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On the friable Turán–Kubilius inequality*

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To the memory of Jonas Kubilius,
whose vision will enlighten our integers for ages

1. Introduction

An additive function, i.e. $f : \mathbb{N}^* \to \mathbb{C}$ with

$$f(n) = \sum_{p^\nu \| n} f(p^\nu) \qquad (n \in \mathbb{N}^*),$$

is the arithmetical analogue of a sum of independent random variables in probability theory. However, in the arithmetical framework, independence is only partly fulfilled. A quantitative measure of this tendency is furnished by Kubilius' gauge, which measures the gap between the probabilistic space $\Omega_x := \{n \in \mathbb{N}^* : n \leqslant x\}$ equipped with the uniform law ν_x and its canonical probabilistic model. It may be equivalently stated in terms of distance between formal probability spaces (see [16], section III.6.5) or as a bound for the total variation distance (see in particular [13]) between the truncation

$$f_y(n) := \sum_{\substack{p^\nu \| n \\ p \leqslant y}} f(p^\nu)$$

and its probabilistic model $Z_{f,y} := \sum_{p \leqslant y} \xi_p$ where the ξ_p are independent geometric random variables with laws

$$\mathbb{P}(\xi_p = f(p^\nu)) := (1 - 1/p)p^{-\nu} \qquad (\nu \geqslant 0).$$

In this latter setting, the best known uniform estimate, due to the second author [15], states that

$$K(x, y) := \sup_f \sup_{A \subset \mathbb{R}} |\nu_x(f_y \in A) - \mathbb{P}(Z_{f,y} \in A)| \ll u^{-u} + x^{-1+\varepsilon} \qquad (x \geqslant 2,\ y \geqslant 2)$$

where $\varepsilon > 0$ is arbitrary and $u := (\log x)/\log y$.

Kubilius' historical result ([8], [11]) yielded the optimal, qualitative statement that $K(x, y)$ approaches zero as x and u tend to infinity, and indeed provided the effective bound $K(x, y) \ll \mathrm{e}^{-cu}$ for some suitable positive constant c. This upper bound was later improved from the quantitative viewpoint by Barban and Vinogradov [1]—see Elliott [4], ch. 3, for details. More precise results, including an asymptotic formula for $K(x, y)$, may be given in wide subregions [15]. In particular, $K(x, y)$ approaches a strictly positive limit when x and y tend to infinity in such a way that u remains fixed.

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* Some corrections with respect to the published version are included here.

These results show how friable integers, namely integers all of whose prime factors do not exceed a given bound (y in the above), naturally occur in probabilistic number theory.

In the last twenty years the study of the distribution of friable integers has been intensively developed. Letting $P(n)$ denote the largest prime factor of an integer n, with the convention that $P(1) = 1$, we say that n is y-friable if $P(n) \leqslant y$ and we write

$$S(x, y) := \{n \leqslant x : P(n) \leqslant y\}$$

the set of friable integers not exceeding x.

Another classical result in probabilistic number theory marks the discrepancy between the empirical arithmetic situation and its probabilistic model based on independence assumptions. This is the celebrated Turán–Kubilius inequality which may be stated (see in particular [9], [10], [6], [14], [3]) as

$$\limsup_{x\to\infty} \sup_f \mathcal{V}_f(x)/\mathbb{V}(Z_{f,x}) = 2.$$

where $\mathcal{V}_f(x)$ denotes the variance of the random variable variable f with respect to the measure ν_x and the inner supremum is taken over all complex-valued additive functions f. The constant 2 appearing in the above formula may be thought of as a quantitative measure of the distance of analytic number theory and its probabilistic model. In all but the last quoted result, one will actually observe a constant with value 3/2: this is due to the fact that corresponding authors used various other normalisation terms instead of $\mathbb{V}(Z_{f,x})$. While the finiteness of the left-hand side proved to be a very fruitful tool in probabilistic number theory, mainly because it is uniform with respect to f, the fact that the constant is not equal to unity is quite significant from a theoretical viewpoint: probabilistic number theory cannot be reduced to probability theory.

In [3] and [12], the friable version of this inequality is studied, thereby providing another quantitative description of the propensity to independence as the parameter u is growing. Thus, in accord to the Kubilius model, is is shown in [3] that, if $V_f(x, y)$ denotes the variance of the additive function f with respect to the uniform measure on the set $S(x, y)$ of y-friable integers not exceeding x, then, for a suitable model $Z_{f,x,y}$ defined as a sum of independent geometric random variables on an abstract probability space (see (1·1) below for a precise definition),

$$C(x, y) := \sup_f \mathcal{V}_f(x, y)/\mathbb{V}(Z_{f,x,y}) = 1 + o(1)$$

provided $1/u + (\log x)/y \to 0$. In [12], it is shown that $C(x, y) = C(u) + o(1)$ when x and y tend to infinity and u remains fixed. Moreover, the limit $C(u)$ is exactly determined in a computable way—see [5].

All results described above rest on the saddle-point method, as developed by Hildebrand and Tenenbaum [7]. Indeed, its specific feature of providing so-called 'semi-asymptotic' formulae yields simple estimates for ratios of the type

$$\frac{\Psi_m(x/d, y)}{\Psi(x, y)} \qquad (1 \leqslant d, y \leqslant x,\ m \in \mathbb{N}^*)$$

where $\Psi(x, y) := |S(x, y)|$ and

$$\Psi_m(x, y) := |\{n \in S(x, y) : (n, m) = 1\}|.$$

These estimates depend on the parameter $\alpha = \alpha(x, y)$ defined as the unique solution to the equation

$$\sum_{p\leqslant y} \frac{\log p}{p^\alpha - 1} = \log x \qquad (2 \leqslant y \leqslant x)$$

and which incidentally turns out to be the saddle-point corresponding to the Perron integral for $\Psi(x,y)$. Writing $g_m(s) := \prod_{p|m}(1-p^{-s})$, it turns out that

$$\Psi_m(x/d,y)/\Psi(x,y) \approx g_m(\alpha)/d^\alpha$$

in a wide region for d,m,x,y [2].

The Kubilius probabilistic model $Z_{f,x,y}$ for a complex additive function f restricted to $S(x,y)$ is defined as

$$Z_{f,x,y} = \sum_{p\leqslant y} \xi_{p,\alpha} \tag{1·1}$$

where the $\xi_{p,\alpha}$ are abstract independent geometric random variables with laws

$$\mathbb{P}\big(\xi_{p,\alpha} = f(p^\nu)\big) = \frac{g_p(\alpha)}{p^{\nu\alpha}} \qquad (\nu = 0,1,2,\ldots).$$

It is then a standard computation to show that

$$\mathbb{V}\big(Z_{f,x,y}\big) = \sum_{p^\nu\in S(x,y)} \frac{g_p(\alpha)}{p^{\nu\alpha}}|f(p^\nu)|^2 - \sum_{p\leqslant y} g_p(\alpha)^2\bigg|\sum_{1\leqslant\nu\leqslant\frac{\log x}{\log p}} \frac{f(p^\nu)}{p^{\nu\alpha}}\bigg|^2.$$

A natural upper bound for this is

$$B_f(x,y)^2 := \sum_{p^\nu\in S(x,y)} \frac{g_p(\alpha)}{p^{\nu\alpha}}|f(p^\nu)|^2.$$

Indeed, we have $B_f(x,y)^2 = \mathbb{V}(Z_{f,x,y})$ when the $\xi_{p,\alpha}$ are centred and we have in full generality

$$\big(1-2^{-\alpha}\big)B_f(x,y)^2 \leqslant B_f^-(x,y)^2 \leqslant \mathbb{V}(Z_{f,x,y}) \leqslant B_f(x,y)^2, \tag{1·2}$$

with

$$B_f^-(x,y)^2 := \sum_{p^\nu\in S(x,y)} \frac{g_p(\alpha)^2}{p^{\nu\alpha}}|f(p^\nu)|^2. \tag{1·3}$$

When $y \geqslant \sqrt{\log x}\,\log_2 x$, we may replace the variance $\mathcal{V}_f(x,y)$ by its semi-empirical variant, namely

$$V_f(x,y) := \frac{1}{\Psi(x,y)} \sum_{n\in S(x,y)} |f(n) - \mathbb{E}(Z_{f,x,y})|^2.$$

Indeed, writing $E_f(x,y)$ for the empiric expectation of f over the set $S(x,y)$ and $\overline{u} := \min(u, y/\log y)$, we have, under the above condition on x and y,

$$V_f(x,y) - \mathcal{V}_f(x,y) = |E_f(x,y) - \mathbb{E}(Z_{f,x,y})|^2 \ll B_f^2(x,y)/\overline{u} \ll \mathbb{V}(Z_{f,x,y}),$$

where the last estimate follows from (1·2), standard estimates for α and theorem 2.4 of [2].

In [3], we show that

$$V_f(x,y) \ll B_f^2(x,y) \tag{1·4}$$

holds uniformly for all additive f and $x \geqslant y \geqslant 2$. However, from the estimates given in [3], we can only infer that the bound

$$V_f(x,y) \ll \mathbb{V}(Z_{f,x,y}) \tag{1·5}$$

holds in the region $c\log x \leqslant y \leqslant x$ where c is an arbitrary positive constant.

Even if applications—see [3]—usually only require (1·4), the theoretical study necessitates to analyse the status of (1·5) in complete generality.

We now provide a wider region for the validity of (1·5).

Theorem 1.1. *The upper bound* (1·5) *holds uniformly for all complex, additive f provided $x \geqslant y \geqslant \sqrt{\log x}\,\log_2 x$.*

2. Proof

We first note that, with no loss of generality, we may restrict to the case of real functions. Indeed, if $f = f_1 + if_2$, then

$$V_f(x,y) = V_{f_1}(x,y) + V_{f_2}(x,y), \qquad \mathbb{V}(Z_{f,x,y}) = \mathbb{V}(Z_{f_1,x,y}) + \mathbb{V}(Z_{f_2,x,y}).$$

We could also assume that $f \geqslant 0$ since, defining $f^{\pm}(p^\nu) := \max(0, \pm f(p^\nu))$, we plainly have

$$\begin{aligned} V_f(x,y) &\leqslant 2V_{f^+}(x,y) + 2V_{f^-}(x,y),\\ \mathbb{V}(Z_{f,x,y}) &= \mathbb{V}(Z_{f^+,x,y}) + \mathbb{V}(Z_{f^-,x,y}) + 2\mathbb{E}(Z_{f^+,x,y})\mathbb{E}(Z_{f^-,x,y})\\ &\geqslant \mathbb{V}(Z_{f^+,x,y}) + \mathbb{V}(Z_{f^-,x,y}). \end{aligned}$$

However, we shall not need this extra assumption.

As noted above, we may assume $y \leqslant \log x$: in the opposite case, the required estimate follows from theorem 1.1 of [3] and the computations appearing in the proof of corollary 5.2.

Setting

$$R_m(x/d,y) := \frac{\Psi_m(x/d,y)}{\Psi(x,y)} - \frac{g_m(\alpha)}{d^\alpha},$$

we have

$$\begin{aligned} E_{f^2}(x,y) &= \frac{1}{\Psi(x,y)} \sum_{n\in S(x,y)} f(n)^2\\ &= \sum_{p^\nu\in S(x,y)} f(p^\nu)^2 \frac{\Psi_p(x/p^\nu,y)}{\Psi(x,y)} + \sum_{p\neq q} f(p^\nu)f(q^\mu)\frac{\Psi_{pq}(x/p^\nu q^\mu,y)}{\Psi(x,y)}\\ &= \mathbb{E}(Z_{f,x,y}^2) + \sum_{p^\nu\in S(x,y)} f(p^\nu)^2 R_p\Big(\frac{x}{p^\nu},y\Big) + \sum_{\substack{p\neq q\\ p^\nu q^\mu\in S(x,y)}} f(p^\nu)f(q^\mu)R_{pq}\Big(\frac{x}{p^\nu q^\mu},y\Big) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(Z_{f,x,y})E_f(x,y) &= \frac{\mathbb{E}(Z_{f,x,y})}{\Psi(x,y)} \sum_{n\in S(x,y)} f(n)\\ &= \mathbb{E}(Z_{f,x,y}) \sum_{p^\nu\in S(x,y)} f(p^\nu)\Big\{\frac{g_p(\alpha)}{p^{\nu\alpha}} + R_p\Big(\frac{x}{p^\nu},y\Big)\Big\}\\ &= \mathbb{E}(Z_{f,x,y})^2 + \sum_{p^\nu q^\mu\in S(x,y)} \frac{g_q(\alpha)}{q^{\mu\alpha}} f(p^\nu)f(q^\mu)R_p\Big(\frac{x}{p^\nu},y\Big). \end{aligned}$$

Write, when $p \neq q$,

$$D_{p^\nu,q^\mu}(x,y) = R_{pq}\Big(\frac{x}{p^\nu q^\mu},y\Big) - \frac{g_q(\alpha)}{q^{\mu\alpha}}R_p\Big(\frac{x}{p^\nu},y\Big) - \frac{g_p(\alpha)}{p^{\nu\alpha}}R_q\Big(\frac{x}{q^\mu},y\Big) = \frac{\Delta_{p^\nu,q^\mu}(x,y)}{\Psi(x,y)p^{\nu\alpha}q^{\mu\alpha}}.$$

We may infer from the above that

$$V_f(x,y) = \mathbb{V}(Z_{f,x,y}) + T_f(x,y) + V_f^*(x,y) - U_f(x,y), \tag{2·1}$$

where

$$\begin{aligned} V_f^*(x,y) &:= \sum_{\substack{p\neq q\\ p^\nu q^\mu\in S(x,y)}} f(p^\nu)f(q^\mu)D_{p^\nu,q^\mu}(x,y)\\ T_f(x,y) &:= \sum_{p^\nu\in S(x,y)} f(p^\nu)^2R_p\Big(\frac{x}{p^\nu},y\Big)\\ U_f(x,y) &:= 2\sum_{p\leqslant y}\sum_{\mu,\nu\leqslant\frac{\log x}{\log p}}\frac{f(p^\nu)f(p^\mu)g_p(\alpha)}{p^{\nu\alpha}}R_p\Big(\frac{x}{p^\mu},y\Big). \end{aligned}$$

In theorem 3.8 of [3], we give an estimate of $\Delta_{k,\ell}(x,y)$. We now improve on this. We put

$$v_k(\alpha) := \log k - g_k'(\alpha)/g_k(\alpha) = \log k - \sum_{p\mid k}\frac{\log p}{p^\alpha-1}\qquad (k\geqslant 1).$$

We also write

$$\sigma_k := \left|\left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}\sum_{p\leqslant y}\frac{\log p}{p^s-1}\right]_{s=\alpha}\right| \asymp \frac{(u\log y)^k}{\overline{u}^{k-1}}. \tag{2·2}$$

Lemma 2.1. *Let $K>0$. There is a constant C, depending only on K such that, uniformly for $x\geqslant y\geqslant 2$, $u>\sqrt{\log y}$, $k\ell\leqslant x$, $(k,\ell)=1$, $\omega(k\ell)\leqslant K$, $P(k\ell)\leqslant y$, we have*

$$\Delta_{k,\ell}(x,y) = g_{k\ell}(\alpha)\Psi(x,y)\Big\{-\frac{v_k(\alpha)v_\ell(\alpha)}{\sigma_2}+O\Big(\frac{1}{\overline{u}^2}+\frac{\overline{u}^2(\kappa+\lambda)^4}{u^4}+\frac{\overline{u}^{5/2}(\kappa+\lambda)^5}{u^5}\Big)\Big\}, \tag{2·3}$$

where $\kappa := (\log k)/\log y$, $\lambda := (\log \ell)/\log y$.

Proof. Let

$$G_m(s) := \frac{g_m(s)m^\alpha}{g_m(\alpha)m^s}-1\qquad (m\geqslant 1),\quad T_0 := \frac{\overline{u}^{2/3}}{u\log y},\quad \delta := \frac{u\log y}{\overline{u}}\cdot$$

For $P(m)\leqslant y$, $\omega(m)\ll 1$, $s=\alpha+i\tau$, $|\tau|\leqslant T_0$, an expansion of $G_m(\alpha+i)$ to the third order furnishes that

$$G_m(s) = -i\tau v_m(\alpha) - \tfrac12\tau^2G_m''(\alpha) + O\big(\tau^3(\delta+\log m)^3\big).$$

and $G_m''(\alpha)\ll(\delta+\log m)^2$. This implies that

$$G_k(s)G_\ell(s)\zeta(s,y)\frac{x^s}{s} = -\frac{x^\alpha\zeta(\alpha,y)}{\alpha}\tau^2\mathrm{e}^{-\tau^2\sigma_2/2}\Big\{v_k(\alpha)v_\ell(\alpha)+I(\tau)+O\Big(b^2D'(\tau)\Big)\Big\}$$

where $I(\tau)$ is an odd function of τ, $b:=\delta+\log(k\ell)\ll(\log y)\big\{(u/\overline{u})+\kappa+\lambda\big\}$ and

$$D'(\tau) := \tau^2\delta^2+\tau^6\sigma_3^2+\tau^4\sigma_4+b^2\tau^2(1+b|\tau|).$$

The required estimate now follows from the formulae

$$\int_{-T_0}^{T_0}\tau^{1+2k}\mathrm{e}^{-\sigma_2\tau^2/2}\,\mathrm{d}\tau=0,\ \int_{-T_0}^{T_0}|\tau|^k\mathrm{e}^{-\sigma_2\tau^2/2}\,\mathrm{d}\tau\ll_k\frac{1}{\sigma_2^{(k+1)/2}}\quad (k\in\mathbb{N}).$$

□

We may now embark the final part of the proof of our theorem.

In the domain under consideration for x, y, we have $1/\overline{u} \ll \alpha \asymp \overline{u}/(u\log y)$, hence it follows from (1·2) that

$$\mathbb{V}(Z_{f,x,y}) \gg B_f(x,y)^2/\overline{u}.$$

Inserting the estimate of Lemma 2.1 into the proof of formula (4.24) of [3] yields, *mutatis mutandis*,

$$V_f^*(x,y) = S^* + O\Big(\frac{B_f(x,y)^2}{\overline{u}}\Big) = S^* + O(\mathbb{V}(Z_{f,x,y})),$$

with

$$S^* := \frac{-1}{\sigma_2} \sum_{\substack{p^\nu, q^\mu \in S(x,y)\\ p\neq q}} \frac{g_{pq}(\alpha)f(p^\nu)f(q^\mu)v_{p^\nu}(\alpha)v_{q^\mu}(\alpha)}{p^{\nu\alpha}q^{\mu\alpha}}. \tag{2·4}$$

Moreover, for a suitable constant $C>0$, we have

$$S^* \leqslant C\mathbb{V}(Z_{f,x,y}). \tag{2·5}$$

Indeed

$$S^* = \frac{-1}{\sigma_2}\Big(\sum_{p^\nu\in S(x,y)} \frac{g_p(\alpha)f(p^\nu)v_{p^\nu}(\alpha)}{p^{\nu\alpha}}\Big)^2 + \frac{1}{\sigma_2}\sum_{p\leqslant y}\Big(\sum_{\nu\leqslant (\log x)/\log p} \frac{g_p(\alpha)f(p^\nu)v_{p^\nu}(\alpha)}{p^{\nu\alpha}}\Big)^2.$$

Since, by the Cauchy-Schwarz inequality, (2·2) and the estimate $\alpha \asymp \overline{u}/(u\log y)$, valid for $y \leqslant \log x$, the second term is $\ll \mathbb{V}(Z_{f,x,y})$, we get (2·5).

Put $v := Au/\overline{u}$ where A is a large constant. It follows from lemma 3.5 of [3] that, when $p^\nu \leqslant y^v$, we have $R_p(x/p^\nu, y) \ll g_p(\alpha)/(\overline{u}p^{\nu\alpha})$. Therefore those prime powers p^ν not exceeding y^v contribute to $T_f(x,y)$ a quantity $\ll B_f(x,y)^2/\overline{u} \ll \mathbb{V}(Z_{f,x,y})$. Moreover, the same bound for R_p implies that the complementary contribution is negative.

It remains to deal with $U_f(x,y)$, that we bound appealing to lemma 3.5 of [3] which furnishes, uniformly for $p^\mu \leqslant x$,

$$R_p\Big(\frac{x}{p^\mu}, y\Big) \ll \frac{g_p(\alpha)}{p^{\mu\alpha}}\Big(\frac{1}{\overline{u}} + \frac{t^2\overline{u}}{u^2}\Big)$$

where $t := (\mu\log p)/\log y$. From the Cauchy–Schwarz inequality, we then get that the contribution arising from any fixed prime $p \leqslant y$ is

$$\begin{aligned}
&\ll \frac{1}{\overline{u}} \sum_{\nu\leqslant \frac{\log x}{\log p}} \frac{g_p(\alpha)|f(p^\nu)|}{p^{\nu\alpha}} \sum_{\mu\leqslant \frac{\log x}{\log p}} \frac{g_p(\alpha)|f(p^\mu)|}{p^{\mu\alpha}}\Big(1+\Big(\frac{\overline{u}\log p^\mu}{u\log y}\Big)^2\Big)\\
&\ll \frac{1}{\overline{u}} \sum_{\nu\leqslant \frac{\log x}{\log p}} \frac{g_p(\alpha)f(p^\nu)^2}{p^{\nu\alpha}}\Big(\sum_{\nu\leqslant \frac{\log x}{\log p}}\frac{g_p(\alpha)}{p^{\nu\alpha}}\sum_{\mu\leqslant \frac{\log x}{\log p}} g_p(\alpha)\frac{1+((\overline{u}\log p^\mu)/u\log y)^4}{p^{\mu\alpha}}\Big)^{1/2}\\
&\ll \frac{1}{\overline{u}} \sum_{\nu\leqslant \frac{\log x}{\log p}} \frac{g_p(\alpha)f(p^\nu)^2}{p^{\nu\alpha}}\Big(1+\frac{\overline{u}^4}{(u\alpha\log y)^4}\Big)^{1/2} \ll \frac{1}{\overline{u}} \sum_{\nu\leqslant \frac{\log x}{\log p}} \frac{g_p(\alpha)f(p^\nu)^2}{p^{\nu\alpha}}.
\end{aligned}$$

Summing over $p \leqslant y$, we get

$$U_f(x,y) \ll B_f(x,y)^2/\overline{u} \ll \mathbb{V}(Z_{f,x,y}).$$

This completes the proof.

References

[1] M.B. Barban & A.I. Vinogradov, On the number theoretic basis of probabilistic number theory. Dokl. Akad. Nauk SSSR 154 (1964), 495-496 ; = Soviet Math. Doklady 5 (1964), 96–98.

[2] R. de la Bretèche & G. Tenenbaum, Propriétés statistiques des entiers friables, *Ramanujan J.* **9** (2005), 139–202.

[3] R. de la Bretèche & G. Tenenbaum, Entiers friables : inégalité de Turán–Kubilius et applications, *Invent. Math.* **159** (2005), 531–588.

[4] P.D.T.A. Elliott, Probabilistic number theory : mean value theorems (Grundlehren der Math. Wiss. 239). Springer-Verlag, New York, Berlin, Heidelberg 1979.

[5] G. Hanrot, B. Martin & G. Tenenbaum, Constantes de Turán-Kubilius friables : étude numérique, *Exp. Math.* Experiment. Math. **19, n° 3** (2010), 345-361.

[6] A. Hildebrand , An asymptotic formula for the variance of an additive function, *Math. Z.* **183** (1983), 145–170.

[7] A. Hildebrand & G. Tenenbaum, On integers free of large prime factors, *Trans. Amer. Math. Soc.* **296** (1986), 265–290.

[8] J. Kubilius, Probabilistic methods in the theory of numbers. Uspehi Mat. Nauk (N.S.) 11 (1956), 2(68), 31–66 ; = Amer. Math. Soc. Translations, vol. 19 (1962), 47–85.

[9] J. Kubilius, Estimation of the central moment for strongly additive arithmetic functions (en russe, résumés en anglais et lituanien.), *Litovsk. Mat. Sb.* **23** (n° 1), 122–133.

[10] J. Kubilius, Estimate of the second central moment for any additive arithmetic functions (en russe, résumés en anglais et lituanien), *Litovsk. Mat. Sb.* **23** (n° 2), 110–117.

[11] P.D.T.A. Elliott, *Probabilistic methods in the theory of numbers*, Amer. Math. Soc. Monographs 11, Providence 1964.

[12] B. Martin & G. Tenenbaum, Sur l'inégalité de Turán-Kubilius friable, *J. reine angew. Math.* **647** (2010), 175-234.

[13] I.Z. Ruzsa, Effective results in probabilistic number theory, in: *Théorie élémentaire et analytique des nombres* (éd. par J. Coquet), 107–130. Dép. Math. Univ. Valenciennes 1982.

[14] *On the Turán–Kubilius inequality*, Technical Report n° 220 (1984), Stanford University.

[15] G. Tenenbaum, Crible d'Ératosthène et modèle de Kubilius, in: K. Győry, H. Iwaniec, J. Urbanowicz (eds.), *Number Theory in Progress*, Proceedings of the conference in honor of Andrzej Schinzel, Zakopane, Poland 1997, 1099–1129, Walter de Gruyter, Berlin, New York, 1999.

[16] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, troisième édition, coll. Échelles, Belin, Paris 2008, 592 pp.

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