*Sci. China Math.* **66,**
no. 12 (2023), 2683–2692.

# Two upper bounds for the Erdős–Hooley Delta-function

Régis de la Bretèche & Gérald Tenenbaum

**Abstract.** For integer $n \geqslant 1$ and real $u$, let $\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|$. The Erdős–Hooley Delta-function is then defined by $\Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u)$. We improve the current upper bounds for the average and normal orders of this arithmetic function.

**Keywords:** Concentration of divisors, average order, normal order, Waring's problem.
**2020 Mathematics Subject Classification:** primary 11N37; secondary 11K65.

## 1. Introduction and statement of results

For integer $n \geqslant 1$ and real $u$, put

$$\Delta(n,u) := |\{d : d \mid n,\, \mathrm{e}^u < d \leqslant \mathrm{e}^{u+1}\}|, \quad \Delta(n) := \max_{u\in\mathbb{R}} \Delta(n,u).$$

Introduced by Erdős [1] (see also [2]) and studied by Hooley [6], the $\Delta$-function proved very useful in several branches of number theory — see, e.g., [5] and [13] for further references. If $\tau(n)$ denotes the total number of divisors of $n$, then $\Delta(n)/\tau(n)$ coincides with the concentration of the numbers $\log d$, $d|n$. In this work, we aim at improving the current upper bounds for the average and normal orders. In the former case, we consider weighted versions.

For $A > 0$, $y \geqslant 1$, $c > 0$, $\eta \in ]0,1[$, we define the class $\mathcal{M}_A(y,c,\eta)$ comprising those arithmetic functions $g$ that are multiplicative, non-negative, and satisfy the conditions

$$g(p^\nu) \leqslant A^\nu \qquad (\nu \geqslant 1) \tag{1·1}$$

$$(\forall\varepsilon > 0) \quad g(n) \ll_\varepsilon n^\varepsilon \quad (n \geqslant 1) \tag{1·2}$$

$$\sum_{p\leqslant x} g(p) = y\,\mathrm{li}(x) + O\big(x\mathrm{e}^{-c(\log x)^\eta}\big) \quad (x \geqslant 2). \tag{1·3}$$

Here and in the sequel we reserve the letter $p$ to designate a prime number. Note for the sake of further reference that by a theorem of Shiu [11; th. 1], we have

$$\sum_{n\leqslant x} g(n) \ll x(\log x)^{y-1} \tag{1·4}$$

for any $g$ in $\mathcal{M}_A(y,c,\eta)$.

Regarding average values of the $\Delta$-function, we consider the weighted sum

$$S(x;g) := \sum_{n\leqslant x} g(n)\Delta(n).$$

Here and throughout we let $\log_k$ denote the $k$-fold iterated logarithm.

**Theorem 1.1.** *Let $A > 0$, $y \geqslant 1$, $c > 0$, $\eta \in ]0,1[$, $g \in \mathcal{M}_A(y,c,\eta)$, $a > \sqrt{2}\log 2 \approx 0.980258$. We have*

$$S(x;g) \ll x(\log x)^{2y-2}\mathrm{e}^{a\sqrt{\log_2 x}} \qquad (x \geqslant 3). \tag{1·5}$$

We note that by a different approach Koukoulopoulos (private communication) obtained a similar estimate for $g = \mathbf{1}$ with $a = 2.1$

When $y \geqslant 1$, Theorem 1.1 provides a small improvement over known estimates, for instance [5; th. 70] stating that, for any $\varepsilon > 0$,

$$S(x;\mathbf{1}) \ll x\mathrm{e}^{(1+\varepsilon)\sqrt{2\log_2 x \log_3 x}} \qquad (x \to \infty).$$

When $y \leqslant \frac{1}{2}$, the proof of [5; th. 64] may be readily generalized to any $g \in \mathcal{M}_A(y,c,\eta)$ to yield

$$S(x;g) \ll x(\log x)^{y-1}(\log_2 x)^{\delta(y,1/2)}, \tag{1·6}$$

where $\delta(u,v) := 1$ if $u = v$, and $:= 0$ otherwise.

For $y \geqslant 1 + \frac{1}{2}\sqrt{2}$, we may adapt *mutatis mutandis* [5; th. 67] and derive

$$S(x;g) \ll x(\log x)^{2y-2}(\log_2 x)^{\delta(y,1+\sqrt{2}/2)}. \tag{1·7}$$

Finally, we note that the proof of [5; th. 71] may be extended to get, for any fixed $y < 1$,

$$S(x;g) \ll x(\log x)^{y-1}\exp\Big\{\frac{4\log 3}{1-y}(\log_3 x)^2\Big\}. \tag{1·8}$$

We omit further details since the relevant approaches are straightforward.

As put forward by Hooley [6], among other applications, average bounds such as (1·5) may be employed to count solutions of certain Diophantine equations. For given $k \in \mathbb{N}^*$ and positive integers $c_j$, $\ell_j$ $(0 \leqslant j \leqslant k)$ with $\ell_0 = \min \ell_j = 2$, we consider as in [10] the number $N(x)$ of solutions $(\boldsymbol{m},\boldsymbol{n}) = (m_0,\dots,m_k,n_0,\dots,n_k) \in \mathbb{N}^{2k+2}$ of the system

$$\sum_{0\leqslant j\leqslant k} c_j m_j^{\ell_j} = \sum_{0\leqslant j\leqslant k} c_j n_j^{\ell_j} \leqslant x,\ m_0 \neq n_0,$$

and let $V(x)$ denote the number of integers $n \leqslant x$ that are representable in the form

$$n = \sum_{0\leqslant j\leqslant k} c_j n_j^{\ell_j}.$$

The case $k = 2$, $c_0 = c_1 = c_2 = 1$, $\ell_1 = \ell_2 = 4$ has been studied by the second author [13].

Applying Theorem 1.1 with $y = 1$ and following the approach displayed in [10], we derive the next corollary.

**Corollary 1.2.** *Assume* $\sum_{1\leqslant j\leqslant k} 1/\ell_j = \frac{1}{2}$ *and let* $a > \sqrt{2}\log 2$. *Then, as* $x \to \infty$, *we have*

$$N(x) \ll x\mathrm{e}^{a\sqrt{\log_2 x}}, \tag{1·9}$$

$$V(x) \gg x\mathrm{e}^{-a\sqrt{\log_2 x}}. \tag{1·10}$$

We omit the details of the proof, since they are almost identical to those in [10].

We next turn our attention to the normal order of the $\Delta$-function. We employ the mention pp to indicate that a formula holds on a sequence of natural density 1. Improving on estimates of Maier & Tenenbaum [7], [9], Ford, Green & Koukoulopoulos [3] recently claimed

$$\Delta(n) > (\log_2 n)^{\gamma_1} \qquad \text{pp}$$

for any $\gamma_1 < 0.35332$. Regarding upper bounds, Maier & Tenenbaum (see [8], [9]) proved that, given any $\gamma_2 > \log 2 \approx 0.693147$, we have

$$\Delta(n) \leqslant (\log_2 n)^{\gamma_2} \qquad \text{pp}. \tag{1·11}$$

We are now able to improve on this result by reducing the exponent further.

**Theorem 1.3.** *Let* $\gamma_3 > (\log 2)/(\log 2 + 1/\log 2 - 1) \approx 0.6102495$. *We have*

$$\Delta(n) \leqslant (\log_2 n)^{\gamma_3} \qquad \text{pp}.$$

## 2. Average order: proof of Theorem 1.1

### *2·1. Reductions*

Let us start by a technical reduction similar to that of the proof of [12; (25)] and enabling to substitute the evaluation of a logarithmic mean to that of a Cesàro mean. Considering the inequality (see [5; lemma 61.1])

$$\Delta(mn) \leqslant \tau(m)\Delta(n) \qquad (m \geqslant 1,\, n \geqslant 1), \tag{2·1}$$

we may write, for any function $g$ in $\mathcal{M}_A(y,c,\eta)$,

$$\sum_{n\leqslant x} g(n)\Delta(n)\log n \leqslant \sum_{m\leqslant x} g(m)\Delta(m) \sum_{\substack{p^\nu\leqslant x/m\\ p\nmid m}} (\nu+1)g(p^\nu)\log p^\nu \ll x\sum_{m\leqslant x}\frac{g(m)\Delta(m)}{m},$$

where the second bound is obtained by invoking (1·2) in the form $g(p^\nu) \ll_A (3/2)^\nu$ for $p \leqslant 2A$. Since we trivially have

$$\sum_{n\leqslant x} g(n)\Delta(n)\log\Big(\frac{x}{n}\Big) \ll x\sum_{n\leqslant x}\frac{g(n)\Delta(n)}{n},$$

it follows that

$$\sum_{n\leqslant x} g(n)\Delta(n) \ll \frac{x}{\log x}\sum_{n\leqslant x}\frac{g(n)\Delta(n)}{n}. \tag{2·2}$$

Moreover, the canonical representation $n = md$ where $m$ is squarefree and $d$ is squarefull implies

$$\sum_{n\leqslant x}\frac{g(n)\Delta(n)}{n} \ll \sum_{n\leqslant x}\frac{\mu(n)^2 g(n)\Delta(n)}{n}. \tag{2·3}$$

Next we observe that proving (1·5) for $y = 1$ implies the required bound for $y \geqslant 1$. Indeed, any $g$ in $\mathcal{M}_A(y,c,\eta)$ is representable as $g(n) = y^{\omega(n)}h(n)$ with $h \in \mathcal{M}_A(1,c,\eta)$. The identity

$$y^{\omega(n)} = \sum_{d\mid n}\mu(d)^2(y-1)^{\omega(d)},$$

already used in [4], and the inequality (2·1) hence imply

$$\sum_{n\leqslant x}\frac{\mu(n)^2 g(n)\Delta(n)}{n} \leqslant \sum_{d\leqslant x}\frac{\mu(d)^2 h(d)(2y-2)^{\omega(d)}}{d}\sum_{m\leqslant x/d}\frac{\Delta(m)h(m)}{m} \ll \mathrm{e}^{a\sqrt{\log_2 x}}(\log x)^{2y-1},$$

by applying (1·5) to $h$. The required estimate follows by (2·2).

In the sequel, we hence consider a function $g \in \mathcal{M}_A(1,c,\eta)$ and aim at estimating the right-hand side of (2·3).

Let $\{p_j(n) : 1 \leqslant j \leqslant \omega(n)\}$ denote the increasing sequence of distinct prime factors of a generic integer $n$ and define $n_k := p_1(n)\cdots p_k(n)$ if $\omega(n) \geqslant k$, $n_k = n$ otherwise. Our final estimate will be obtained from a bound for

$$D_k(x;g) := \sum_{n\leqslant x}\frac{\mu(n)^2 g(n)\Delta(n_k)}{n}$$

obtained by induction on $k$. To determine a suitable size for $k$, we appeal to a straightforward variant of [5; th. 72] providing

$$\sum_{n\leqslant x}\Delta(n)g(n)y^{\omega(n)} = x(\log x)^{2y-2+o(1)} \quad (y \geqslant 1,\, x\to\infty),$$

and hence, for any fixed $y > 1$,

$$\sum_{\substack{n\leqslant x\\ \omega(n)>2y\log_2 x}} g(n)\Delta(n) \ll \sum_{n\leqslant x} g(n)\Delta(n)y^{\omega(n)-2y\log_2 x} \ll x(\log x)^{-2(y\log y-y+1)+o(1)} = o(x).$$

Therefore, we see that it will be sufficient to bound $D_k(x;g)$ for $k \leqslant K_x := (2+\varepsilon)\log_2 x$.

A last reduction is described as follows. Given $\xi(x)$ tending to infinity arbitrarily slowly, let $\mathcal{A}_x$ denote the set of those integers $n \geqslant 1$ that are squarefree and satisfy

$$\omega(n) \geqslant k \Rightarrow \log_2 p_k(n) > k/5 \qquad (\xi(x) \leqslant k \leqslant K_x). \tag{2·4}$$

Let

$$D(x;g) := \sum_{n\leqslant x} \frac{\mu(n)^2 g(n)\Delta(n)}{n}, \qquad D^-(x;g) := \sum_{n\in[1,x]\smallsetminus\mathcal{A}_x} \frac{\mu(n)^2 g(n)\Delta(n)}{n}\cdot$$

Put $\omega(n,t) := \sum_{p|n,\, p\leqslant t} 1$, $r_k := \exp\exp(k/5)$. Letting $h_{v,k}$ denote the multiplicative function supported on squarefree integers and defined by $h_{v,k}(p) := (v-1)\mathbf{1}_{[2,r_k]}(p)$, we have, for any $v \geqslant 1$

$$\begin{aligned} D^-(x;g) &\leqslant \sum_{\xi(x)\leqslant k\leqslant K_x} v^{-k} \sum_{n\leqslant x} \frac{\mu(n)^2 g(n) v^{\omega(n,r_k)}\Delta(n)}{n} \\ &\leqslant \sum_{\xi(x)\leqslant k\leqslant K_x} v^{-k} \sum_{\substack{d\leqslant x\\ P^+(d)\leqslant r_k}} \frac{\mu(d)^2 g(d) h_{v,k}(d) 2^{\omega(d)}}{d} \sum_{n\leqslant x} \frac{\mu(n)^2 g(n)\Delta(n)}{n} \\ &\ll D(x;g) \sum_{\xi(x)\leqslant k\leqslant K_x} \mathrm{e}^{2(v-1)k/5-k\log v} = o(D(x;g)), \end{aligned}$$

by selecting $v = \frac{3}{2}$ since $1/5 - \log(3/2) < 0$.

Thus we obtain that suitable averages over $\mathcal{A}_x$ will imply (1·5) as stated.

## *2·2. A lemma*

Write

$$M_q(n) := \int_{\mathbb{R}} \Delta(n,u)^q \,\mathrm{d}u \quad (n \geqslant 1,\ q \geqslant 1). \tag{2·5}$$

The following estimate will play a crucial role in the proof.

**Lemma 2.1.** *Let $A > 0$, $c > 0$, $\eta > 0$, and $g \in \mathcal{M}_A(1,c,\eta)$. We have*

$$\sum_{\omega(n)\geqslant k} \frac{\mu(n)^2 g(n) M_2(n_k)}{n^\sigma} \ll \frac{k2^k}{\sigma-1} \qquad (k \geqslant 1,\ 1 < \sigma \leqslant 2). \tag{2·6}$$

*Proof.* Let $S_2(\sigma)$ denote the left-hand side of (2·6). Put $\tau(n,\vartheta) := \sum_{d|n} d^{i\vartheta}$ $(n \geqslant 1,\ \vartheta \in \mathbb{R})$. Plainly,

$$\begin{aligned} S_2(\sigma) &\ll \frac{1}{\sigma-1} \sum_{\substack{m\geqslant 1\\ \omega(m)=k}} \frac{\mu(m)^2 g(m) M_2(m)}{m\log P^+(m)} \\ &\ll \frac{1}{\sigma-1} \sum_p \frac{g(p)}{p\log p} \sum_{\substack{P^+(m)<p\\ \omega(m)=k-1}} \frac{\mu(m)^2 g(m)}{m} \int_{\mathbb{R}} \frac{|\tau(m,\vartheta)|^2}{1+\vartheta^2}\,\mathrm{d}\vartheta \end{aligned}$$

by Parseval's formula in view of [5; (3.2)]. The last integral is classically dominated by the contribution of the interval $[-1,1]$—see the Montgomery-Wirsing lemma as stated e.g. in [14; lemma III.4.10]. For $|\vartheta| \leqslant 1$, $t \geqslant 2$, we have

$$\sum_{\substack{r\leqslant t\\ r\in\mathcal{P}}} \frac{g(r)|\tau(r,\vartheta)|^2}{r} = 4\log_2 t - 2\log(1+|\vartheta|\log t) + O(1),$$

whence

$$\sum_{\substack{P^+(m)<p\\ \omega(m)=k-1}} \frac{\mu(m)^2 g(m)|\tau(m,\vartheta)|^2}{m} \ll \frac{\{4\log_2 p - 2\log(1+|\vartheta|\log p)+O(1)\}^{k-1}}{(k-1)!}.$$

We therefore get

$$\sum_p \frac{g(p)}{p\log p} \sum_{\substack{P^+(m)<p\\ \omega(m)=k-1}} \frac{\mu(m)^2 g(m)}{m^\sigma} \int_{\mathbb{R}} \frac{|\tau(m,\vartheta)|^2}{1+\vartheta^2}\,\mathrm{d}\vartheta \ll \frac{2^k}{(k-1)!}\int_0^1 \{T_1(\vartheta)+T_2(\vartheta)\}\,\mathrm{d}\vartheta,$$

with, for a suitable absolute constant $c_0$,

$$T_1(\vartheta) := \sum_{p\leqslant \exp(1/\vartheta)} \frac{g(p)(2\log_2 p + c_0)^{k-1}}{p\log p},$$
$$T_2(\vartheta) := \sum_{p> \exp(1/\vartheta)} \frac{g(p)\{\log_2 p + \log(1/\vartheta) + c_0\}^{k-1}}{p\log p}.$$

It follows that

$$\int_0^1 T_1(\vartheta)\,\mathrm{d}\vartheta \ll \sum_p \frac{g(p)(2\log_2 p + c_0)^{k-1}}{p(\log p)^2} \ll (k-1)!,$$
$$\int_0^1 T_2(\vartheta)\,\mathrm{d}\vartheta \ll \int_0^1 \sum_{p>\exp(1/\vartheta)} \frac{g(p)\{\log_2 p+\log(1/\vartheta)+c_0\}^{k-1}}{p\log p}\,\mathrm{d}\vartheta$$
$$\ll \int_0^1 \frac{1}{\vartheta}\int_{2\log(1/\vartheta)}^\infty (v+c_0)^{k-1}\mathrm{e}^{-v}\,\mathrm{d}v\,\mathrm{d}\vartheta \ll k!.$$

This yields (2·6) as required. □

### *2·3. Completion of the proof*

With notation (2·5) and

$$L(n) := \mathrm{meas}\{u\in\mathbb{R} : \Delta(n,u)>0\},$$

we introduce the series

$$F_{k,q}(\sigma) := \sum_{\omega(n)\geqslant k}^{*} \frac{g(n)M_q(n_k)L(n_k)^{(q-1)/2}}{2^k M_2(n_k)^{(q-1)/2} n^\sigma}, \quad G_{k,q}(\sigma) := \sum_{\omega(n)\geqslant k}^{*} \frac{g(n)M_q(n_k)^{1/q}}{n^\sigma}$$

for $\sigma>1$, $k\geqslant 1$. Here and throughout the asterisk indicates that the summation domain is restricted to $\mathcal{A}_x$.

Since by [5; th. 72] we have

(2·7) $$\Delta(n)\leqslant 2M_q(n)^{1/q} \qquad (n\geqslant 1,\ q\geqslant 1),$$

the validity of the bound

(2·8) $$\sum_{k\leqslant K_x} G_{k,q(k)}(1+1/\log x) \ll_a \mathrm{e}^{a\sqrt{\log_2 x}}\log x$$

for any $a>\sqrt{2}\log 2$ and suitable $q(k)$ implies the same estimate for the right-hand side of (2·2). An appropriate choice $q(k)$ will be given later.

Put

$$N_{j,q}(n,p) := \int_{\mathbb{R}} \Delta(n,u)^j \Delta(n,u-\log p)^{q-j}\,\mathrm{d}u, \quad W_q(n,p) := \sum_{1\leqslant j\leqslant q-1} \binom{q}{j} N_{j,q}(n,p).$$

The inequality

$$M_q(n_{k+1}) \leqslant 2M_q(n_k) + W_q(n_k,p_{k+1})\mathbf{1}_{\{\omega(n)\geqslant k+1\}},$$

is an equality if $\omega(n) \geqslant k+1$ and holds trivially otherwise. Since $M_2(n_{k+1}) \geqslant 2M_2(n_k)$ and $L(n_{k+1}) \leqslant 2L(n_k)$, it follows that

$$F_{k+1,q}(\sigma) \leqslant F_{k,q}(\sigma) + \sum_{\substack{m\in\mathcal{M}_k\\ \omega(m)=k}} \sum_{\substack{p>P^+(m)\\ \log_2 p\geqslant k/5}} \frac{W_q(m,p)L(m)^{(q-1)/2}}{2^k M_2(m)^{(q-1)/2}} \sideset{}{^*}\sum_{n_{k+1}=mp} \frac{g(n)}{n^\sigma}, \tag{2·9}$$

where $\mathcal{M}_k := \{m \geqslant 1 : \exists n \in \mathcal{A}_x : n_k = m\}$.

Let $\mathcal{H}_k := \{h \geqslant 1 : \mu(h)^2 = 1, \omega(h) \geqslant j \Rightarrow \log_2 p_j(h) \geqslant (j+k+1)/5\}$. The inner sum in (2·9) does not exceed

$$\frac{\mu(mp)^2 g(mp)}{(mp)^\sigma} \sum_{\substack{P^-(h)>p\\ h\in\mathcal{H}_k}} \frac{g(h)}{h^\sigma},$$

hence

$$\begin{aligned} &F_{k+1,q}(\sigma) - F_{k,q}(\sigma)\\ &\leqslant \sum_{\substack{m\in\mathcal{M}_k\\ \omega(m)=k}} \frac{\mu(m)^2 g(m) L(m)^{(q-1)/2}}{2^k M_2(m)^{(q-1)/2} m^\sigma} \sum_{\substack{P^-(h)>P^+(m)\\ h\in\mathcal{H}_k}} \frac{g(h)}{h^\sigma} \sum_{\substack{p>P^+(m)\\ \log_2 p\geqslant k/5}} \frac{g(p)W_q(m,p)}{p}. \end{aligned} \tag{2·10}$$

Now, observe that, for all $z > 1$ and $1 \leqslant j < q$,

$$\begin{aligned} (\log z)\sum_{p>z} \frac{g(p)N_{j,q}(m,p)}{p} &\leqslant \sum_{p>z} \frac{g(p)N_{j,q}(m,p)\log p}{p}\\ &= \int_{\mathbb{R}} \Delta(m,u)^j \sum_{d_1,\dots,d_{q-j}\mid m} \sum_{\substack{p>z\\ \mathrm{e}^u \max_h d_h<p\leqslant \mathrm{e}^{u+1}\min_h d_h}} \frac{g(p)\log p}{p}\,\mathrm{d}u. \end{aligned}$$

The inner $p$-sum does not exceed

$$\Big\{1 - \log\Big(\frac{\max_h d_h}{\min_h d_h}\Big) + O\Big(\mathrm{e}^{-c(\log z)^\eta}\Big)\Big\}\mathbf{1}_{\{\max_h d_h/\min_h d_h<\mathrm{e}\}},$$

and so

$$\begin{aligned} (\log z)\sum_{p>z} \frac{g(p)N_{j,q}(m,p)}{p} &\leqslant \sum_{p>z} \frac{g(p)N_{j,q}(m,p)\log p}{p}\\ &\leqslant AM_j(m)M_{q-j}(m) + O\Big(\mathrm{e}^{-c(\log z)^\eta} M_j(m)M^*_{q-j}(m)\Big), \end{aligned}$$

where

$$M^*_\ell(n) := \sum_{\substack{d_h\mid n\ (1\leqslant h\leqslant \ell)\\ \max d_h\leqslant \mathrm{e}\min d_h}} 1 \leqslant 2^\ell M_\ell(n) \qquad (\ell \geqslant 1,\ n\geqslant 1),$$

by [8; (6)].

At this stage we note that Hölder's inequality furnishes for $2 \leqslant \ell \leqslant q-2$, $m \geqslant 1$

$$M_\ell(m) \leqslant M_2(m)^{(q-\ell-2)/(q-4)} M_{q-2}(m)^{(\ell-2)/(q-4)}.$$

Applying twice for $\ell = j$ and $\ell = q-j$, we get for any $2 \leqslant j \leqslant q-2$

$$M_j(m)M_{q-j}(m) \leqslant M_2(m)M_{q-2}(m)$$

so that for any $2 \leqslant j \leqslant q-2$

$$\sum_{p>z} \frac{g(p)N_{j,q}(m,p)\log p}{p} \leqslant AM_2(m)M_{q-2}(m)\big\{1+R_{j,q}(m,z)\big\}$$

where

$$R_{j,q}(m,z) \ll 2^{q-j}\mathrm{e}^{-c(\log z)^\eta}.$$

Carrying back into (2·10) and taking into account the fact that, when $m \in \mathcal{M}_k$, $h \in \mathcal{H}_k$, $P^-(h) > P^+(m)$, $\mu(mh)^2 = 1$, the integer $mh$ belongs to $\mathcal{A}_x$, we obtain

$$F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \ll q\big(1+2^q\varepsilon_k\big)H_{k,q}(\sigma) + \big(2^q+3^q\varepsilon_k\big)J_{k,q}(\sigma),$$

with

$$\varepsilon_k := \mathrm{e}^{-c\exp(\eta k/5)},$$

$$H_{k,q}(\sigma) := \sideset{}{^*}\sum_{\omega(n)\geqslant k} \frac{g(n_k)M_{q-1}(n_k)L(n_k)^{(q-1)/2}}{M_2(n_k)^{(q-1)/2}n^\sigma \log p_k(n)},$$

$$J_{k,q}(\sigma) := \sideset{}{^*}\sum_{\omega(n)\geqslant k} \frac{g(n_k)M_{q-2}(n_k)L(n_k)^{(q-1)/2}}{2^k M_2(n_k)^{(q-3)/2}n^\sigma \log p_k(n)}.$$

From the inequalities

$$\frac{L(n_k)^{(q-1)/2}}{\log p_k(n)} \leqslant kL(n_k)^{(q-3)/2}, \qquad \frac{1}{L(n_k)} \leqslant \frac{M_2(n_k)}{4^k} \qquad (\omega(n) \geqslant k),$$

we deduce that

$$H_{k,q}(\sigma) \leqslant \sideset{}{^*}\sum_{\omega(n)\geqslant k} \frac{kg(n_k)M_{q-1}(n_k)L(n_k)^{(q-2)/2}}{2^k M_2(n_k)^{(q-2)/2}n^\sigma} = kF_{k,q-1}(\sigma),$$

$$J_{k,q}(\sigma) \leqslant \sideset{}{^*}\sum_{\omega(n)\geqslant k} \frac{kg(n_k)M_{q-2}(n_k)L(n_k)^{(q-3)/2}}{2^k M_2(n_k)^{(q-3)/2}n^\sigma} = kF_{k,q-2}(\sigma),$$

whence

$$F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \ll kq\big(1+2^q\varepsilon_k\big)F_{k,q-1}(\sigma) + k\big(2^q+3^q\varepsilon_k\big)F_{k,q-2}(\sigma).$$

Let $k_0(q) := B\log q$, where $B$ is sufficiently large to ensure $\varepsilon_k 2^q \leqslant 1$ whenever $q \geqslant 2$, $k \geqslant k_0(q)$. We thus have, for a suitable constant $C > 0$,

$$F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \leqslant CkqF_{k,q-1}(\sigma) + Ck2^qF_{k,q-2}(\sigma) \qquad (k \geqslant k_0(q),\ \sigma > 1). \tag{2·11}$$

We now show by induction on $k$ that this implies, for a suitable constant $D$,

$$F_{k,q}(\sigma) \leqslant \frac{Dq^{Bq}k^{3q}2^{q^2/4}}{\sigma-1} \prod_{1\leqslant j\leqslant k}\Big(1+\frac{4C}{j^2}\Big) \qquad (q \geqslant 2,\ k \geqslant 1,\ 1 < \sigma \leqslant 2). \tag{2·12}$$

For $k \leqslant k_0(q)$, this follows from the trivial bound $(\sigma-1)F_{k,q}(\sigma) \ll 2^{k(q-1)}$, in view of the inequalities $1/M_2(n) \leqslant L(n)/\tau(n)^2$, $L(n) \leqslant \tau(n)$, $M_q(n) \leqslant \tau(n)^q$. Assuming that (2·12) holds for $k \geqslant k_0(q)$, we deduce from (2·11) that

$$F_{k+1,q}(\sigma) \leqslant \frac{Dq^{Bq}(k+1)^{3q}2^{q^2/4}}{\sigma-1} \prod_{1\leqslant j\leqslant k+1}\Big(1+\frac{4C}{j^2}\Big),$$

since $q\leqslant 2^{(q+1)/2}$ and

$$k^{3q}2^{q^2/4}+Ck^{3q-2}2^{(q+1)/2+(q-1)^2/4}+Ck^{3q-2}2^{q+(q-2)^2/4}\leqslant (k+1)^{3q}2^{q^2/4}\Big\{1+\frac{4C}{(k+1)^2}\Big\}.$$

Therefore, we may state that

$$F_{k,q}(\sigma)\ll\frac{q^{Bq}k^{3q}2^{q^2/4}}{\sigma-1}\qquad(q\geqslant 2,\,k\geqslant 1,\,1<\sigma\leqslant 2).\tag{2·13}$$

Now invoking once more the inequality $4^k=\tau(n_k)^2\leqslant L(n_k)M_2(n_k)$ we get via Hölder's inequality that

$$\begin{aligned}G_{k,q}(\sigma)&\leqslant 2^{k/q}F_{k,q}(\sigma)^{1/q}\Big\{\sum_{\omega(n)\geqslant k}\frac{\mu(n)^2g(n)\sqrt{M_2(n_k)}}{n^\sigma\sqrt{L(n_k)}}\Big\}^{1-1/q}\\&\leqslant\frac{q^Bk^32^{k/q+q/4}}{(\sigma-1)^{1/q}}\Big\{\sum_{\omega(n)\geqslant k}\frac{\mu(n)^2g(n)M_2(n_k)}{2^kn^\sigma}\Big\}^{1-1/q}\ll\frac{q^Bk^42^{k/q+q/4}}{\sigma-1},\end{aligned}\tag{2·14}$$

by (2·6).

Applying this for all $k\leqslant K_x$, with $q=q(k)=\lceil 2\sqrt{k}\,\rceil$ and $\sigma=1+1/\log x$, yields (2·8) and thus finishes the proof.

## 3. Normal order: proof of Theorem 1.3

This is a reappraisal of [9; th. 1.3]. By (2·7), it is sufficient to bound $M_q(n)$.

Let $\lambda\in]1,2[$ and let $\gamma$, $\delta$ be real numbers satisfying

$$\delta(\log 2)/\lambda<\gamma<1,\qquad 1<\delta<\lambda(\gamma-1)+1/\log 2.\tag{3·1}$$

We shall show by induction on $k$ that

$$M_q(n_k)\leqslant 2^{\delta k}(q!)^\gamma\qquad(1\leqslant q\leqslant\lambda k)\quad\text{pp}x.\tag{3·2}$$

Here, as in [5], the mention pp$x$ indicates that a formula holds for all but at most $o(x)$ integers $n\leqslant x$ as $x\to\infty$.

Given an integer-valued function $\xi=\xi(x)$ tending to infinity arbitrarily slowly, we put

$$K=K(n,x):=\max\{k:1\leqslant k\leqslant\omega(n),\,\log_2p_k(n)<\log_2x-\xi(x)\}$$

and redefine

$$n_k:=\begin{cases}\prod_{\xi<j\leqslant k}p_j(n)&\text{if }k\leqslant K,\\ n_K&\text{if }k>K.\end{cases}$$

Assuming (3·2) holds for $k$ with $\xi<k\leqslant K$ we aim at showing that this bound persists at rank $k+1$.

Let $\mathrm{e}_1<\mathrm{e}$. By [9; (3.2)], for $\xi<k\leqslant K$, $q\geqslant 1$, we have

$$M_q(n_{k+1})\leqslant 2M_q(n_k)+\mathrm{e}_1^{-k}\sum_{1\leqslant j\leqslant q-1}\binom{q}{j}M_j(n_k)M_{q-j}(n_k)\quad\text{pp}x.\tag{3·3}$$

Note that Hölder's inequality implies

$$\sum_{1\leqslant j\leqslant q-1}\binom{q}{j}^{1-\gamma}\leqslant 2^{(1-\gamma)q}(q-1)^\gamma.$$

When $q\leqslant q_k:=\lfloor\lambda k\rfloor$, we appeal to the induction bound (3·2) to majorize the right-hand side of (3·3). This yields

$$\begin{aligned}M_q(n_{k+1})&\leqslant 2^{\delta(k+1)}(q!)^\gamma\Big\{2^{1-\delta}+(2^\delta/\mathrm{e}_1)^k\sum_{1\leqslant j\leqslant q-1}\binom{q}{j}^{1-\gamma}\Big\}\\&\leqslant 2^{\delta(k+1)}(q!)^\gamma\Big\{2^{1-\delta}+(2^\delta/\mathrm{e}_1)^k2^{(1-\gamma)q}q^\gamma\Big\}\\&\leqslant 2^{\delta(k+1)}(q!)^\gamma\Big\{2^{1-\delta}+(2^{\delta+(1-\gamma)\lambda}/\mathrm{e}_1)^kq^\gamma\Big\}\leqslant 2^{\delta(k+1)}(q!)^\gamma,\end{aligned}\tag{3·4}$$

for sufficiently large $\xi$, by the second condition (3·1).

When $q_k < q \leqslant \lfloor\lambda(k+1)\rfloor$, we apply [9; (3.3)]: given $\alpha > 0$ and $r > 1/\alpha$ we have

$$\Delta(n_k) \leqslant r + \mathrm{e}^{\alpha k/q} M_q(n_k)^{1/q} \qquad (\xi < k \leqslant K, q \geqslant 1) \quad \mathrm{pp}x.$$

Since $\lambda < 2$, we have $q = q_k + 1$ or $q = q_k + 2$. If $r$ is sufficiently large and $\alpha > 1/r$, our induction hypothesis (3·2) furnishes

$$\begin{aligned} M_{q_k+1}(n_k) &\leqslant \Delta(n_k) M_{q_k}(n_k) \leqslant \big\{\mathrm{e}^{\alpha/\lambda} M_{q_k}(n_k)^{1/q_k} + r\big\} M_{q_k}(n_k) \\ &\leqslant 2^{\delta k}\{(q_k+1)!\}^\gamma \left\{\frac{2^{\delta/\lambda}\mathrm{e}^{\alpha/\lambda}(q_k!)^{\gamma/q_k}}{(q_k+1)^\gamma} + \frac{r}{(q_k+1)^\gamma}\right\} \\ &\leqslant 2^{\delta k}\{(q_k+1)!\}^\gamma \big\{2^{\delta/\lambda}\mathrm{e}^{\alpha/\lambda-\gamma} + o(1)\big\}. \end{aligned} \tag{3·5}$$

Carrying back into (3·3) and writing

$$b := 2^{1-\delta+\delta/\lambda}\mathrm{e}^{\alpha/\lambda-\gamma} < 1,$$

we get, taking (3·4) into account,

$$\begin{aligned} M_{q_k+1}(n_{k+1}) &\leqslant 2M_{q_k+1}(n_k) + \mathrm{e}_1^{-k} \sum_{1\leqslant j\leqslant q_k} \binom{q_k+1}{j} M_j(n_k) M_{q_k+1-j}(n_k) \\ &\leqslant 2^{\delta(k+1)}\{(q_k+1)!\}^\gamma \left\{b + o(1) + (2^\delta/\mathrm{e}_1)^k \sum_{1\leqslant j\leqslant q_k} \binom{q_k+1}{j}^{1-\gamma}\right\} \\ &\leqslant 2^{\delta(k+1)}\{(q_k+1)!\}^\gamma \left\{b + o(1) + (2^\delta/\mathrm{e}_1)^k 2^{(1-\gamma)(\lambda k+1)}(q_k+1)^\gamma\right\} \\ &\leqslant 2^{\delta(k+1)}\{(q_k+1)!\}^\gamma \big\{b + o(1)\big\} \leqslant 2^{\delta(k+1)}\{(q_k+1)!\}^\gamma, \end{aligned} \tag{3·6}$$

by (3·1).

If $q_k + 2 \leqslant \lambda(k+1)$, we also need to bound $M_{q_k+2}(n_{k+1})$. Put $g := 2^{\delta/\lambda}\mathrm{e}^{\alpha/\lambda-\gamma} < 1$. By (3·2), [9; (3.3)] and (3·6), we have, for large $\xi$,

$$\begin{aligned} M_{q_k+2}(n_{k+1}) &\leqslant \Delta(n_{k+1}) M_{q_k+1}(n_{k+1}) \\ &\leqslant \big\{r + \mathrm{e}^{\alpha/\lambda} M_{q_k+1}(n_{k+1})^{1/(q_k+1)}\big\} M_{q_k+1}(n_{k+1}) \\ &\leqslant 2^{\delta(k+1)}\{(q_k+2)!\}^\gamma \frac{b+o(1)}{(q_k+2)^\gamma} \left\{r + \mathrm{e}^{\alpha/\lambda} 2^{\delta/\lambda+\delta/\lambda k}\{(q_k+1)!\}^{\gamma/(q_k+1)}\right\} \\ &\leqslant 2^{\delta(k+1)}\{(q_k+2)!\}^\gamma \{b+o(1)\}\big\{g+o(1)\big\} \leqslant 2^{\delta(k+1)}\{(q_k+2)!\}^\gamma, \end{aligned}$$

still by (3·1).

Selecting $\alpha$ sufficiently small, we see that the induction hypothesis is still valid at rank $k+1$. We may take $\delta$ arbitrarily close to 1, and so $\gamma$ arbitrarily close to $\gamma_3$. This yields

$$\Delta(n_K) \leqslant 2^{\delta K/q_K}(q_K!)^{\gamma/q_K} \ll K^\gamma.$$

Since we have classically $K(n,x) \sim \log_2 x$ pp$x$ as $x \to \infty$, we may conclude as in [9] by invoking the bound

$$\Delta(n) \leqslant \Delta(n_K) 2^{\Omega(n/n_K)} \ll \Delta(n_K) 4^\xi \qquad \mathrm{pp}x.$$

## References

[1] P. Erdős, Problem 218, Solution by the proposer, *Can. Math. Bull.* **16** (1973), 621-622.
[2] P. Erdős & J.-L. Nicolas, Répartition des nombres superabondants, *Bull. Soc. math. France* **103** (1975), 65-90.
[3] K. Ford, B. Green, D. Koukoulopoulos, Equal sums in random sets and the concentration of divisors, *Inventiones Math.* **232**, (2023),1027–1160.

[4] R.R. Hall & G. Tenenbaum, The average orders of Hooley's $\Delta_r$-functions, II, *Compositio Math.* **60** (1986), 163–186.
[5] R.R. Hall & G. Tenenbaum, *Divisors*, Cambridge tracts in mathematics, no 90, Cambridge University Press (1988).
[6] C. Hooley, On a new technique and its applications to the theory of numbers, *Proc. London Math. Soc.* (3) **38** (1979), 115–151.
[7] H. Maier & G. Tenenbaum, On the set of divisors of an integer, *Invent. Math.* **76** (1984), 121–128.
[8] H. Maier & G. Tenenbaum, On the normal concentration of divisors, *J. London Math. Soc.* (2) **31** (1985), 393–400.
[9] H. Maier and G. Tenenbaum, On the normal concentration of divisors, 2, *Math. Proc. Camb. Phil. Soc.* **147** n° 3 (2009), 593-614.
[10] O. Robert, Sur le nombre des entiers représentables comme somme de trois puissances, *Acta Arith.* **149** (2011), n° 1, 1–21.
[11] P. Shiu, A Brun–Titchmarsh theorem for multiplicative functions, *J. reine angew. Math.* **313** (1980), 161–170.
[12] G. Tenenbaum, Sur la concentration moyenne des diviseurs, *Comment. Math. Helvetici* **60** (1985), 411-428.
[13] G. Tenenbaum, Fonctions $\Delta$ de Hooley et applications, Séminaire de théorie des nombres, Paris 1984-85, Prog. Math. 63 (1986), 225-239.
[14] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, 3rd ed., Graduate Studies in Mathematics 163, Amer. Math. Soc. (2015).

Régis de la Bretèche
Université Paris Cité, Sorbonne Université, CNRS
Institut de Math. de Jussieu-Paris Rive Gauche
F-75013 Paris
France
regis.de-la-breteche@imj-prg.fr

Gérald Tenenbaum
Institut Élie Cartan
Université de Lorraine
BP 70239
54506 Vandœuvre-lès-Nancy Cedex
France
gerald.tenenbaum@univ-lorraine.fr