4) it follows that, if ε is sufficiently small and $x_0(\varepsilon)$ is suitably chosen, then the inequality $|H(a_n) - \alpha| \leq \varepsilon$ holds, provided $x > x_0(\varepsilon)$, for at least $\frac{1}{2}\delta x$ integers $n \leq x$. Denote by $\mathcal{A} = \mathcal{A}(\varepsilon)$ the sequence of all integers a such that $|H(a) - \alpha| \leq \varepsilon$. By partial summation, the above property implies that

$$\log x \ll \sum_{\substack{n \leq x \\ a_n \in \mathcal{A}}} \frac{1}{n} \leq \sum_{\substack{P^+(a) \leq y \\ a \in \mathcal{A}}} \sum_{\substack{n \leq x \\ a_n = a}} \frac{1}{n}.$$

The inner sum is equal to

$$\sum_{\substack{b \leq x/a \\ P^-(b) > y}} \frac{1}{ab} \leq \frac{1}{a} \prod_{y < p \leq x} \Big(1 - \frac{1}{p}\Big)^{-1} \ll \frac{\log x}{a \log y}.$$

Hence

$$\sum_{\substack{P^+(a)\le y\\ a\in\mathcal{A}}} \frac{1}{a} \gg \log y. \tag{5.5}$$

Set $\theta := \frac{1}{\log y}$. For $t > 1$, we have

$$\sum_{\substack{a>y^t\\ P^+(a)\le y}} \frac{1}{a} \le \sum_{P^+(a)\le y} a^{\theta-1}y^{-\theta t} = e^{-t}\prod_{p\le y}(1-p^{\theta-1})^{-1}$$
$$\ll e^{-t}\log y.$$

Thus it follows from (5.5) that there exists a constant t such that

$$\sum_{\substack{a\le y^t\\ a\in\mathcal{A}}} \frac{1}{a} \gg \log y. \tag{5.6}$$

Lemma 5.1 enables us to deduce from (5.6) that $\mathcal{A}$ is not primitive for small ε. In this case there is at least one pair a, a' of elements of $\mathcal{A}$ such that

(i) $a \mid a'$, $a' \le y^t$

(ii) $\alpha - \varepsilon \le H(a) \le H(a') \le \alpha + \varepsilon$.

We are going to show that the extra condition

(iii) $\exists p \mid \frac{a'}{a} :\ p \nmid a,\ p \le z := y^{\frac14}$.

can also be imposed.

Indeed, suppose that (iii) fails to hold for all pairs a, a' satisfying (i) and (ii). Let $\mathcal{A}_0$ be the sequence of those elements of $\mathcal{A}$ which are divisible by no other element of $\mathcal{A}$. Then $\mathcal{A}_0$ is primitive — see e.g. [5], Chap. V, § 1. The hypothesis that (iii) never holds when (i) and (ii) are fulfilled implies that $\mathcal{A}\cap[1,y^t]$ is contained in the set

$$\{a_0 m : a_0 \in \mathcal{A}_0,\ p \mid m \Rightarrow p \mid a_0 \text{ or } p > z\}.$$

Hence

$$\sum_{\substack{a\le y^t\\ a\in\mathcal{A}}} \frac{1}{a} \le \sum_{\substack{a_0\le y^t\\ a_0\in\mathcal{A}_0}} \frac{1}{a_0} \prod_{\substack{p\mid a_0\\ p\le z}} \Big(1-\frac{1}{p}\Big)^{-1} \prod_{z<p\le y^t} \Big(1-\frac{1}{p}\Big)^{-1}$$
$$\ll \Big\{ \sum_{\substack{a_0\le y^t\\ a_0\in\mathcal{A}_0}} \frac{1}{a_0} \cdot \sum_{n\le y^t} \frac{n}{\varphi(n)^2} \Big\}^{\frac12} \ll \frac{\log y}{(\log_2 y)^{\frac14}}$$

by Lemma 5.1, since the second p-product is clearly bounded. This contradicts (5.6) for sufficiently small ε and in turn implies the existence of a, a' in $\mathcal{A}$ satisfying (i), (ii) and (iii).

For these a, a', denote by d_j, d_{j+1} the divisors of a such that $d_j < p < d_{j+1}$, with the convention that $d_{j+1} = d_{\tau(a)+1} = +\infty$ if $p > a$. We have

$$H(pa) - H(a) \geq \frac{1}{p - d_j} + \frac{1}{d_{j+1} - p} - \frac{1}{d_{j+1} - d_j} \geq \frac{1}{p}$$

whence

$$H(a') - H(a) \geq H(pa) - H(a) \geq \frac{1}{p} \geq \sqrt{\varepsilon}.$$

This is in contradiction to the definition of $\mathcal{A}$ when ε is small enough. The proof of Theorem 4 is thereby completed.

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